Estimation of Interactions among Dynamical Elements by Koopman Operator

Yuka Hashimoto

NTT Network Service Systems Laboratories / RIKEN AIP

Joint work with Masahiro Ikeda, Hiroya Nakao, and Yoshinobu Kawahara

arXiv:2208.06186

Network dynamics

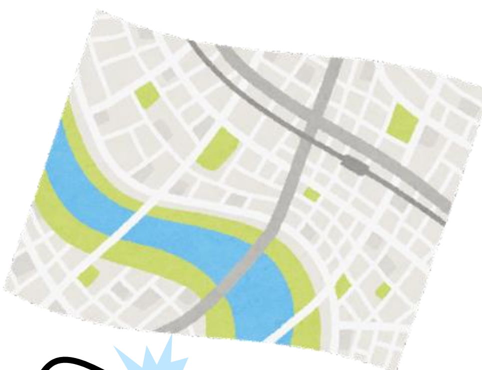
$$\frac{dX_i(t)}{dt} = F_i\big(X_i(t)\big) + \underline{\sum_{j=1}^{N} G_{i,j}(X_i(t), X_j(t))}$$

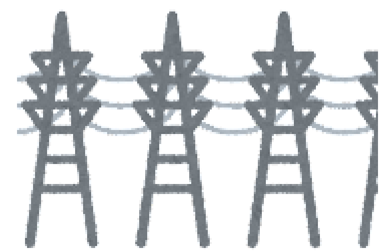
Interaction

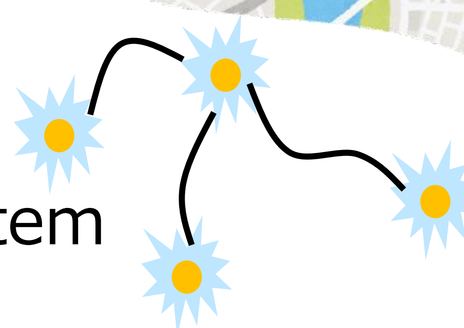
Network dynamics is important for the understanding of various phenomena in nature and human society

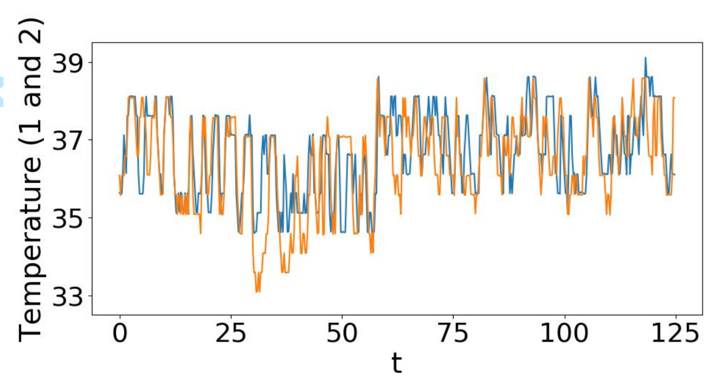
Examples

- Traffic network
- Power grid
- Neural circuit
- Physiological system









Koopman operator

$f: \mathcal{X} \to \mathcal{X}$, $\mathcal{F} \subseteq (\mathbb{C}^N)^{\mathcal{X}}$: function space

Koopman operator K on $\mathcal{F}$ with respect to f is defined as

$$Kv = v \circ f$$

e.x.

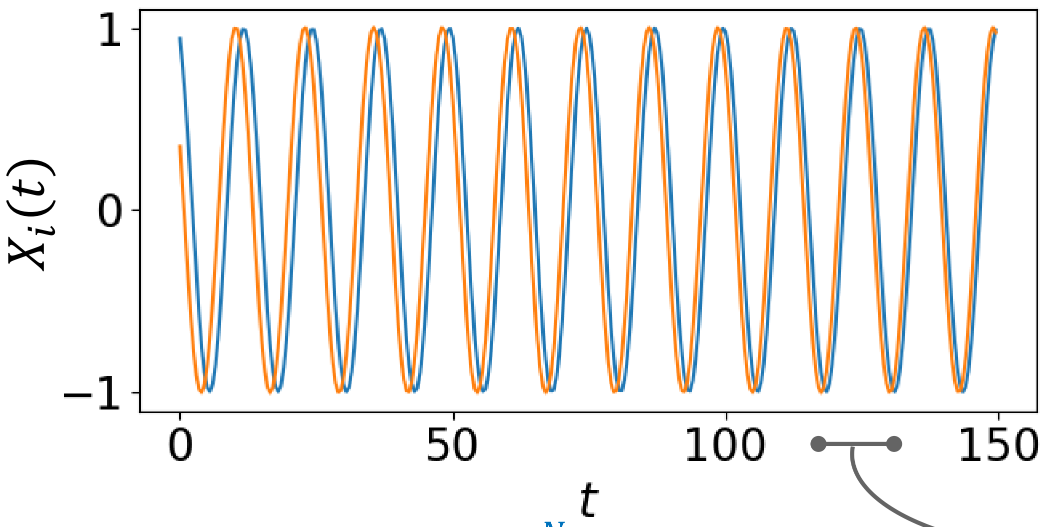
- $\mathcal{X}$: measure space, $\mathcal{F} = L^p(\mathcal{X})$
- $\mathcal{X}$: unit disc in $\mathbb{C}$, $\mathcal{F} = H^p(\mathcal{X})$ (Hardy space)
- $\mathcal{X} = \mathbb{R}^d$,

  $\mathcal{F}$: RKHS associated with Gaussian or Laplacian kernel

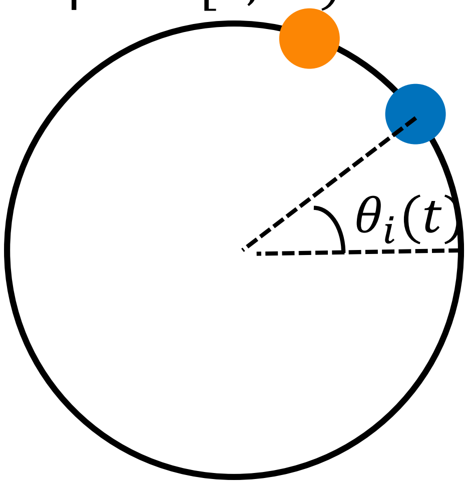
  $k(x, y) = e^{-c\|x-y\|^2}$ $k(x, y) = e^{-c\|x-y\|}$

Phase model (synchronization and interaction)

State space $\mathcal{X} = \mathbb{R}^d$



Phase space $[0,2\pi)$



$$\frac{dX_i(t)}{dt} = F_i(X_i(t)) + \sum_{j=1}^{N} G_{i,j}(X_i(t), X_j(t))$$

$$\frac{d\theta_i}{dt} = \omega + \Gamma_i(\theta_i - \theta_1, \ldots, \theta_i - \theta_N)$$

Transformation by a phase function $\Theta_i: X_i(t) \mapsto \theta_i(t)$

Common frequency

Phase coupling function (interaction)

$$\frac{d\theta_i}{dt} = \underline{\omega} + \underline{\Gamma_i(\theta_i - \theta_1, \ldots, \theta_i - \theta_N)}$$

Goal : To reconstruct the phase model only with given data using a Koopman operator

Connection between Koopman operators and phase models (without interactions)*1

Dynamical system on $\mathcal{X}$

$$\frac{dX(t)}{dt} = F(X(t))$$

$G_{i,j} = 0$

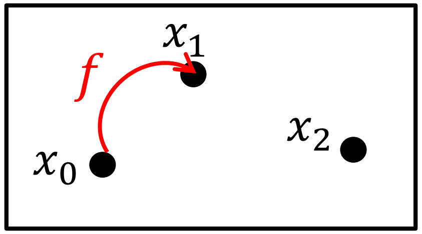
Data

$$x_k = X(k\Delta t)$$

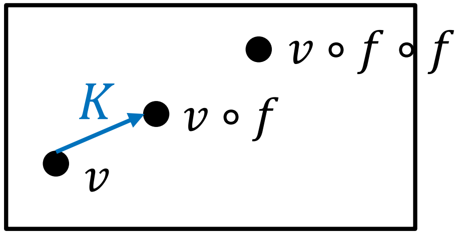
Estimate

Koopman operator (linear operator) on a complex-valued function space

Eigenvalues and eigenvectors of the Koopman operator



Original nonlinear dynamical system



Linear operator on a function space

$$\frac{\theta(t + \Delta t) - \theta(t)}{\Delta t} \approx \omega$$

Phase model

*1 S. Shirasaka, W. Kurebayashi, H. Nakao, Phase-amplitude reduction of transient dynamics far from attractors for limit-cycling systems. Chaos, 27, 023119 (2017).

Connection between Koopman operators and phase models (with interactions)

Network dynamical system on $\mathcal{X}^N$

$$\frac{dX_i(t)}{dt} = F_i\big(X_i(t)\big) + \sum_{j=1}^{N} G_{i,j}(X_i(t), X_j(t))$$

$i = 1, \ldots, N$

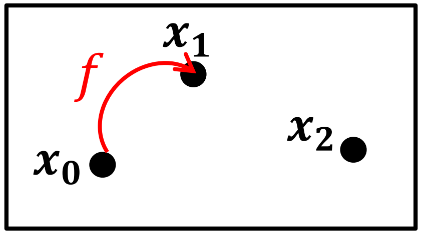
Data

$\boldsymbol{X}(t) = [X_1(t), \ldots, X_N(t)]$

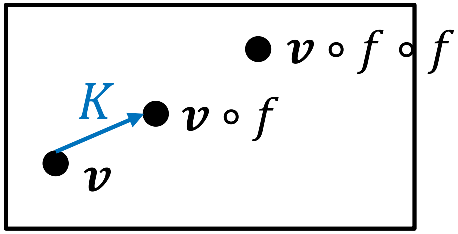
$\boldsymbol{x_k} = \boldsymbol{X}(k\Delta t)$

Estimate

Koopman operator (linear operator) on a vector-valued function space



Original nonlinear dynamical system



Linear operator on a function space

Multiparameter eigenvalues and eigenvectors of the Koopman operator

$$\frac{\theta_i(t+\Delta t) - \theta_i(t)}{\Delta t} \approx \omega + \frac{1}{\Delta t}\sum_{j=1}^{M} \arg\left(\sum_{k=1}^{N} a_{i,k}^{j} e^{-\sqrt{-1}j(\theta_i(t)-\theta_k(t))}\right)$$

Phase model

Multiparameter eigenvalue problem

$\mathcal{H}$: $\mathbb{C}^N$-valued function space (Hilbert space)

$K, B_{i,k}$ $(i, k = 1, \ldots, N)$: linear operators on $\mathcal{H}$

Multiparameter eigenvalue problem :

Find $\{a_{i,k}\}_{i,k=1}^{N} \subseteq \mathbb{C}$, and $\boldsymbol{u} = [u_1, \ldots, u_N] \in \mathcal{H}$ such that

$$K\boldsymbol{u} = \sum_{i,k=1}^{N} a_{i,k} B_{i,k} \boldsymbol{u} \quad (1)$$

Linear interaction

Generalize

Find M $\in \mathbb{N}$ and $\left(\left\{a_{i,k}^{j}\right\}_{i,k=1}^{N}, \boldsymbol{u}^j = \left[u_1^j, \ldots, u_N^j\right] \in \mathcal{H}\right)$ $(j = 1, \ldots, M)$ such that

$$K\boldsymbol{u}^1 \odot \cdots \odot K\boldsymbol{u}^M$$

Nonlinear interaction

$$= \left(\sum_{i,k=1}^{N} a_{i,k}^{1} B_{i,k} \boldsymbol{u}^1\right) \odot \cdots \odot \left(\sum_{i,k=1}^{N} a_{i,k}^{M} B_{i,k} \boldsymbol{u}^M\right) \quad (2)$$

Reconstruction of phase model with interactions

$f^s : \mathcal{X}^N \to \mathcal{X}^N$: Flow of the dynamical system ($f^s(X(t)) = X(s+t)$)

K : Koopman operator on $\mathcal{H}$ defined as $K\boldsymbol{v} = \boldsymbol{v} \circ f^{\Delta t}$

$$B_{i,k}[u_1, \ldots, u_N] := [0, \ldots, 0, \underset{i}{\underline{u_k}}, 0, \ldots, 0]$$

Find $\omega \in [0, 2\pi)$, $\{a_{i,k}\}_{i,k=1}^N \subseteq \mathbb{C}$, and $\boldsymbol{u} = [u_1, \ldots, u_N] \in \mathcal{H}$ such that

$$(\lambda_1)^{-1} K\boldsymbol{u}^1 \odot \cdots \odot (\lambda_M)^{-1} K\boldsymbol{u}^M$$
$$= \left(\sum_{i,k=1}^N a_{i,k}^1 B_{i,k} \boldsymbol{u}^1\right) \odot \cdots \odot \left(\sum_{i,k=1}^N a_{i,k}^M B_{i,k} \boldsymbol{u}^M\right) \quad (3)$$

$$\Rightarrow \quad \prod_{j=1}^M u_i^j(X_i(t+\Delta t)) = \prod_{j=1}^M e^{\sqrt{-1}j\omega\Delta t} \sum_{k=1}^N a_{i,k}^j u_k^j(X_k(t)) \quad (4)$$

where $\lambda_j = e^{\sqrt{-1}j\omega\Delta t}$.

Reconstruction of phase model with interactions

K : Koopman operator, $B_{i,k}[u_1, \ldots, u_N] := [0, \ldots, 0, \underbrace{u_k}_{i}, 0, \ldots, 0]$, $\lambda_j = e^{\sqrt{-1} j \omega \Delta t}$

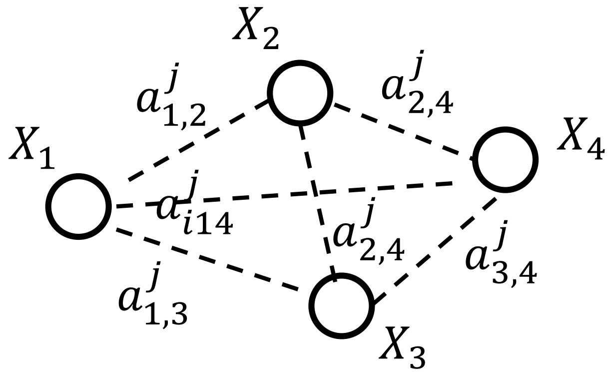
Let $\theta_i^j(t) := \arg\left(u_i^j(X_i(t))\right)$.

Interactions are weak

If $a_{i,k}^j \approx 1\ (i = k)$, $a_{i,k}^j \approx 0\ (i \neq k)$, $K\boldsymbol{u}^j \approx \lambda_j \boldsymbol{u}^j$, and $\theta_i^j \approx j\theta_i^1$,

$$\frac{\theta_i(t+\Delta t) - \theta_i(t)}{\Delta t} \approx \omega + \frac{1}{\Delta t} \sum_{j=1}^{M} \arg\left(\sum_{k=1}^{N} a_{i,k}^j e^{-\sqrt{-1} j (\theta_i(t) - \theta_k(t))}\right) \quad (5)$$

$$= \Gamma_i(\theta_i - \theta_1, \ldots, \theta_i - \theta_N)$$

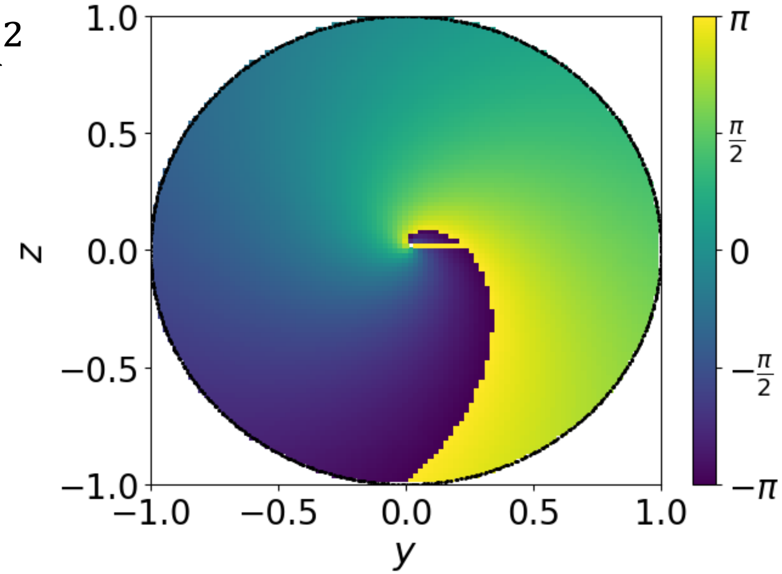


Interactions are weak

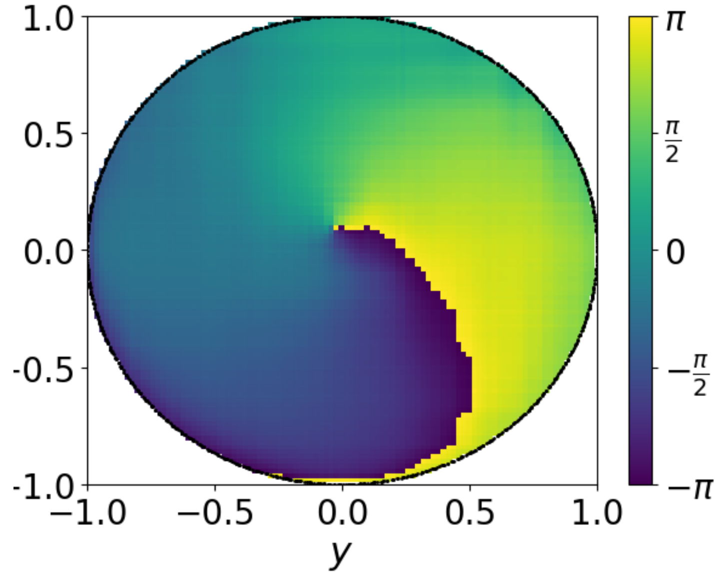
Experiments with Stuart–Landau model

- Generated data from Stuart–Landau model with $N = 2$, $\mathcal{X} = \mathbb{R}^2$, and estimated the Koopman operator.
    - $F_i([y_i, z_i]) = \left[y_i - az_i - \left(y_i^2 + z_i^2\right)(y_i - bz_i), ay_i + z_i - \left(y_i^2 + z_i^2\right)(by_i + z_i)\right]$
    - $G_{i,j}\left(z_i, z_j\right) = \left[\epsilon\left(z_j - z_i\right), 0\right]$
- Computed the phase function $\Theta_i: \mathcal{X} \to [0,2\pi)$, $\Theta_i: X_i(t) \mapsto \theta_i(t)$.

$[y, z] \in \mathbb{R}^2$



Phase function calculated from the model



Phase function estimated by the data (proposed method)

Difference of the phase coupling functions

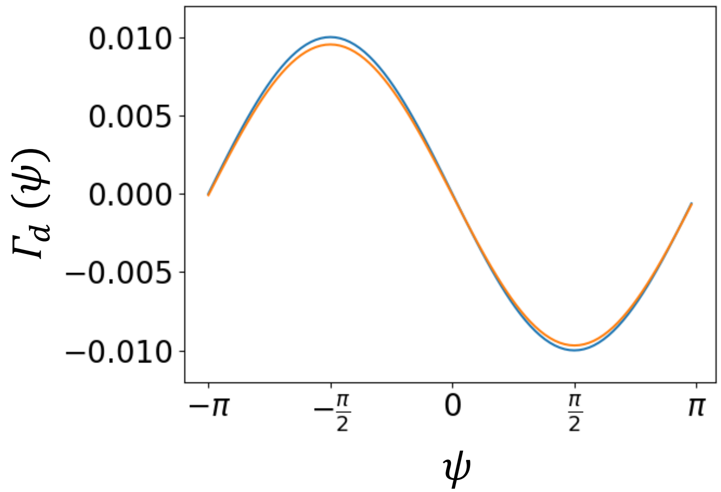
$$\frac{d\theta_1}{dt} = \omega + \Gamma_1(\theta_1 - \theta_2)$$

$$\frac{d\theta_2}{dt} = \omega + \Gamma_2(\theta_2 - \theta_1)$$

$$\frac{d(\theta_1 - \theta_2)}{dt} = \Gamma_1(\theta_1 - \theta_2) - \Gamma_2(\theta_2 - \theta_1)$$

Equation with the phase difference

$$\Gamma_d\ (\psi) := \Gamma_1(\psi) - \Gamma_2(-\psi)$$



$\mathrm{sign}(\psi) \neq \mathrm{sign}(\Gamma_d(\psi))$

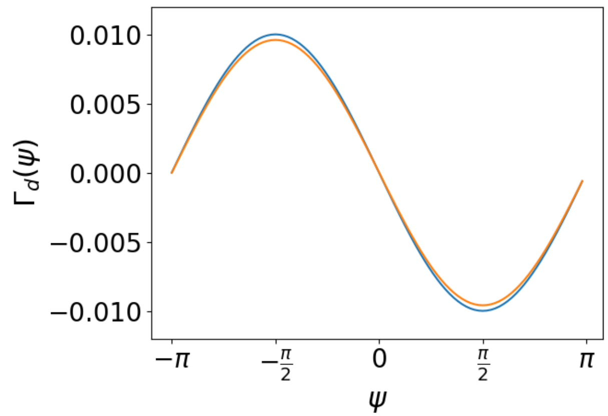
⇒ Synchronization of two oscillator

(The phase difference changes so that it becomes closer to 0.)

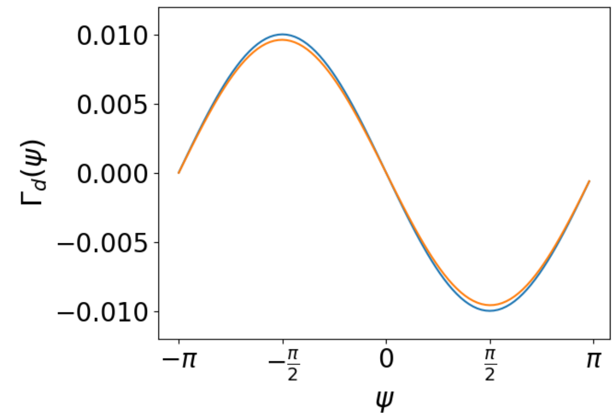
Experiments with Stuart–Landau model

- Computed the phase coupling functions Γ_1, Γ_2 and their difference $\Gamma_d\ (\psi) := \Gamma_1(\psi) - \Gamma_2(-\psi)$.

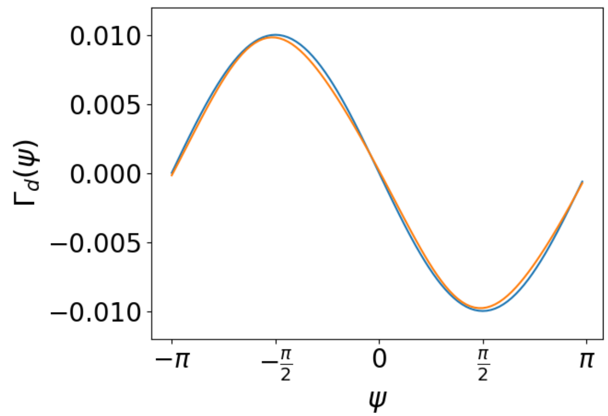
$M = 1$



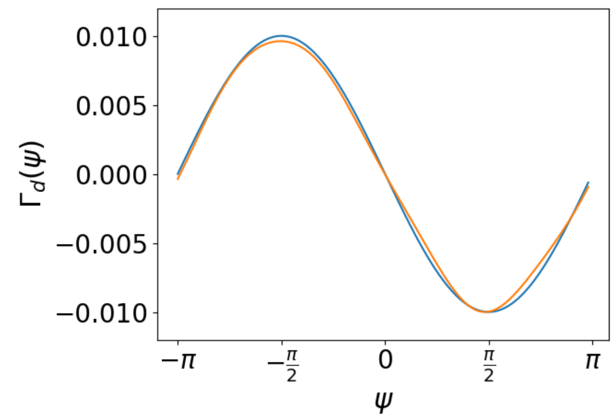
$M = 2$



$M = 3$



$M = 4$



Phase coupling function
Blue : calculated from the model   Orange : estimated by the data

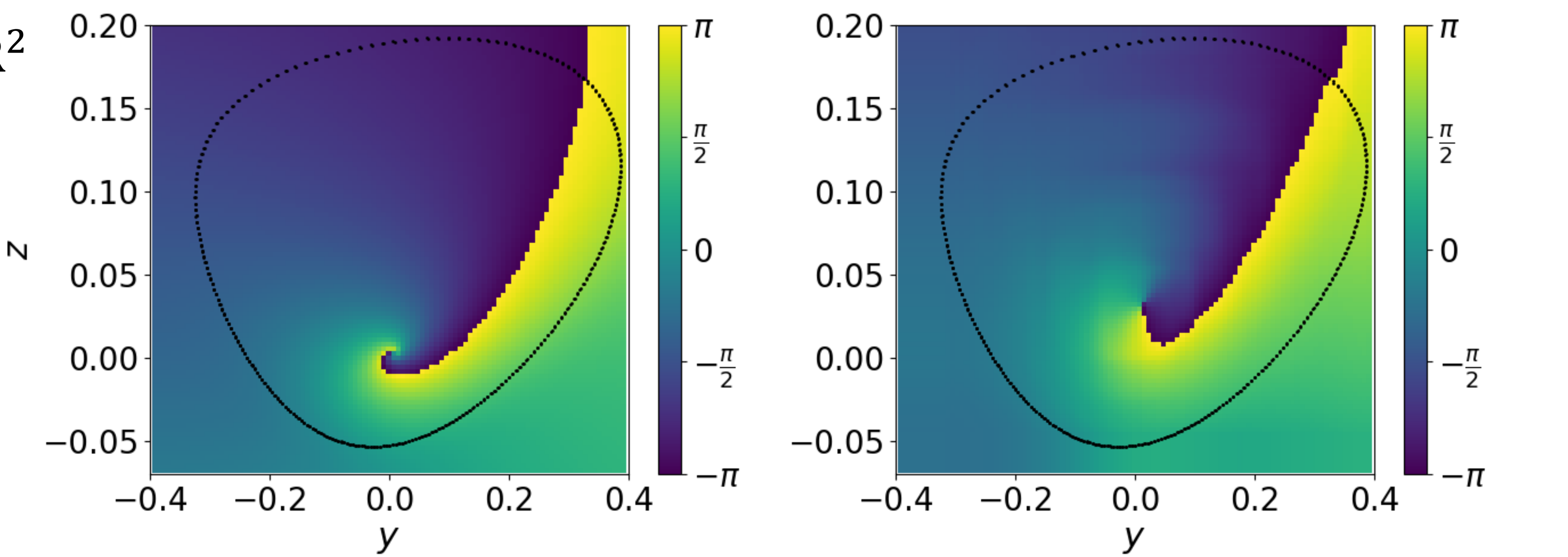
Experiments with FitzHugh-Nagumo model

- Generated data from FitzHugh-Nagumo model with $N = 2$, $\mathcal{X} = \mathbb{R}^2$, and estimated the Koopman operator.
  - $F_i([y_i, z_i]) = [y_i(y_i - c)(1 - y_i) - z_i, \mu^{-1}(y_i - dz_i)]$
  - $G_{i,j}(z_i, z_j) = [\epsilon(z_i - z_j), 0]$

Orbit is more complicated than SL model

- Computed the phase function $\Theta_i: \mathcal{X} \to [0, 2\pi)$, $\Theta_i: X_i(t) \mapsto \theta_i(t)$.

$[y, z] \in \mathbb{R}^2$



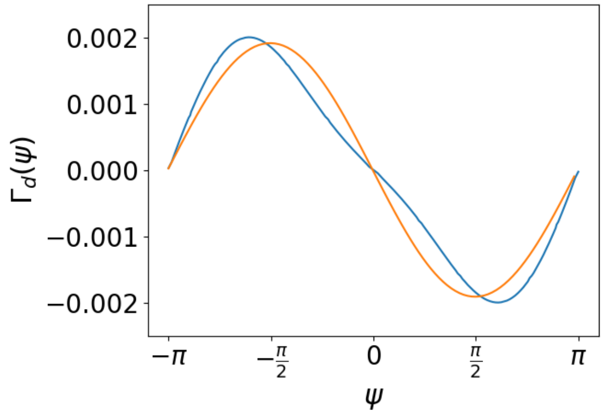
Phase function calculated from the model

Phase function estimated by the data (proposed method)

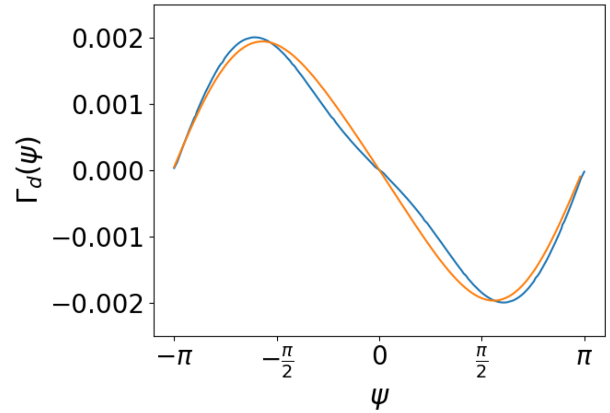
Experiments with FHN model

- Computed the phase coupling functions Γ_1, Γ_2 and their difference $\Gamma_d(\psi) := \Gamma_1(\psi) - \Gamma_2(-\psi)$.

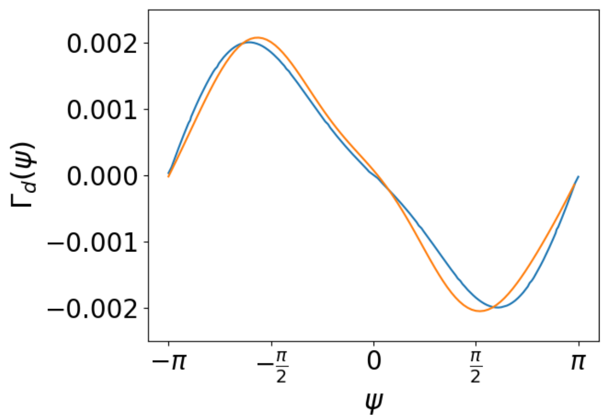
$M = 1$



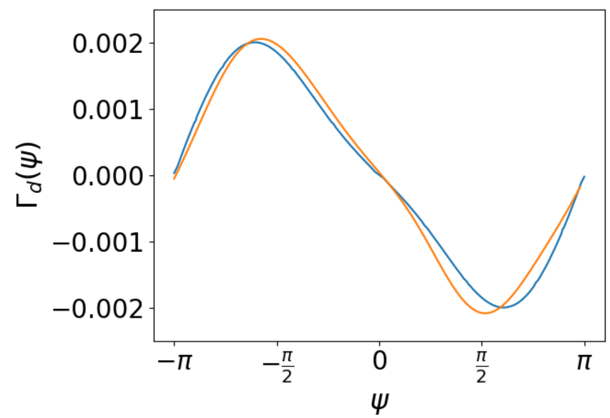
$M = 2$



$M = 3$



$M = 4$



Phase coupling function
Blue : calculated from the model   Orange : estimated by the data

Conclusion

- Considered network dynamical systems and Koopman operators on vector-valued function spaces.
- The phase model is reconstructed by the solution of a generalized multiparameter eigenvalue problem with respect to the Koopman operator.
- Numerical results show the validity of the estimation.