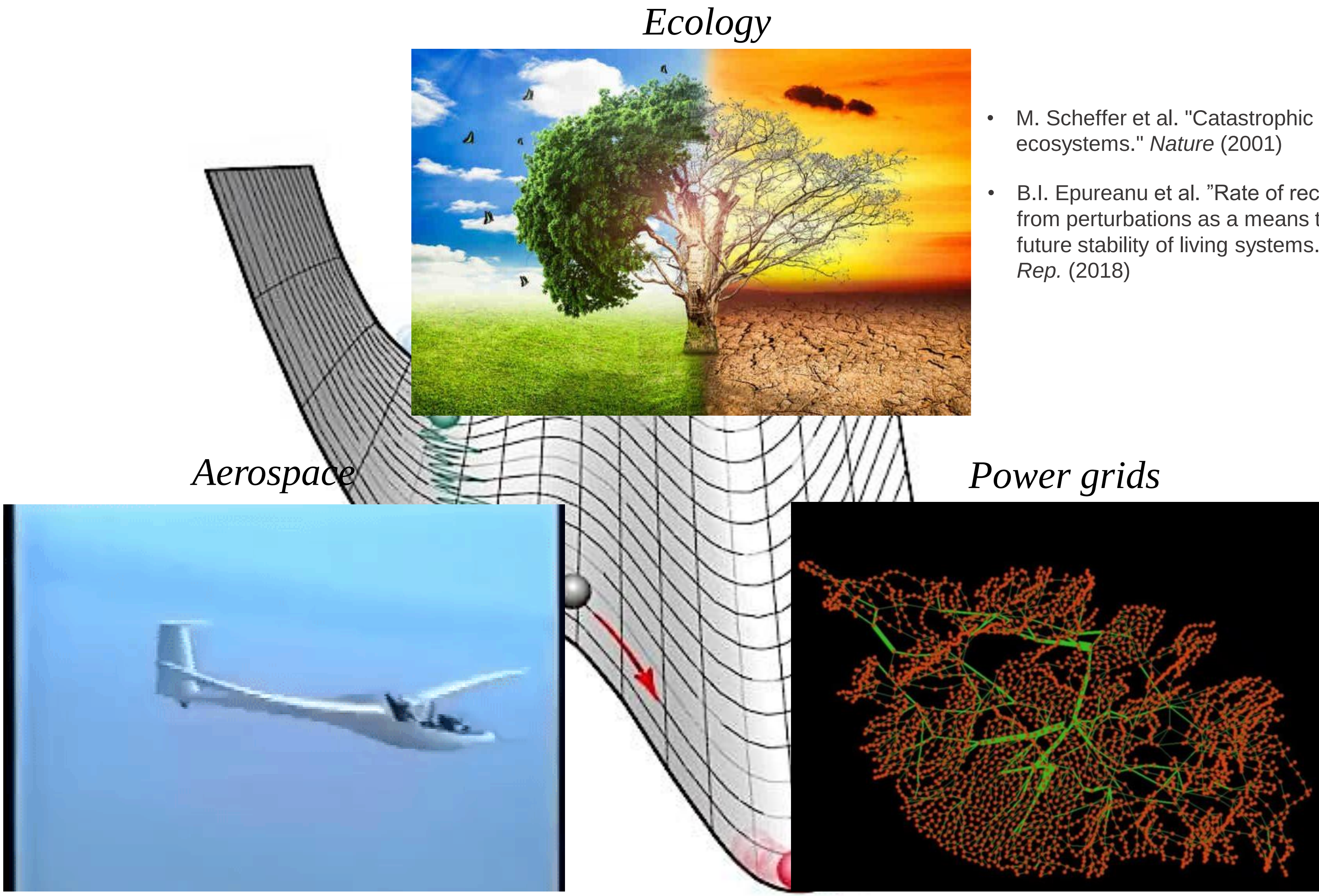
Deep Learning for Nonlinear Stability Analysis in Dynamical Systems

Amin Ghadami
Bogdan I. Epureanu

Department of Mechanical Engineering
University of Michigan, Ann Arbor

epureanu@umich.edu

Introduction: Bifurcations in Nonlinear Systems



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- D. Nedic et al. “Criticality in cascading failure blackout model.” *Int. J. Electr. Power Energy Syst*. (2006)
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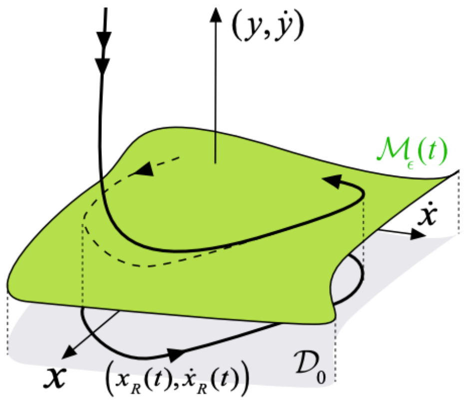
Motivation

- Characterizing nonlinear behavior near stability boundary
- Determination of the criticality of a bifurcation: catastrophic or non-catastrophic?
- Prediction and design of the dynamics

Traditional analytical methods

- Require knowledge of the governing equations
- Nonlinear analysis of the equations is very expensive
- Model errors in real-world applications can be very large



Can modern data-driven approaches address these challenges?

Nonlinear Stability Analysis of Dynamical Systems

Motivations

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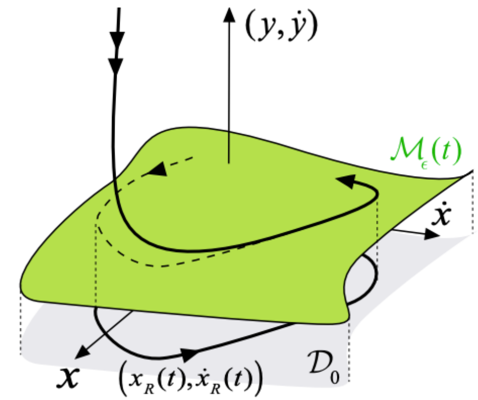

Traditional analytical methods

- Require knowledge of the governing equations
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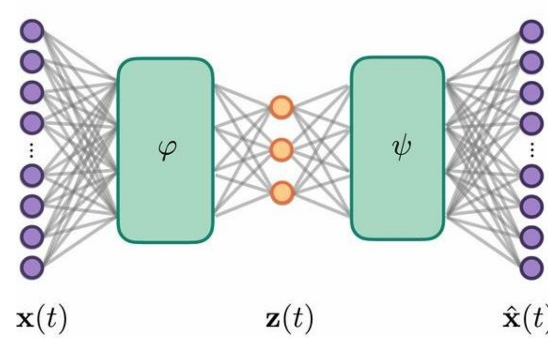
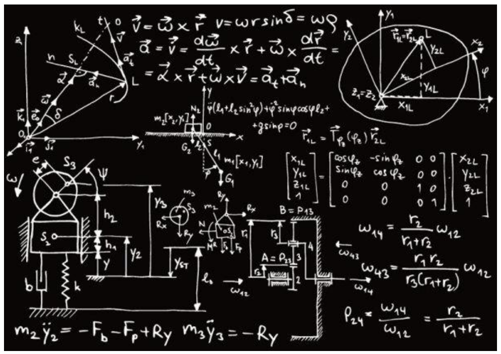
Data-driven approaches

- Typically focus on a single parameter regime
- Do not result in explicit parameterizations of bifurcations
- Require extensive amount of data



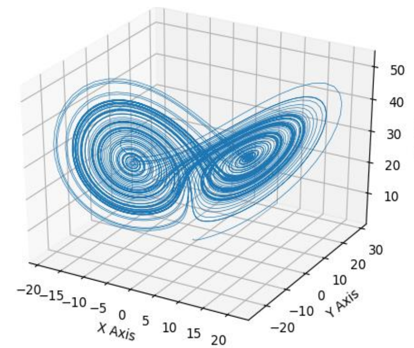
Nonlinear Stability Analysis of Dynamical Systems

Embedding invariants of bifurcation analysis into data-driven methods



Model-based

Mechanistic Data-driven Method



Data-driven

Inaccurate & costly

Data demanding

There exist **generic features** associated with bifurcations that can be incorporated into data-driven methods

Nonlinear Stability Analysis: Center manifold theorem

Center manifold: a low-dimensional invariant manifold on which the essential dynamics in the neighborhood of the critical point is determined

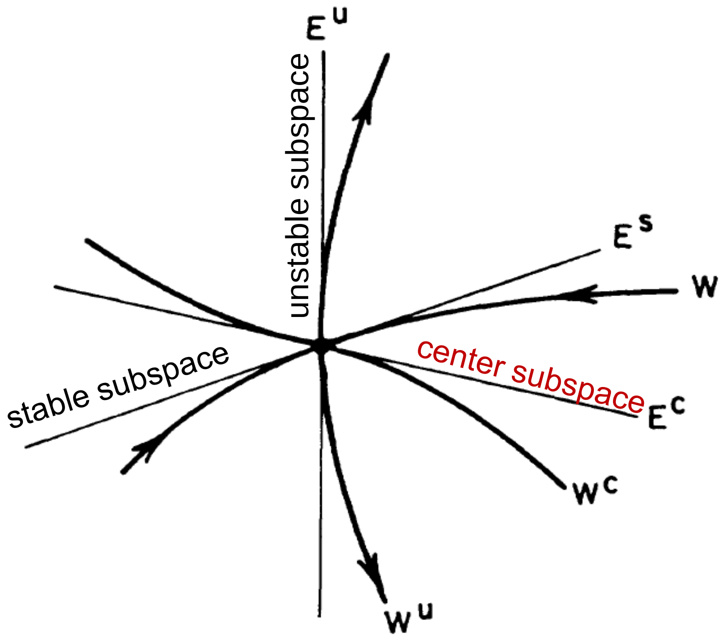
Center manifold reduction theorem:
The bifurcating system $\dot{x} = F(x, \mu)$ is locally topologically equivalent to

$$\begin{cases} \boxed{\dot{u} = f(u, \mu)} \\ \dot{v} = -v \\ \dot{w} = w \\ \boxed{\dot{\mu} = 0} \end{cases}$$

Reduced dominant vector field

$$(u, v, w) \in W^c \times W^s \times W^u$$

Extended theorem



Challenge: Systematic **nonlinear transformations are needed** to express nondominant states as functions of the dominant states

Center manifold theorem: Analytical approach

- Center manifold analysis starts by decomposing system dynamics

$$\dot{x}_1 = Ax_1 + f_1(x_1, x_2, \mu),$$
$$\dot{x}_2 = Bx_2 + f_2(x_1, x_2, \mu),$$
$$\dot{\mu} = 0$$

$$(x_1, x_2) \in R^{n_c} \times R^m$$
$$\mu \in R^m$$

$A_{n\times n}$ and $B_{m\times m}$ have eigenvalues with zero and non-zero real parts, respectively

- Center manifold approximation

$$W^c = \{(x_1, x_2, \mu) | x_2 = g(x_1, \mu)\}, \qquad g(0,0) = Dg(0,0) = 0$$

- Center manifold dynamics

$$\dot{x}_1 = Ax_1 + f_1(x_1, g(x_1, \mu), \mu),$$
$$\dot{y}_2 = \Lambda y_2,$$
$$\dot{\mu} = 0$$

➡ Reduced dominant vector field

➡ Off-manifold dynamics $y_2 = y_2(x_1, x_2)$

➡ Extended (with parameter) center manifold theorem

Center manifold theorem: Analytical approach

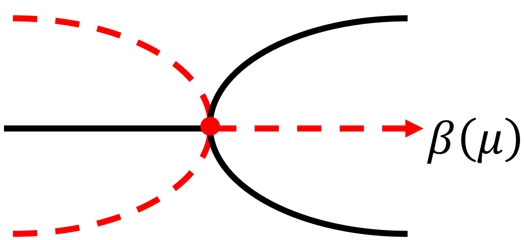
- Normal form transformation
  A coordinate change such that the system becomes "as simple as possible"

Normal form

$$\begin{cases} \dot{x}_1 = Ax_1 + f_1(x_1, g(x_1,\mu), \mu), \\ \dot{y}_2 = \Lambda y_2, \\ \dot{\mu} = 0. \end{cases} \xrightarrow{x_1 = \tilde{x}_1 + p(\tilde{x}_1,\mu)} \begin{cases} \dot{\tilde{x}}_1 = h(\tilde{x}_1, \mu), \\ \dot{y}_2 = \Lambda y_2, \\ \dot{\mu} = 0. \end{cases} \xrightarrow{y_1 = k(\tilde{x}_1,\mu)} \begin{cases} \dot{y}_1 = h(y_1, \beta(\mu)), \\ \dot{y}_2 = \Lambda y_2, \\ \dot{\mu} = 0. \end{cases}$$

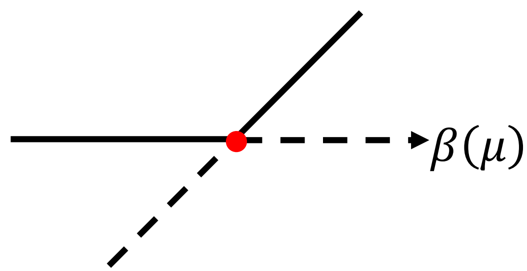
Pitchfork

$$h = \beta(\mu) y_1 \pm y_1^3$$



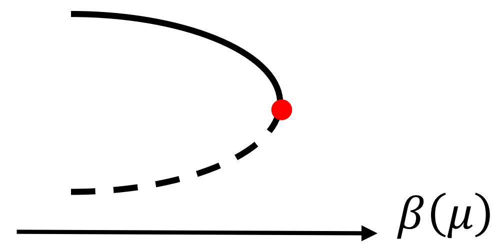
Transcritical

$$h = \beta(\mu) y_1 - y_1^2$$



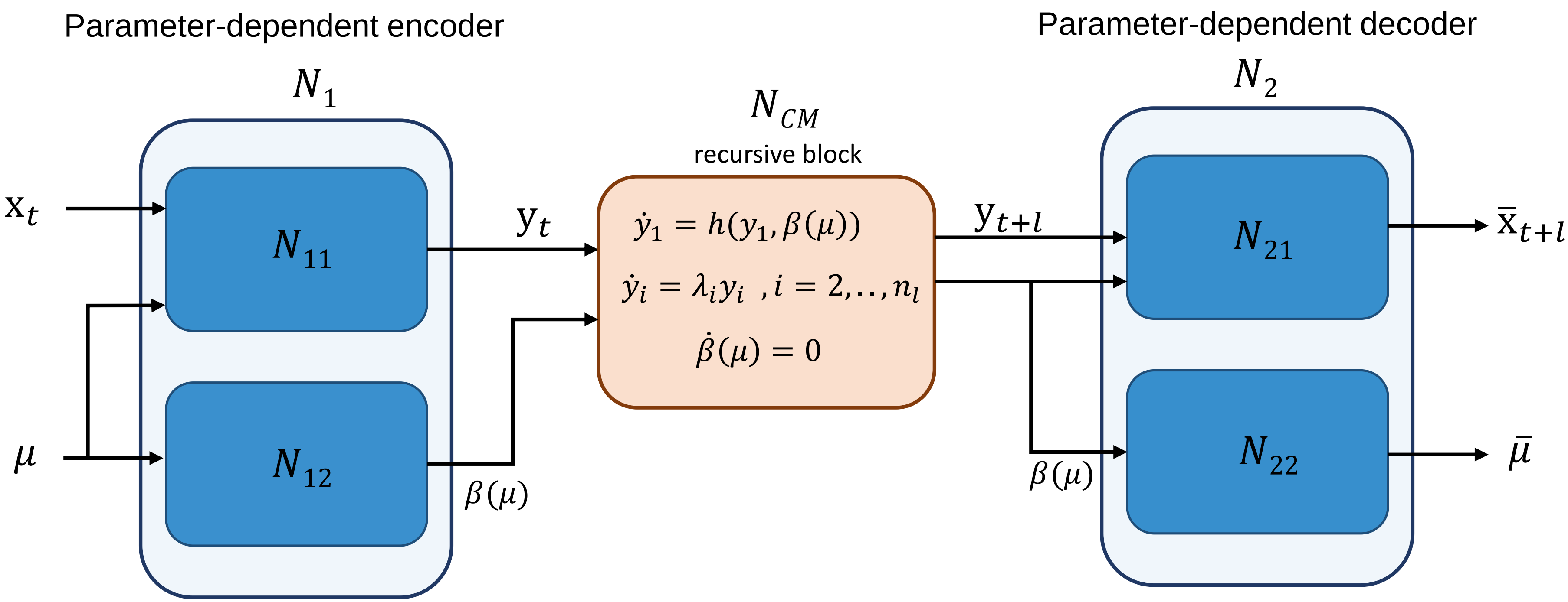
Fold

$$h = \beta(\mu) + y_1^2$$



How to perform this analysis for large-dimensional systems and/or inaccurate models?

Nonlinear Stability Analysis: A deep-learning approach



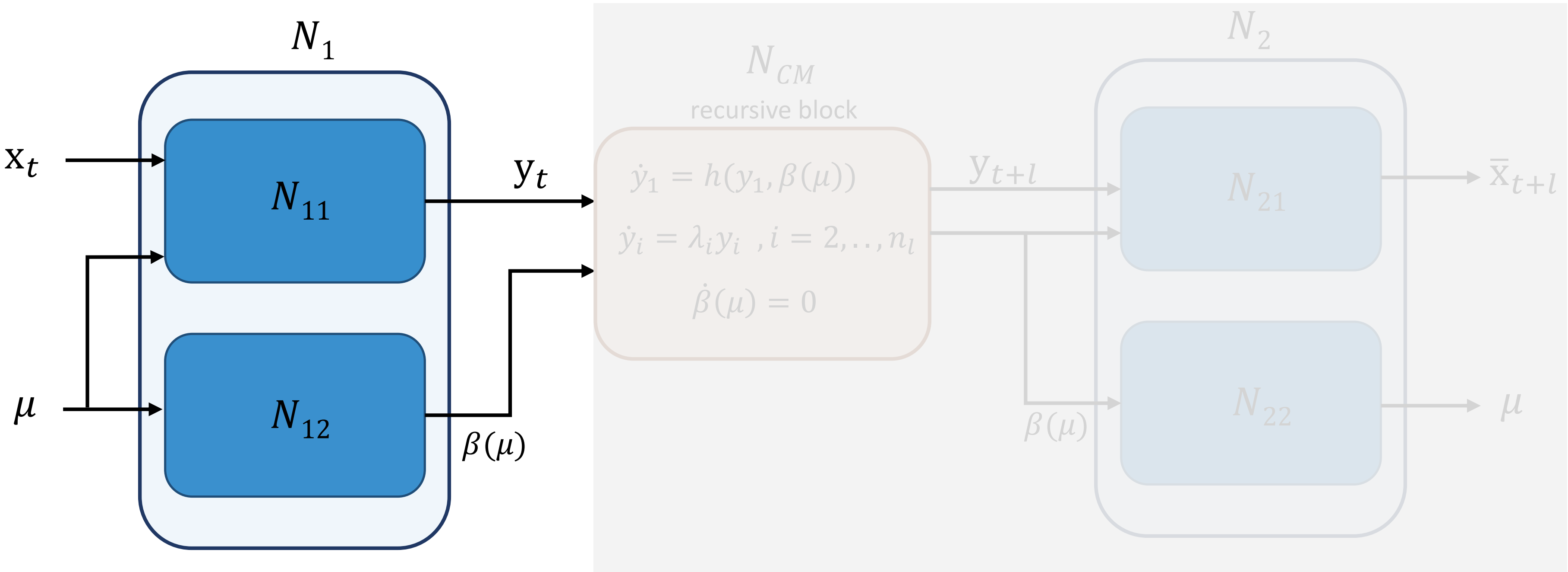
Nonlinear transformations to center manifold

CM theory-informed constraint

Nonlinear transformations to observation space

A. Ghadami & B.I.Epureanu. "Deep learning for centre manifold reduction and stability analysis in nonlinear systems." *Philos. Trans. Royal Soc. A* (2022)

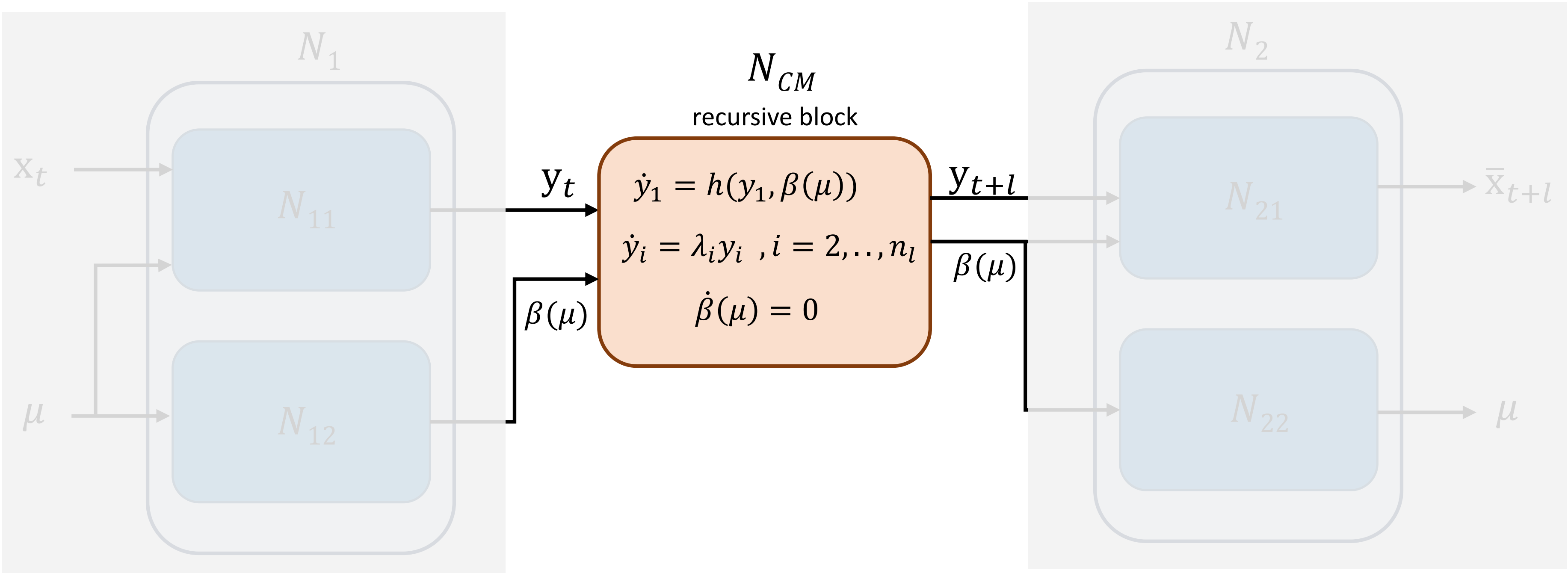
Nonlinear Stability Analysis: A deep-learning approach



Parameter-dependent encoder

- N_{11} simultaneously performs nonlinear center manifold approximation and normal form transformation
- N_{11} is a function of both system states and parameters due to the extended center manifold theorem
- N_{12} maps the physical parameter to the effective bifurcation parameter $\beta(\mu)$

Nonlinear Stability Analysis: A deep-learning approach



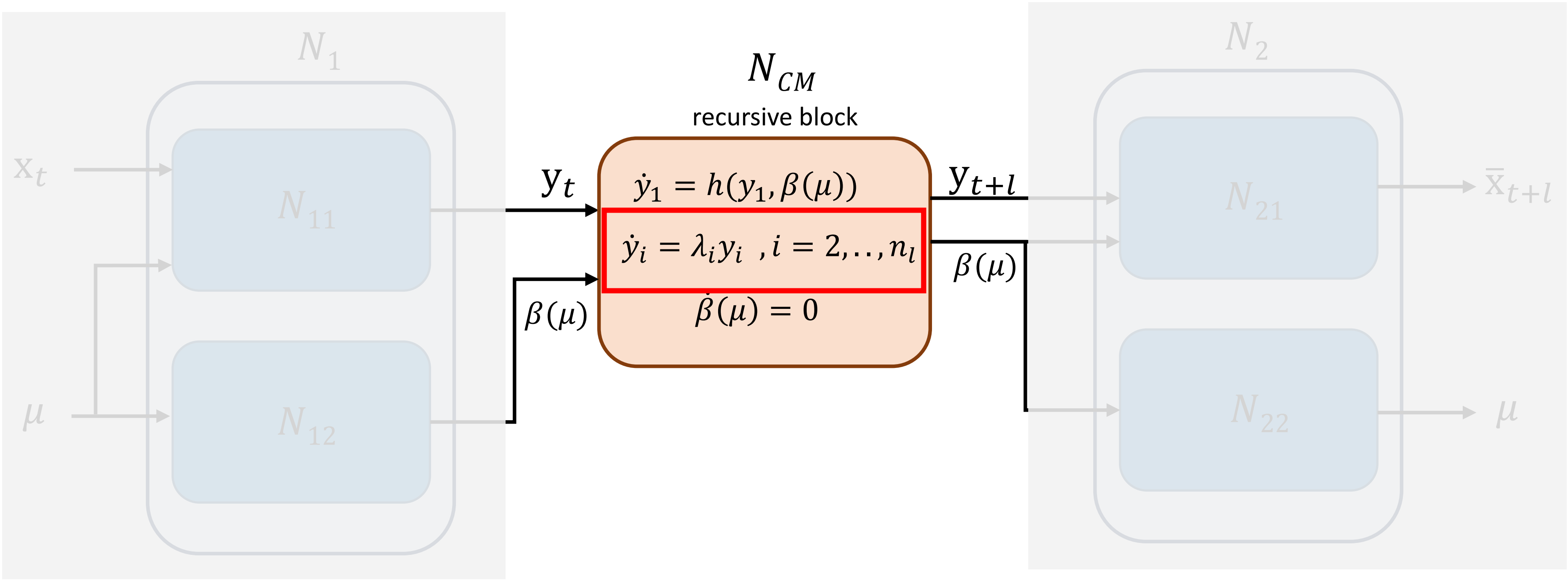
Center manifold block

- Applies constraint on the latent space dynamics
- Forwards the reduced dynamics in time according to the normal forms of bifurcations
- Searches and finds the best normal form that fits the data while optimizing its parameters

e.g., $h = \beta(\mu)y_1 + wy_1^3$

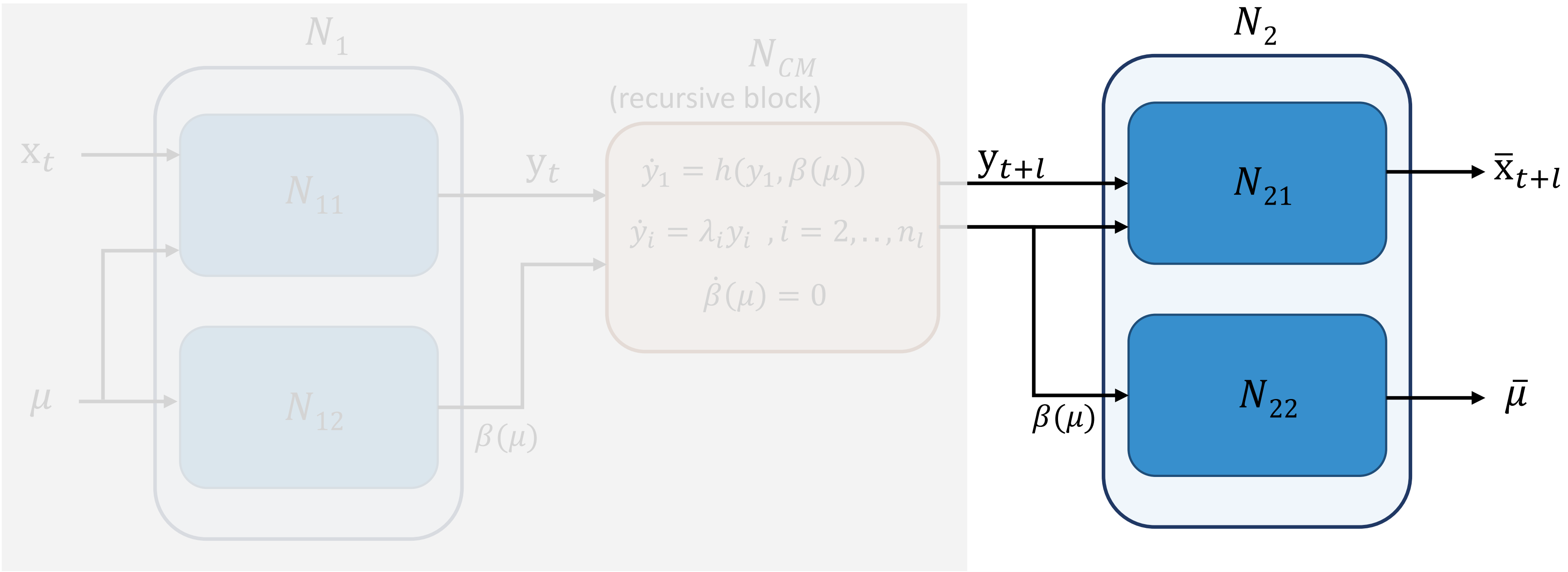
w = 1, subcritical pitchfork
w = -1, supercritical pitchfork

Nonlinear Stability Analysis: A deep-learning approach



Center manifold block

- Applies constraint on the latent space dynamics
- Forwards the reduced dynamics in time according to the normal forms of bifurcations
- Searches and finds the best normal form that fits the data while optimizing its parameters
- Transient dynamics are captured by off-manifold equations in the latent space

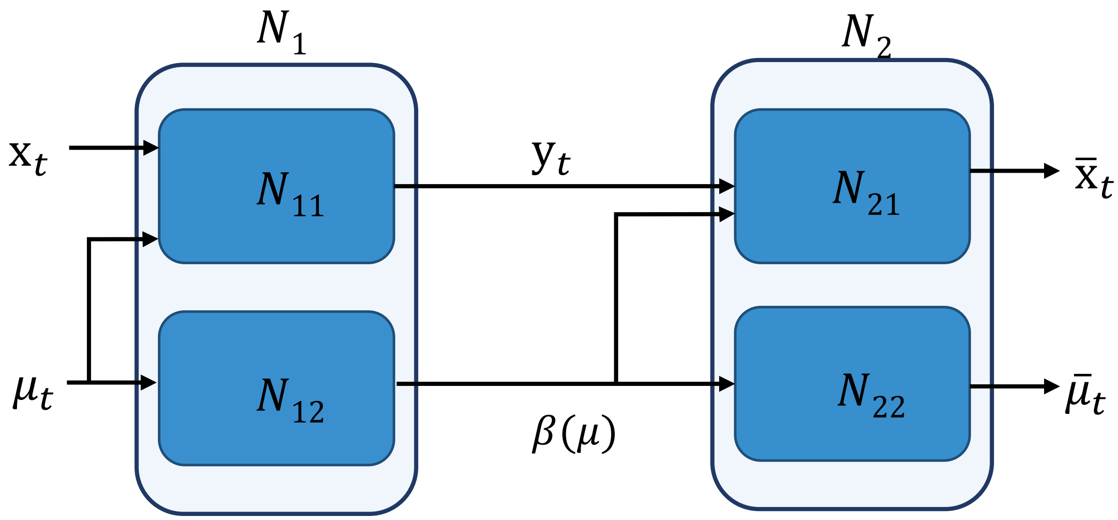


Parameter-dependent decoder

- Transforms the latent space dynamics back to the observation coordinates
- Setting $\beta(\mu) = 0$ and $y_i = 0, \ i = 2, \dots, n_l$, the transformation identifies center manifold of the dynamics in the observation space

Training procedure

1- Autoencoder loss: enforcing reconstruction

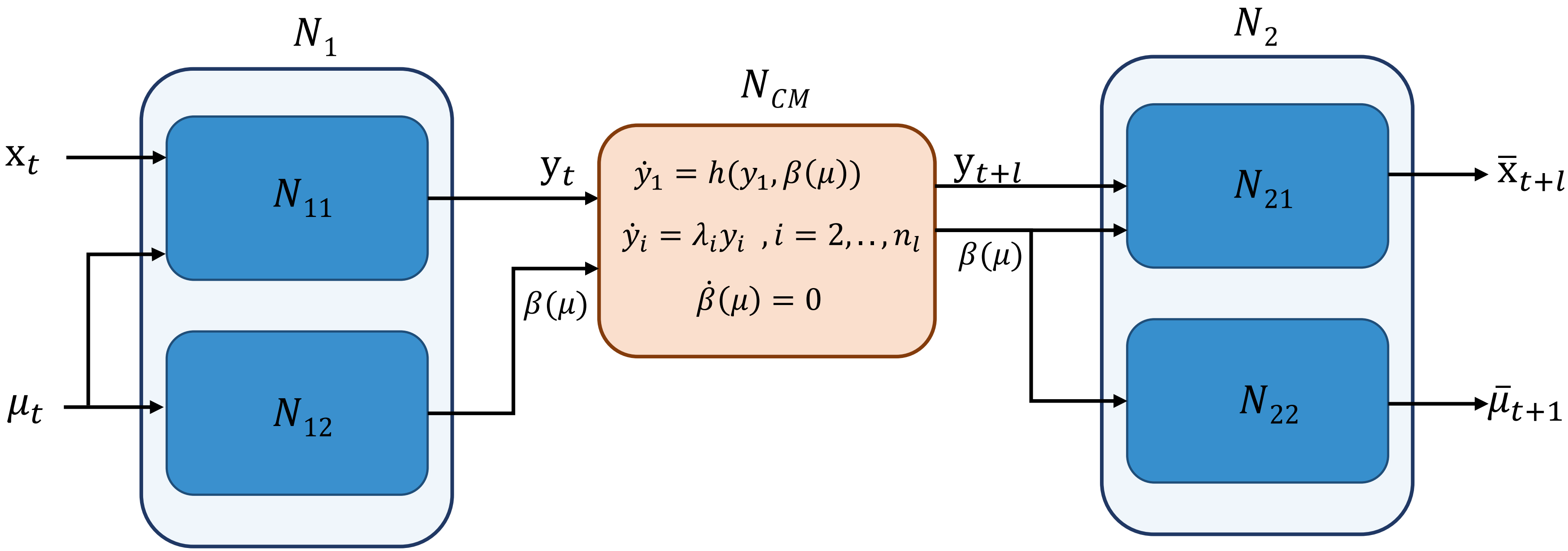


$$L_1 = \frac{1}{N}\sum_{k=1}^{N}\left(\left\|\mathrm{x}_{t_k} - \bar{\mathrm{x}}_{t_k}\right\|^2 + \left\|\mu_{t_k} - \bar{\mu}_{t_k}\right\|^2\right) =$$

$$\frac{1}{N}\sum_{k=1}^{N}\left(\left\|\mathrm{x}_{t_k} - N_{21}(N_{11}(\mathrm{x}_{t_k}, \mu), \beta)\right\|^2 + \left\|\mu_{t_k} - N_{22}\left(N_{12}(\mu_{t_k})\right)\right\|^2\right)$$

Training procedure

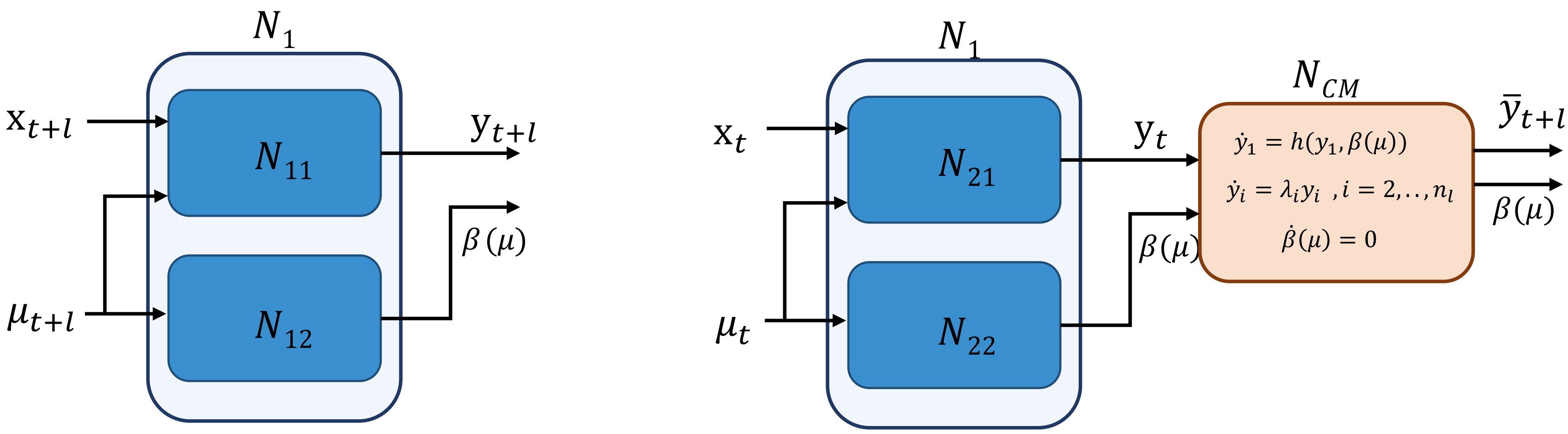
2- State prediction loss: enforcing predictability



$$L_2 = \frac{1}{N}\left(\sum_{k=1}^{N}\left\|x_{t_{k+l}} - \bar{x}_{t_{k+l}}\right\|^2\right) = \frac{1}{N}\left(\sum_{k=1}^{N}\left\|x_{t_{k+l}} - N_2\left(N_{CM}\left(N_1\left(x_{t_k}, \mu\right)\right), \beta\right)\right\|^2\right)$$

Training procedure

3- Code prediction loss: enforcing sequentiality



$$L_3 = \frac{1}{N}\left(\sum_{k=1}^{N}\left\|\mathrm{y}_{t_{k+l}} - \bar{y}_{t_{k+l}}\right\|^2\right) = \frac{1}{N}\left(\sum_{k=1}^{N}\left\|N_1(\mathrm{x}_{t_{k+l}}, \mu) - N_{CM}\big(N_1(\mathrm{x}_{t_k}, \mu), \beta\big)\right\|^2\right)$$

Training procedure: summary

$$Loss = \frac{1}{N}\left(\sum_{k=1}^{N}\left\|\mathrm{x}_{t_k} - \bar{\mathrm{x}}_{t_k}\right\|^2\right)$$

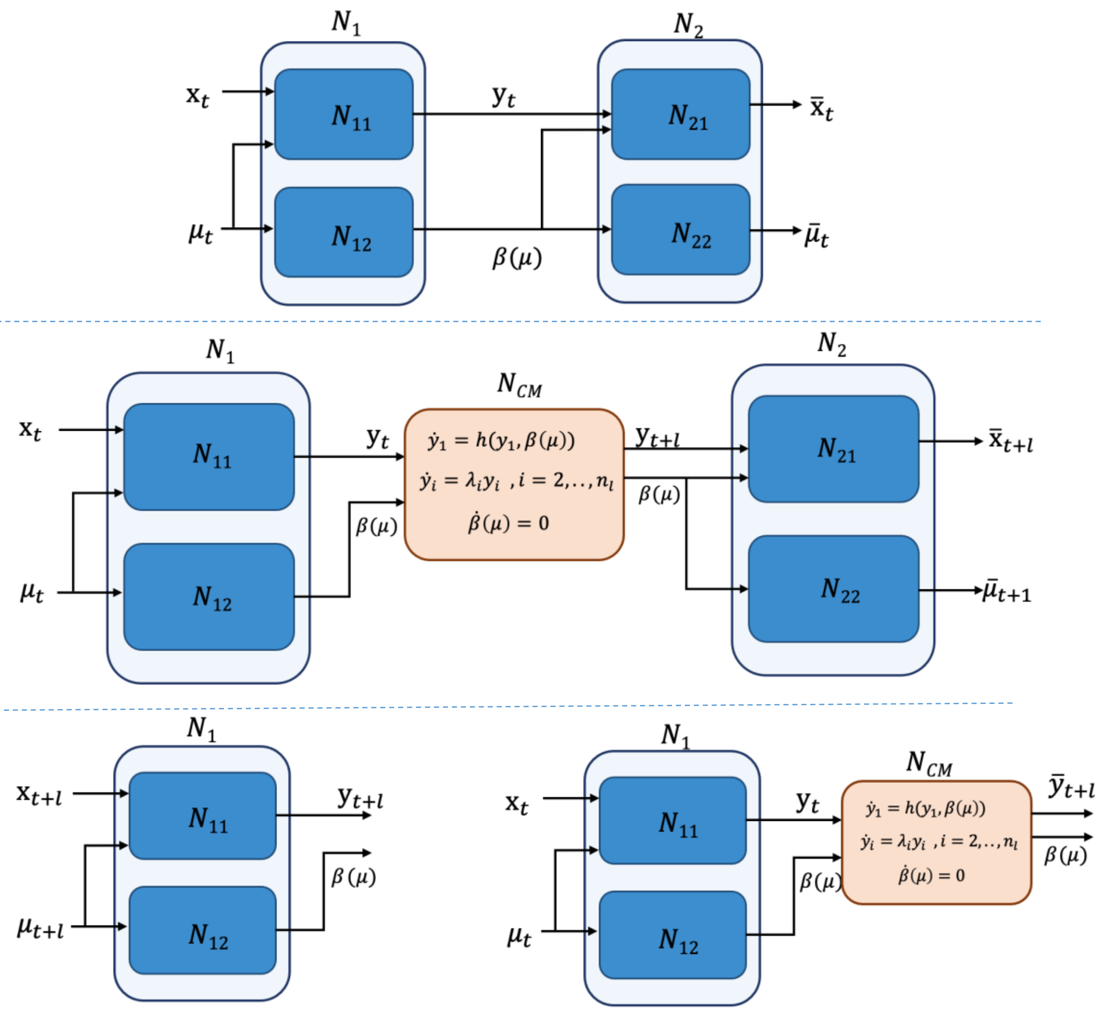
$$+$$

$$\frac{1}{N}\left(\sum_{k=1}^{N}\left\|\mathrm{x}_{t_{k+l}} - \bar{\mathrm{x}}_{t_{k+l}}\right\|^2\right)$$

$$+$$

$$\gamma\,\frac{1}{N}\left(\sum_{k=1}^{N}\left\|\mathrm{y}_{t_{k+l}} - \bar{y}_{t_{k+l}}\right\|^2\right)$$

Adjusting the latent space penalty



Deep Learning for Nonlinear Stability Analysis: Example

- **Lorenz equation**

$$\dot{x}_1 = -\sigma(x_1 - x_2)$$
$$\dot{x}_2 = x_1(\mu - x_3) - x_2$$
$$x_3 = x_1 x_2 - b x_3$$

- Center manifold dynamics

$\beta(\mu)$ instability type

$$\dot{y}_1 = \frac{\sigma}{\sigma + 1}(\mu - 1) y_1 - y_1^3 + O(4)$$
$$\dot{y}_2 = -b y_2$$
$$\dot{y}_3 = -(\sigma + 1) y_3$$

- Center manifold approximation

$$x_2 = x_1 + \frac{1}{\sigma + 1} x_1(\mu - 1) - \frac{1}{b(\sigma + 1)} x_1^3 - \frac{\sigma}{(\sigma + 1)^3} x_1(\mu - 1)^2 + O(4)$$
$$x_3 = \frac{1}{b} x_1^2 + \frac{2\sigma}{b(\sigma+1)} x_1^2(\mu - 1) + O(4)$$

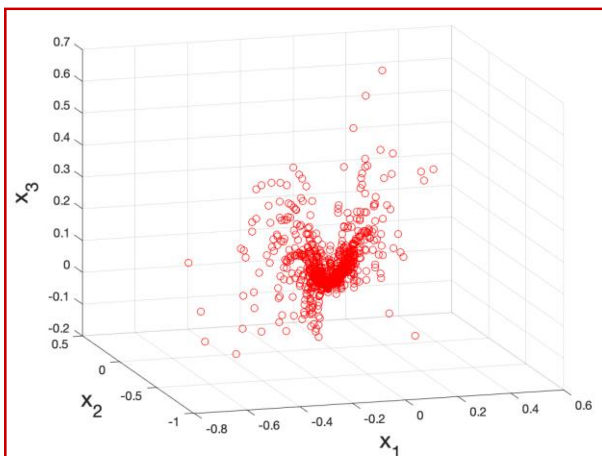
Nonlinear Stability Analysis: Lorenz system

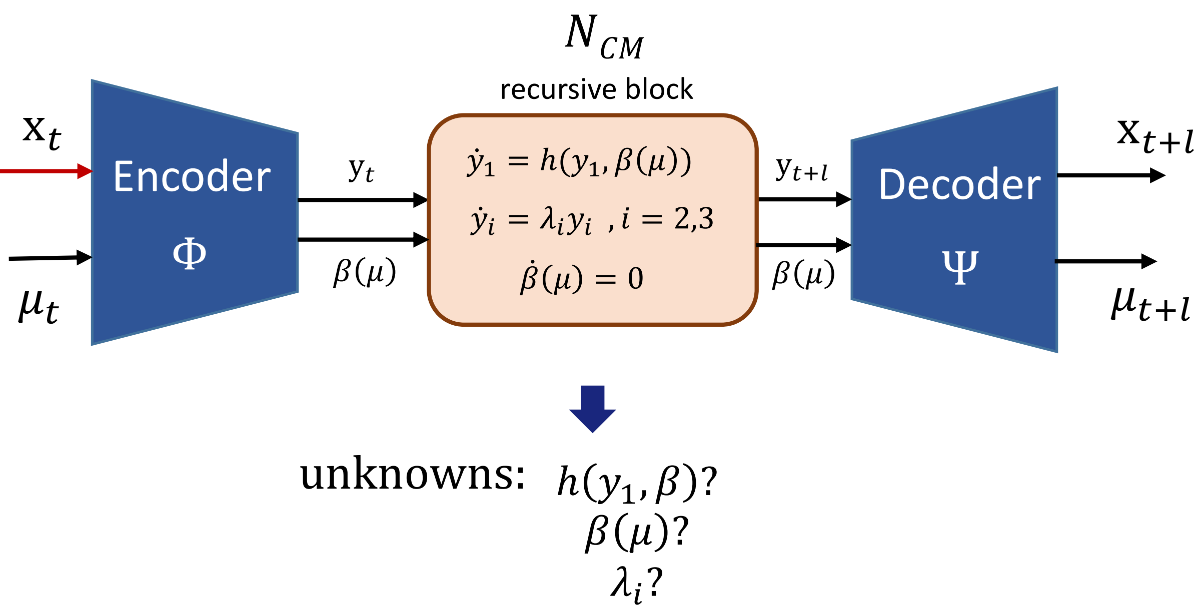
- **Lorenz equation**

$$
\begin{aligned}
\dot{x}_1 &= -4(x_1 - x_2)\\
\dot{x}_2 &= x_1(\mu - x_3) - x_2\\
\dot{x}_3 &= x_1x_2 - 0.25x_3
\end{aligned}
$$

- All measurements are recorded before instability, i.e. $0 \ll \mu < 1$

Randomly sampled dynamics





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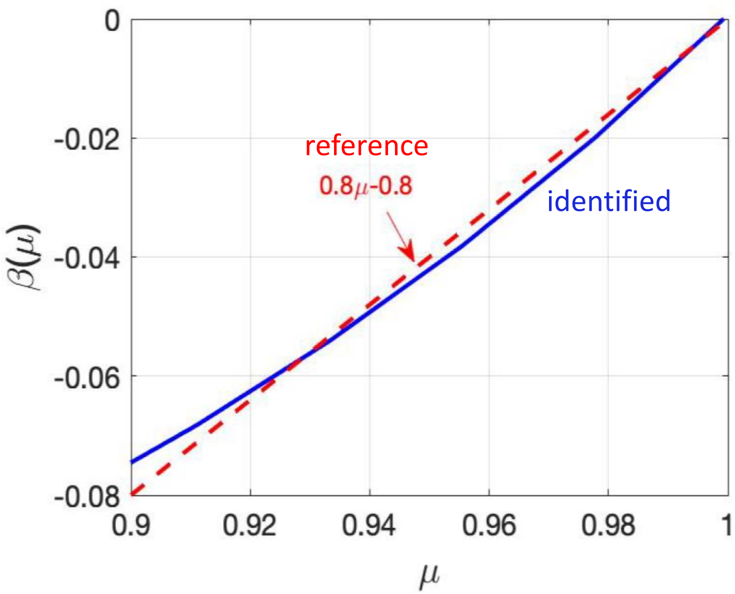
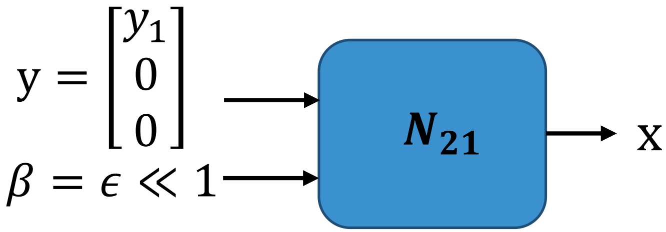
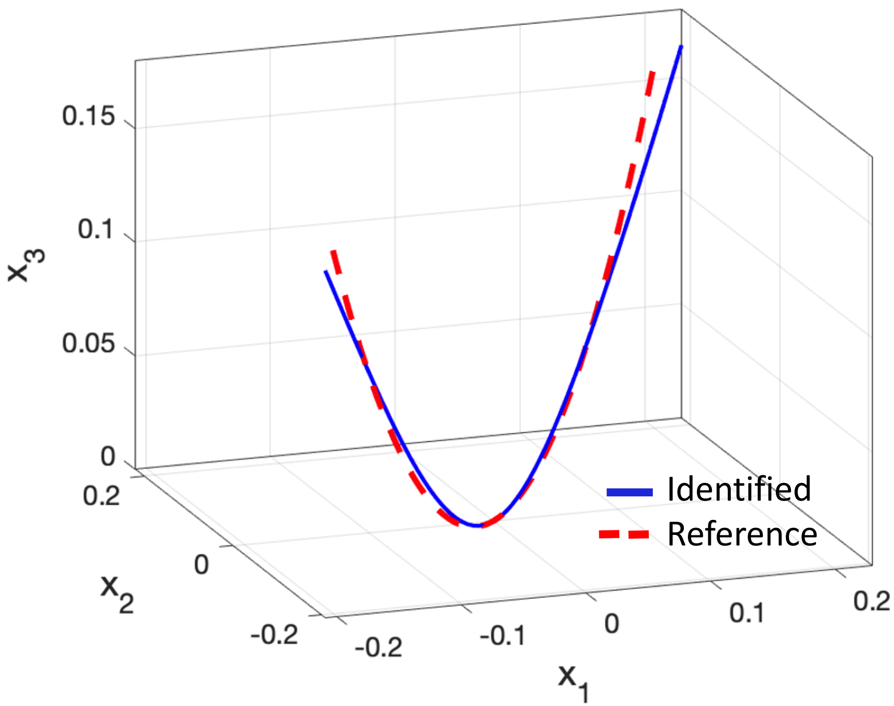


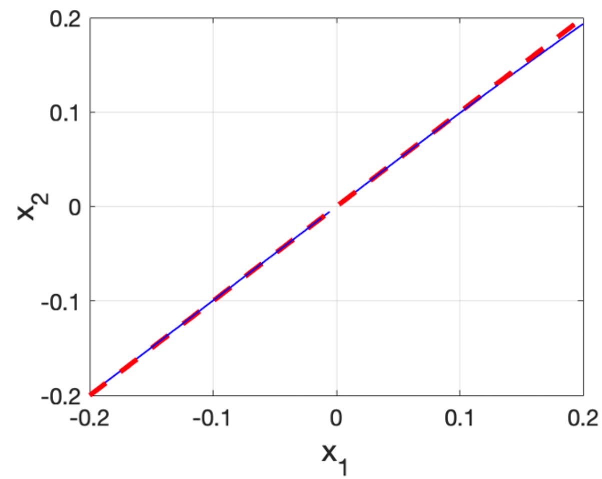
Reference latent space dynamics (CM theory)	Identified latent space dynamics
$\dot{y}_1 = 0.8(\mu - 1)y_1 - y_1^3$ $\dot{y}_2 = -0.25y_2$ $\dot{y}_3 = -5y_3$ Supercritical pitchfork with one center manifold and two stable manifolds	$\dot{\tilde{y}}_1 = \beta(\mu)\tilde{y}_1 - \tilde{y}_1^3$ $\dot{\tilde{y}}_2 = -0.23\tilde{y}_2$ $\dot{\tilde{y}}_3 = -4.14\tilde{y}_3$ Supercritical pitchfork with one center manifold and two stable manifolds

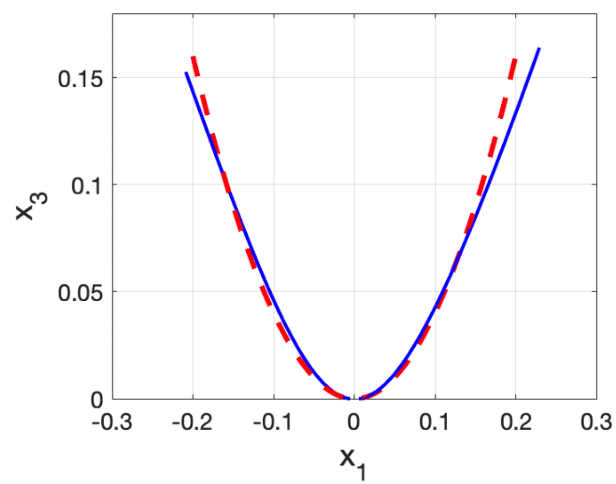
Nonlinear Stability Analysis: Lorenz system

- **Center manifold identification**









Nonlinear Stability Analysis: Large-dimensional observations

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- System dynamics

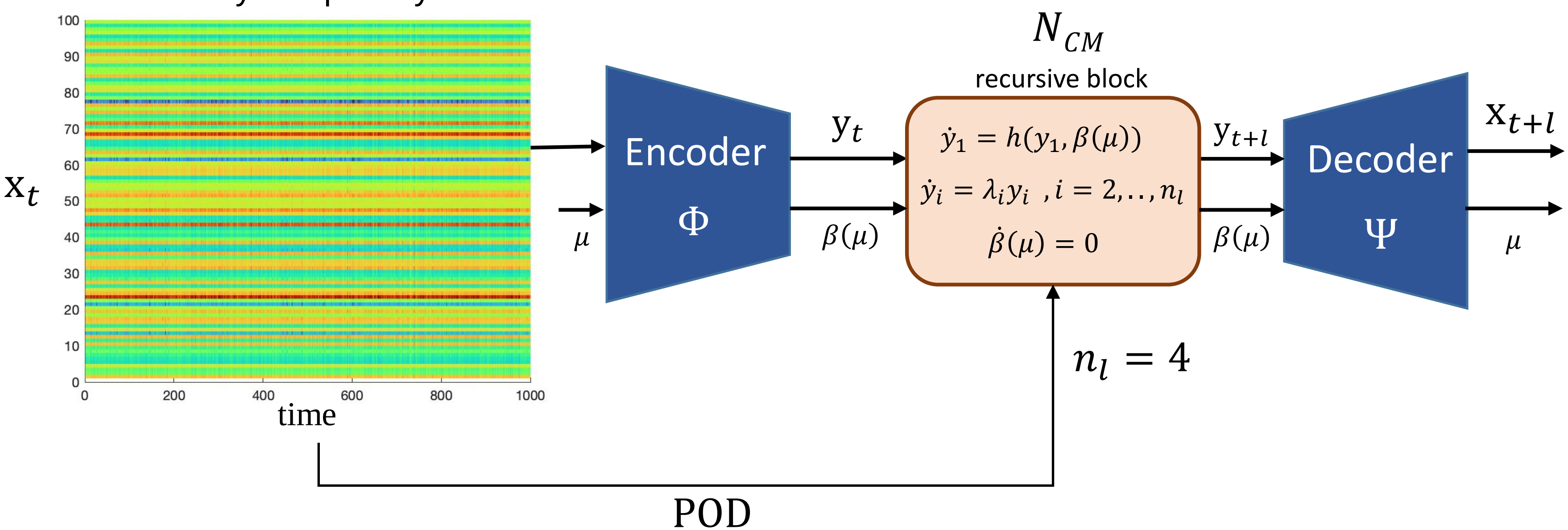
$$\begin{aligned}
\dot{z}_1 &= 6\mu z_1 + 10\mu z_2 - 2a z_1 z_2 - 2a z_2^2, \\
\dot{z}_2 &= 2z_2 + b\mu - 5\mu z_1 - 9\mu z_2 + a z_1^2 + 4a z_1 z_2 + 3a z_2^2, \\
\dot{z}_3 &= -0.575 z_3 + 0.425 z_4, \\
\dot{z}_4 &= 0.425 z_3 - 0.575 z_4.
\end{aligned}$$

Eigenvalues at $\mu = 0$
$\lambda_{1,2} = 0, -2$

Constant eigenvalues
$\lambda_{3,4} = -0.15, -1$

- Large dimensional observation vector $\quad \mathrm{x} = \mathrm{p}_1 z_1 + \mathrm{p}_2 z_2 + \cdots + \mathrm{p}_6 z_1 z_3, \quad \mathrm{p}_i \in \mathrm{R}^{100}$

Randomly sampled dynamics



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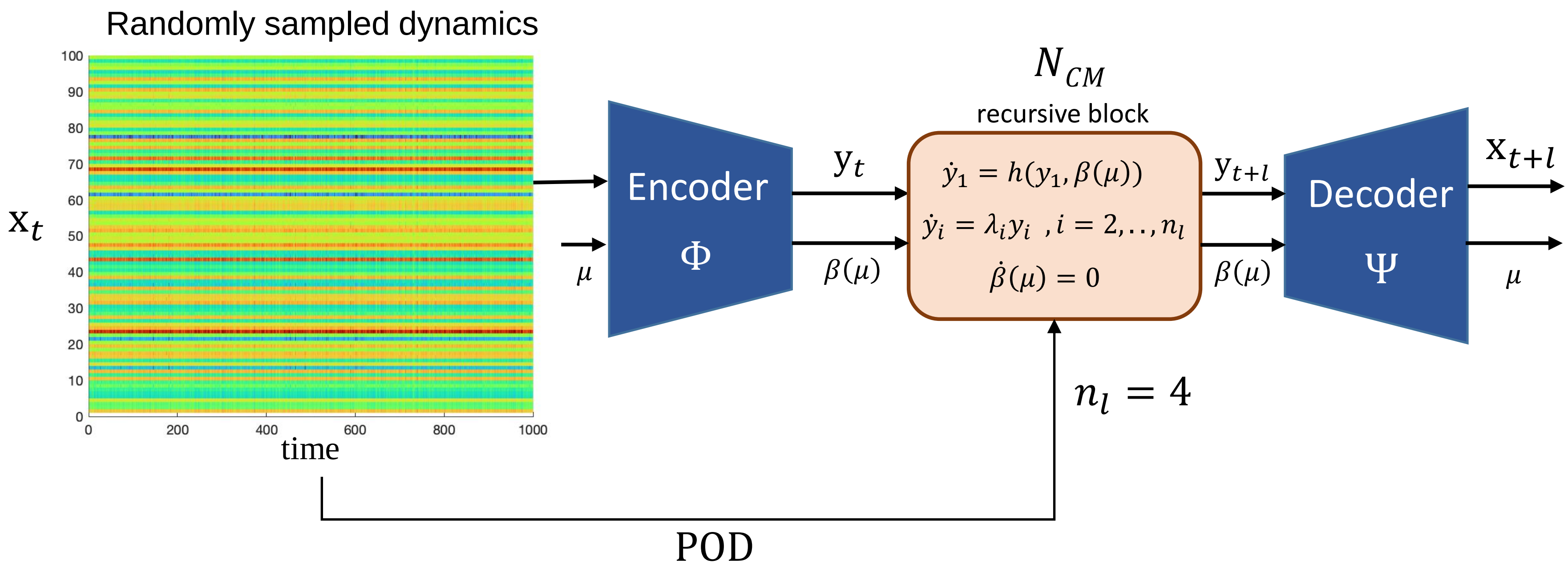
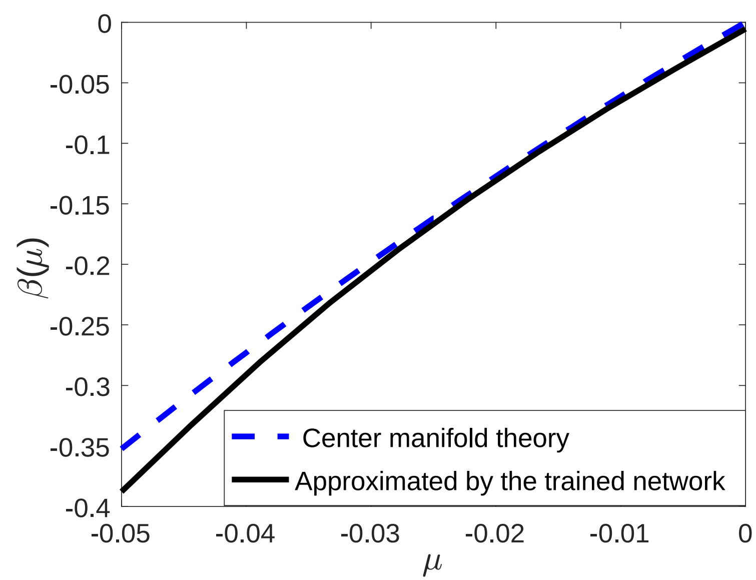
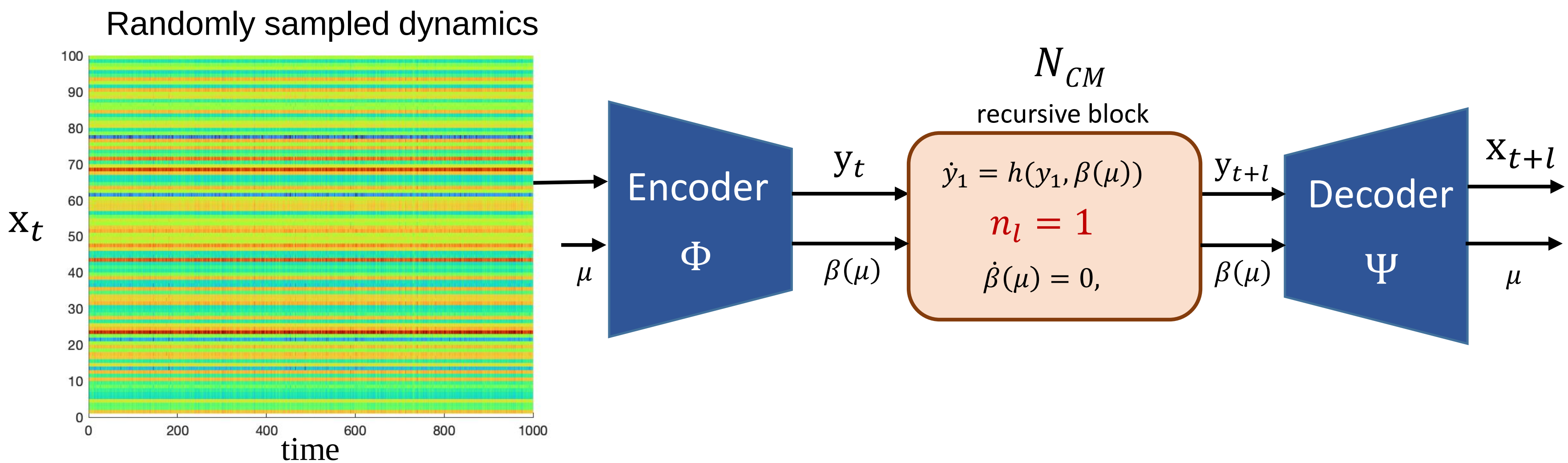
Nonlinear Stability Analysis: Large-dimensional observations



Reference dynamics	Identified dynamics (latent space dimension $n_l = 4$)
$\dot{y}_1 = \beta(\mu) y_1 - y_1^3$ $\dot{y}_2 = -0.15 y_2$ $\dot{y}_3 = -y_3$ $\dot{y}_4 = -2 y_4$ **Bifurcation type:** Supercritical pitchfork bifurcation	$\dot{\tilde{y}}_1 = \beta(\mu)\tilde{y}_1 - \tilde{y}_1^3$ $\dot{\tilde{y}}_2 = -0.148 \tilde{y}_2$ $\dot{\tilde{y}}_3 = -0.955 \tilde{y}_3$ $\dot{\tilde{y}}_4 = -1.967 \tilde{y}_4$ **Bifurcation type:** Supercritical pitchfork bifurcation



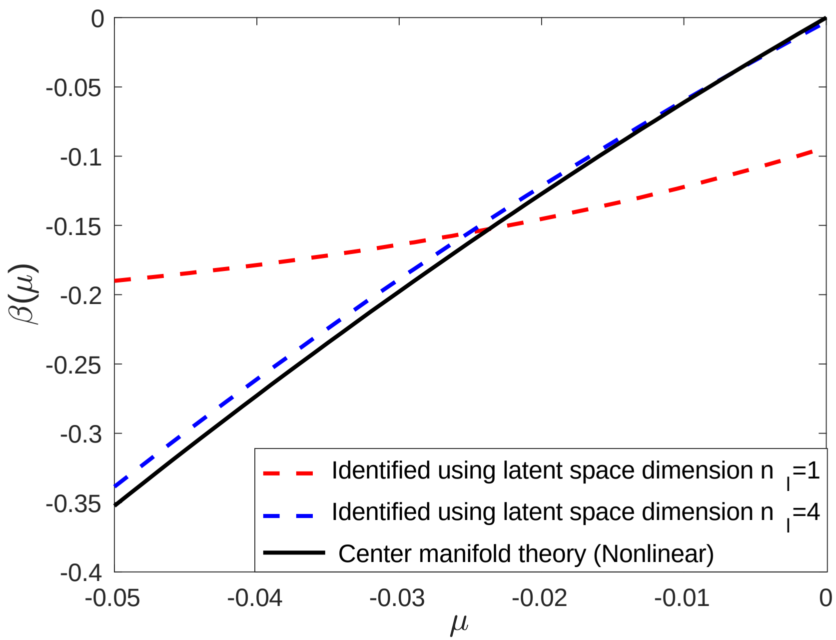
Nonlinear Stability Analysis: Large-dimensional observations

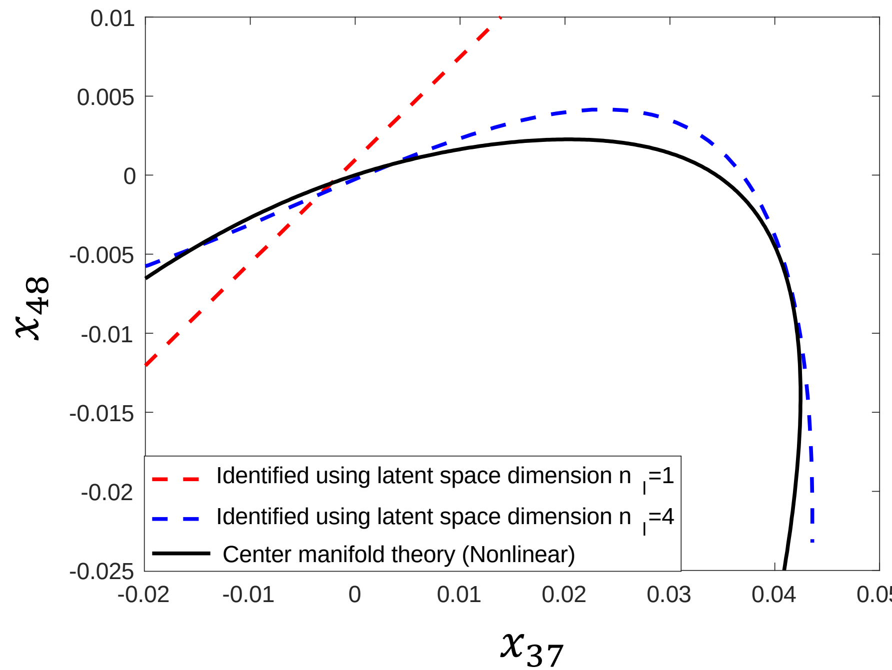


Reference dynamics	Identified dynamics (latent space dimension $n_l = 4$)	Identified dynamics (latent space dimension $n_l = 1$)
$\dot{y}_1 = \beta(\mu) y_1 - y_1^3$ $\dot{y}_2 = -0.15 y_2$ $\dot{y}_3 = -y_3$ $\dot{y}_4 = -2 y_4$	$\dot{\tilde{y}}_1 = \beta(\mu)\tilde{y}_1 - \tilde{y}_1^3$ $\dot{\tilde{y}}_2 = -0.148\tilde{y}_2$ $\dot{\tilde{y}}_3 = -0.955\tilde{y}_3$ $\dot{\tilde{y}}_4 = -1.967\tilde{y}_4$ Error on test dataset: 2.96 x 10^{-6}	$\dot{\tilde{y}}_1 = \beta(\mu)\tilde{y}_1 - \tilde{y}_1^3$ Error on test dataset: 2.11 x 10^{-3}
Bifurcation type: Supercritical pitchfork bifurcation	**Bifurcation type:** Supercritical pitchfork bifurcation	**Bifurcation type:** Supercritical pitchfork bifurcation

Nonlinear Stability Analysis: Large-dimensional observations

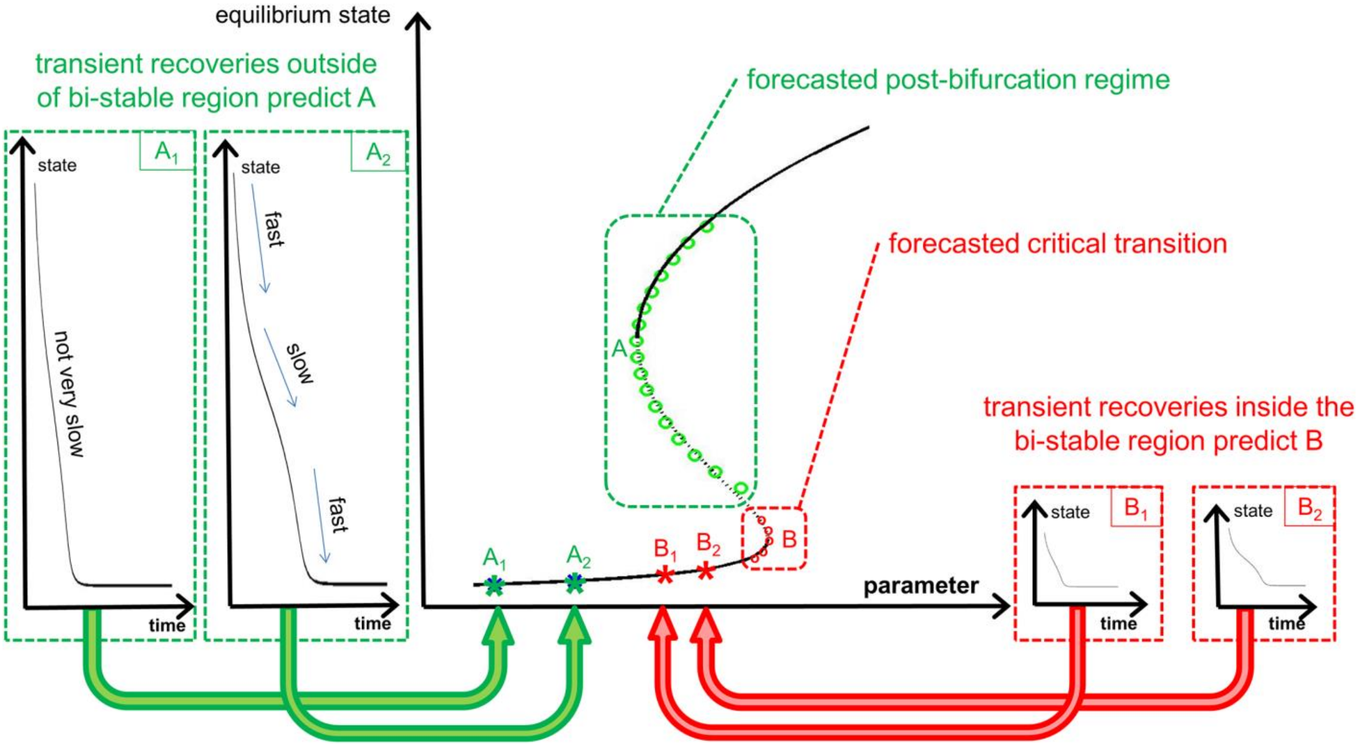
- Identified latent space dynamics and center manifold for different choices of latent space dimensionality





Deep Learning and Forecasting Bifurcations

- **Motivation:** Predicting critical transitions and bifurcation diagrams from safe measurements
- **Bifurcation forecasting method:** rate of recovery from perturbations is an indicator and is correlated to distance to bifurcations



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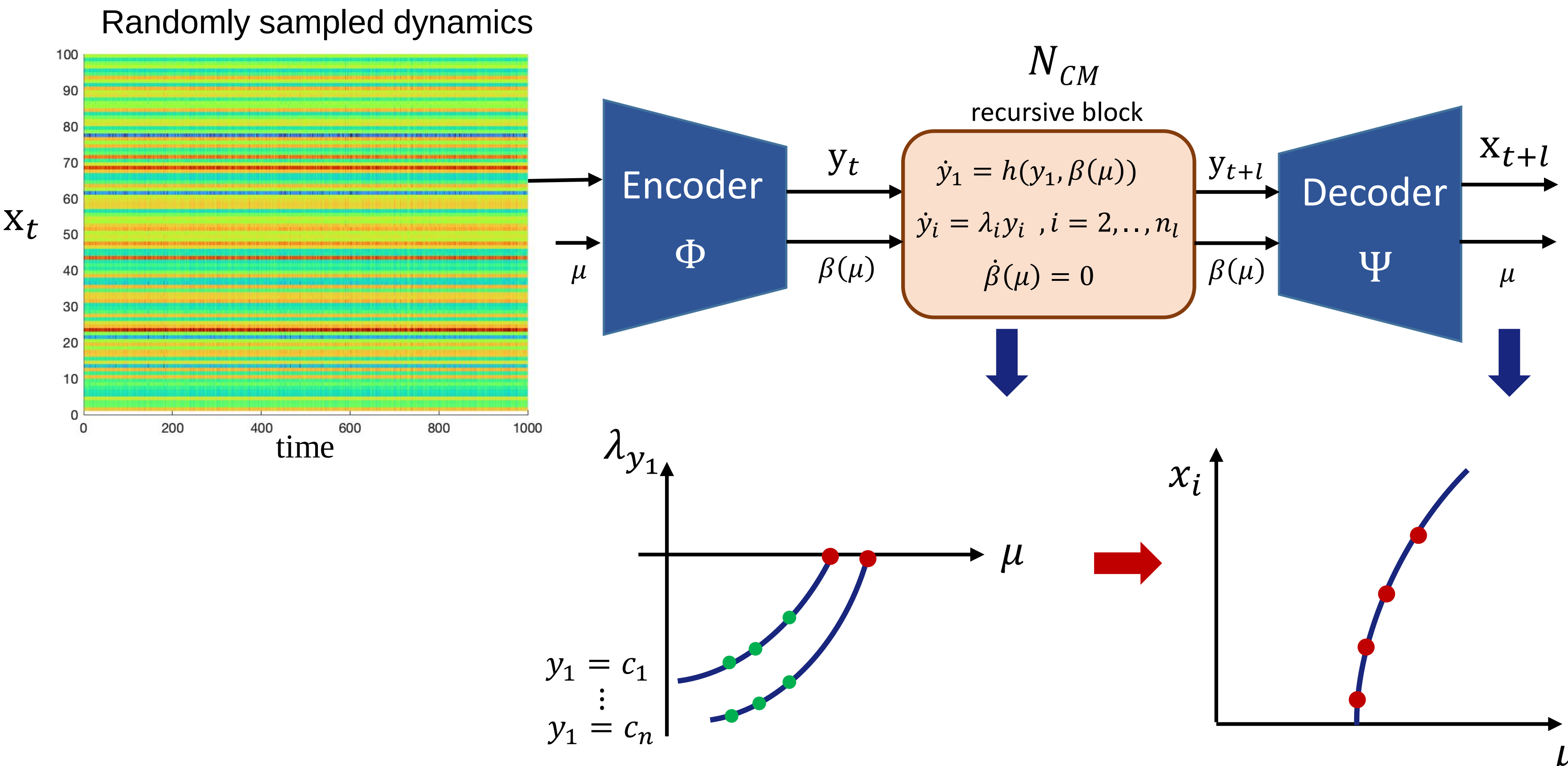
- B. I. Epureanu et al. "Forecasting bifurcations from large perturbation recoveries in feedback ecosystems." *PloS One* (2015)
- A. Ghadami & B. I. Epureanu. "Bifurcation forecasting for large dimensional oscillatory systems." *J. Comput. Nonlinear Dyn.* (2016).
- B.I. Epureanu et al. "Rate of recovery from perturbations as a means to forecast future stability of living systems." *Sci. Rep.* (2018)

- **Challenge:**
  - Recovery rates must be identified on manifold
  - Approximated recovery rates might be affected by the observations
  - Sparce and random sampling affect the recovery rate approximations
- **Deep learning for approximating the recovery rates**



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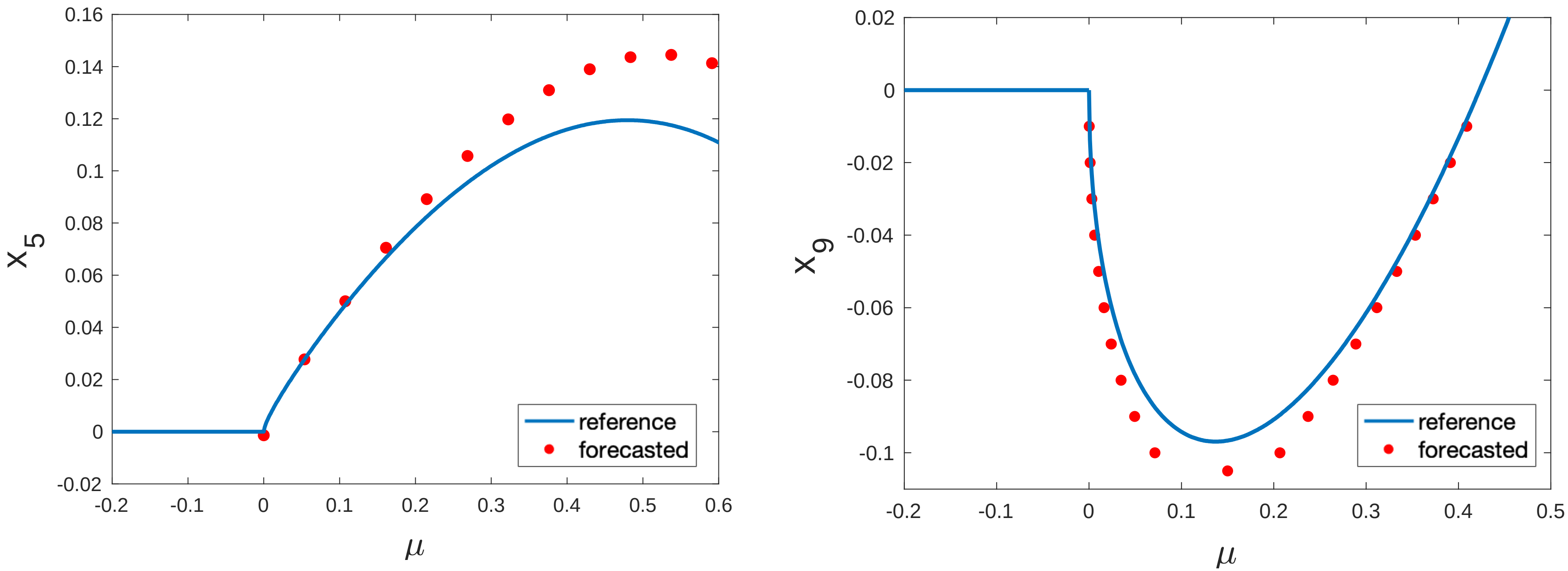
Example

Low-dimensional dynamics $\dot{y} = \mu y - y^3$

Large-dimensional observations $\mathrm{x} = \mathrm{p}_1 y + \mathrm{p}_2(\mu y + y^2) + \mathrm{p}_3(\mu^2 y + y^3),$ $\mathrm{p}_i \in \mathbb{R}^{20}$

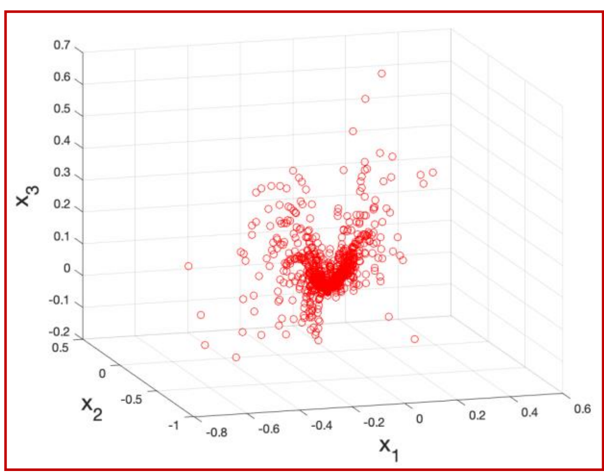


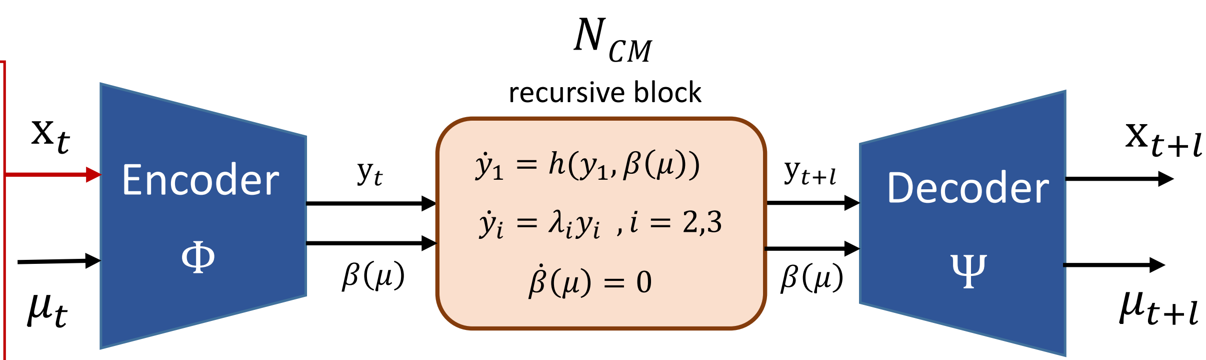
A. Ghadami & B.I. Epureanu, "A deep-learning approach to construct bifurcation diagrams in nonlinear dynamical systems", in preparation.

Summary

- Deep learning approach for bifurcation analysis in dynamical systems
- Rooted in center manifold theory
- Identifies the bifurcation type and its parametric normal form on the center manifold
- Advantageous for data-driven analysis and order reduction with limited information regarding the underlying low-dimensional dynamics is available

Randomly sampled dynamics





Thank you