A stochastic variant of replicator dynamics in zero-sum games and its invariant measures

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Game Theory and Machine Learning

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- Asymptotic stability around/towards Nash equilbria not clear a priori $\rightarrow$ Hamiltonian structures occur [Hofbauer 1996, Balduzzi et al 2018, ..]
- Role of **noise/uncertainty** for dynamics around Nash equilibria?

Two agents model

Two-player game with n (resp. m) pure strategies for the first (resp. second) agent and payoff matrices

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- We denote by $u_i = \{\mathbf{Ay}\}_i$ and $v_j = \{\mathbf{Bx}\}_j$ the utility of the agent for playing strategy i (resp. j) when the opponent chooses $\mathbf{y}$ (resp. $\mathbf{x}$).
- A game is called zero-sum if $\mathbf{A} = -\mathbf{B}^\top$ such that for all $(\mathbf{x}, \mathbf{y}) \in \overline{\mathcal{D}}$

$$\mathbf{x}^\top \mathbf{A}\mathbf{y} + \mathbf{y}^\top \mathbf{B}\mathbf{x} = 0.$$

Equilibria in zero-sum games

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- Otherwise, there is a **unique maximum support** of Nash equilibrium (and anti-equilibrium) strategies.

Replicator dynamics for zero-sum games

Updating the strategies towards improving utility gives the replicator equation [Weibull 1995, Arora et al. 2012]

$$\begin{aligned}\dot{x}_i &= x_i \left(\{\mathbf{Ay}\}_i - \mathbf{x}^\top \mathbf{Ay}\right) \;, \\ \dot{y}_j &= y_j \left(\{\mathbf{Bx}\}_j - \mathbf{y}^\top \mathbf{Bx}\right) \;.\end{aligned}$$

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Lemma ([Piliouras/Shamma 2014])

1. *If there is a fully mixed Nash equilibrium* $(\mathbf{p}, \mathbf{q})$*, then for any starting point* $(x_0, y_0) \in \mathcal{D}$ *the* **cross entropy**

$$V\big((\mathbf{p}, \mathbf{q}); (\mathbf{x}(t), \mathbf{y}(t))\big) = -\sum_i p_i \ln x_i(t) - \sum_j q_i \ln y_j(t)$$

between $(\mathbf{p}, \mathbf{q})$ *and* $(\mathbf{x}(t), \mathbf{y}(t))$ *is a* **constant of motion**.

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   between $(\mathbf{p}, \mathbf{q})$ *and* $(\mathbf{x}(t), \mathbf{y}(t))$ *is a* **constant of motion**.

2. *Otherwise, let* $(\mathbf{p}, \mathbf{q})$ *be a not fully mixed Nash equilibrium of maximal support; then for all* $t' \geq 0$

$$\frac{\mathrm{d}V\big((\mathbf{p}, \mathbf{q}); (\mathbf{x}(t), \mathbf{y}(t))\big)}{\mathrm{d}t}\Big|_{t=t'} < 0,$$

   and reversed for anti-equilibria.

Convergence to maximum support

For index sets I and J, corresponding with the Nash equilibrium of maximum support, we set

$$\Delta_1 := \{(\mathbf{x}, \mathbf{y}) \in \partial\mathcal{D} \,:\, x_i = 0 = y_j \text{ for all } i \in I^c, j \in J^c\},$$

where I^c and J^c denote the complements of I and J respectively.

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Theorem ([Piliouras/Shamma 2014])

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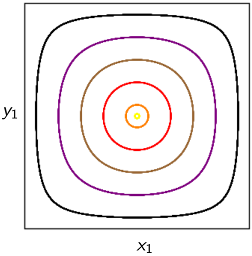
2. *Furthermore, if* $(\mathbf{p}, \mathbf{q})$ *is an equilibrium of maximum support on* $\Delta_1 \subset \partial\mathcal{D}$*, then the omega-limit set satisfies* $\omega(z) \subset int(\Delta_1)$*.*

Example (matching pennies)

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{B} = -\mathbf{A}^T.$$

Orbits around **interior** equilibrium.



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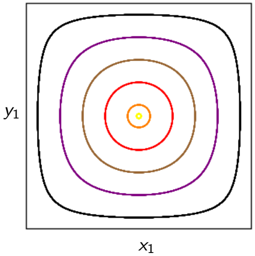
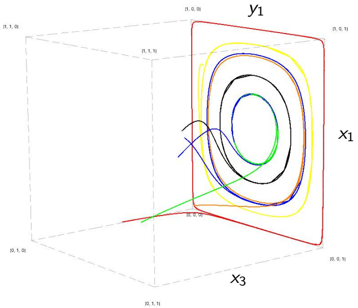


$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -2 \end{pmatrix}, \quad \mathbf{B} = -\mathbf{A}^T.$$

Orbits towards **maximum support**.



A stochastic replicator model with two agents

Our stochastic model (generalizing [Foster/Young 1990]) is the Itô **SDE**

$$\begin{aligned} \mathrm{d}X_i(t) &= X_i(t)\left(\{\mathbf{AY(t)}\}_i - \mathbf{X(t)}^\top \mathbf{AY(t)}\right)\mathrm{d}t + X_i(t)(R(\mathbf{X})\mathrm{d}W(t))_i\,, \\ \mathrm{d}Y_j(t) &= Y_j(t)\left(\{\mathbf{BX(t)}\}_j - \mathbf{Y(t)}^\top \mathbf{BX(t)}\right)\mathrm{d}t + Y_j(t)(S(\mathbf{Y})\mathrm{d}\tilde{W}(t))_j\,, \end{aligned}$$

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- $R : \overline{\mathcal{D}} \to \mathbb{R}^{n\times n}$ and $S : \overline{\mathcal{D}} \to \mathbb{R}^{m\times m}$ are locally Lipschitz continuous and for all $(\mathbf{X}, \mathbf{Y}) \in \overline{\mathcal{D}}$

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- $\exists \xi > 0$ s. t. $\forall i \neq j : \sum_{k=1}^n R_{ik}^2(x) + \sum_{k=1}^n R_{jk}^2(x) \geq \xi$, and $\sum_{k=1}^n R_{ik}^2(x) = 0$ iff $x_i = 1$, (and the same for S).

A specific version

Specific choice of R and S such that (in matrix form)

$$\mathrm{d}\mathbf{X}(\mathbf{t}) = \left(\mathrm{diag}(X_1(t), \dots, X_n(t)) - \mathbf{X}(\mathbf{t})\mathbf{X}(\mathbf{t})^\top\right) \left(\mathbf{A}\mathbf{Y}(\mathbf{t})\,\mathrm{d}t + \mathrm{diag}(\sigma_1, \dots, \sigma_n)\,\mathrm{d}W_t\right),$$

$$\mathrm{d}\mathbf{Y}(\mathbf{t}) = \left(\mathrm{diag}(Y_1(t), \dots, Y_m(t)) - \mathbf{Y}(\mathbf{t})\mathbf{Y}(\mathbf{t})^\top\right) \left(\mathbf{B}\mathbf{X}(\mathbf{t})\,\mathrm{d}t + \mathrm{diag}(\eta_1, \dots, \eta_m)\,\mathrm{d}\tilde{W}_t\right),$$

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- Model describes uncertainty about outcome of the game via random fluctuations around the utilities given by $\mathbf{AY}(\mathbf{t})$ and $\mathbf{BX}(\mathbf{t})$.
- Similar to [Hofbauer/Imhof 2009] for monomatrix games, but with crucially different derivation of noise model.

Generator and Lyapunov functions

Generator $\mathcal{L}$ of the associated Markov semigroup P_t acts as

$$\mathcal{L}h(\mathbf{x},\mathbf{y}) = \lim_{t\downarrow 0} \frac{1}{t}\left(\mathbb{E}_{(\mathbf{x},\mathbf{y})}[h(\mathbf{X_t},\mathbf{Y_t})] - h(\mathbf{x},\mathbf{y})\right)$$
$$= \sum_i x_i \left(\{\mathbf{Ay}\}_i - \mathbf{x}^\top \mathbf{Ay}\right) \partial_{x_i} h(\mathbf{x},\mathbf{y}) + \sum_i y_i \left(\{\mathbf{Bx}\}_i - \mathbf{y}^\top \mathbf{Bx}\right) \partial_{y_i} h(\mathbf{x},\mathbf{y})$$
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Main idea: Use cross entropy functions

$$V(\mathbf{x},\mathbf{y}) = -\sum_{i\in I} p_i \ln x_i - \sum_{j\in J} q_j \ln y_j,$$

as **Lyapunov function** for determining invariant measures on $\mathcal{D}\cup\partial\mathcal{D}$.

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  (ii) *otherwise, for sufficiently "small" noise, the only invariant measures which attract the interior are contained in the subset Δ_1 of $\partial\mathcal{D}$ which* **contains the Nash equilibrium of maximal support**.

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Hence, almost all trajectories accumulate at $\partial\mathcal{D}$ [Khasminskii 2012, Benaim/Strickler 2019], where the only invariant measures lie.

Sketch of proof II

2. Otherwise, consider NE $(\mathbf{p}, \mathbf{q})$ of **maximal support** with index sets I and J, the anti-NE $(\mathbf{p}^*, \mathbf{q}^*)$ and

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2.2 for "small" noise: $H_0(\mathbf{x}, \mathbf{y}) < 0$ on $\Delta_{\partial,2}$ and $H_1(\mathbf{x}, \mathbf{y}) > 0$ on $\Delta_{\partial,1} \Rightarrow$ convergence to $\Delta_{\partial,1}$.

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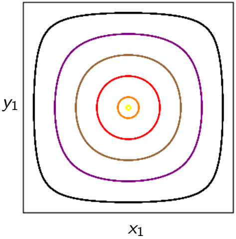

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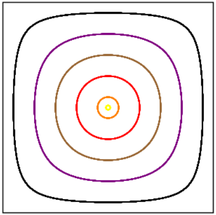
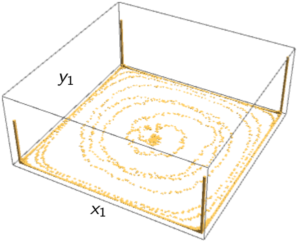


- Ergodic (physical) measure for stochastic case ($\lim_{t\to\infty} \frac{1}{t}\int_0^t f(Z_s)\mathrm{d}s = \int_{\overline{\mathcal{D}}} f(y)\,\mathrm{d}\mu(y)$ for $\mu = \frac{1}{4}\sum_{i,j}\delta_{i,j}$, Lebesgue-almost all $z = Z_0$.)



Matching pennies II: 2×3 with non-interior NE

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -2 \end{pmatrix}, \quad \mathbf{B} = -\mathbf{A}^T = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix},$$

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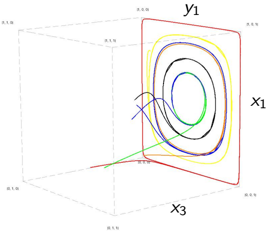

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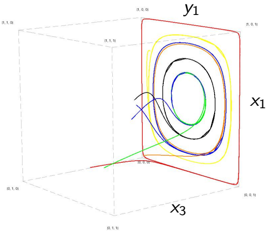

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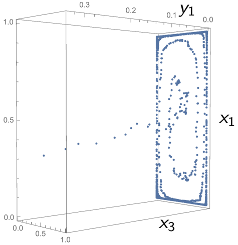

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