NAVAL POSTGRADUATE SCHOOL

Compositional Features and Neural Network Complexity for Dynamical Systems

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Fields Institute

Background and literature

Control and dynamical systems

Modeling: $\mathbf{f} : \mathbf{x} \in \mathbb{R}^{\mathrm{r}} \to \mathbb{R}^{\mathrm{d}}$

Feedback Control $\mathbf{u} : \mathbf{x} \in \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{m}}$

Stability $\mathbf{V} : \mathbf{x} \in \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$

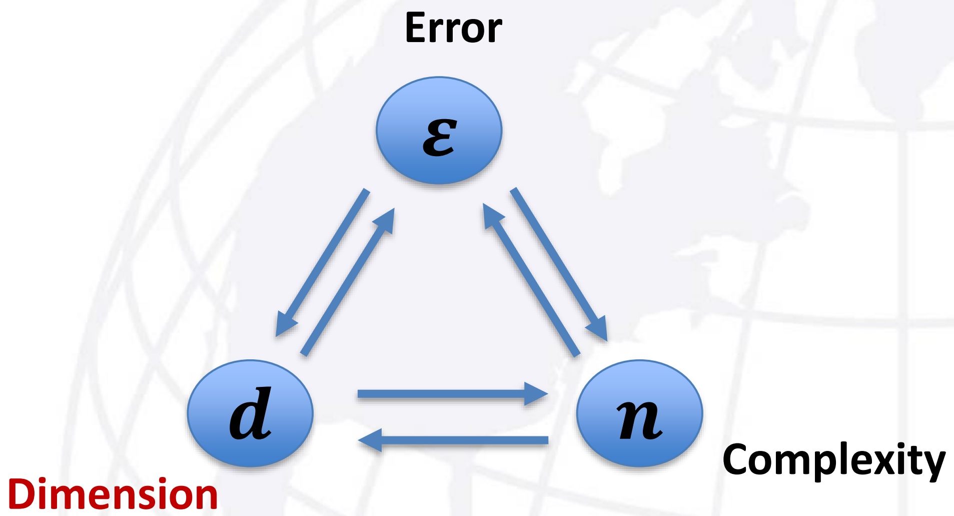
……… ………

Regression

$$f : \mathbb{R}^d \longrightarrow \mathbb{R}$$

$$f^{NN} \approx f\ ?$$

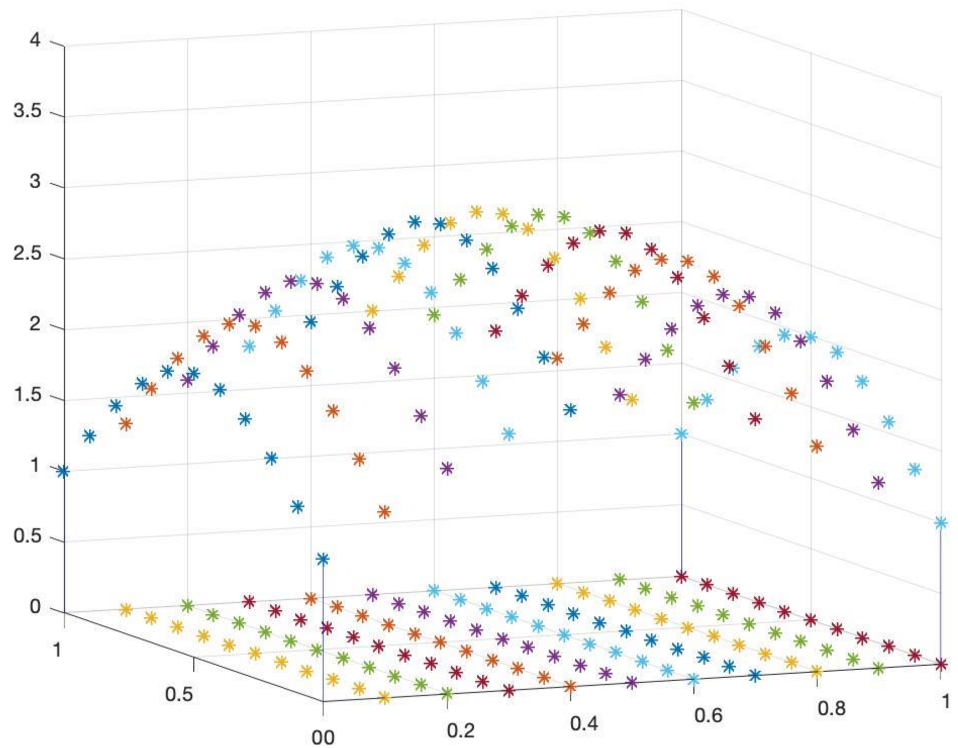
What's the big deal?



Background and literature

Curse-of-dimensionality: Given an error upper bound, the complexity increases exponentially with the dimension.

Dataset Size $= N^d$



Richard Ernest Bellman

* Dynamic programming
* Curse of dimensionality
* HJB equation

……

Background and literature

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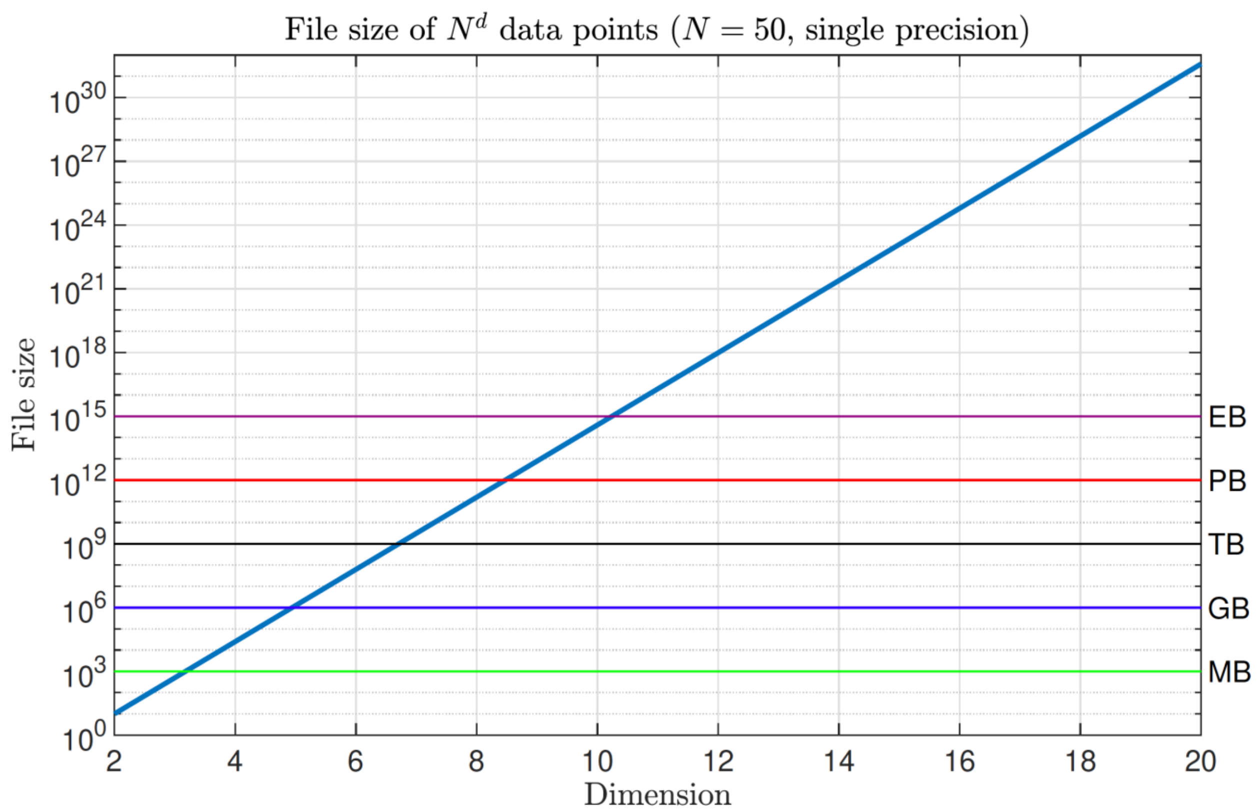
Dataset Size $= N^d$



Background and literature

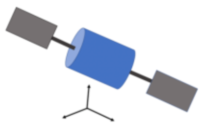
Empirical successes of machine learning

Solving high dimensional problems

- I. E. Lagaris, A. Likas, D. Fotiadis, *Artificial neural networks for solving ordinary and partial differential equations*, IEEE Trans. Neural Networks, 1998. (***d=2***)
- Y. Tassa and T. Erez, *Least square solutions of the HJB equation with neural network value-function approximations*, IEEE Trans. Neural Networks, 2007 (***d=4***)
- J. Han and W. E, *Deep learning approximation for stochastic control problems*, arXiv 2016 (***d=100***)
- J. Han, A. Jentzen, W. E., *Solving high-dimensional PDEs using deep learning*, arXiv 2017 (***d=100****)*
- J. Sirignano and K. Spiliopoulos, *DGM: A deep learning algorithm for solving PDEs*, arXiv 2018 (***d=200****)*
- M. Raissi, P. Perdikaris, G. Karniadakis, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear PDEs*, J. Comput. Phys., 2018 ($d_x=2, d_\lambda=2$)
- T. Nakamura-Zimmerer, Q. Gong, W. Kang, *Adaptive deep learning for high-dimensional HJB equations*, arXiv 2019 (***d=30***)
- D. Izzo, E. Öztürk, and M. Märtens, *Interplanetary transfers via deep representations of the optimal policy and/or of the value function*, arXiv 2019 (***d=7***)
- B. Azmi, D. Kalise, K. Kunisch, Optimal feedback law recovery by gradient-augmented sparse polynomial regression, arXiv 2020. (***d=80****)*

Some examples of dynamical systems

- T. Nakamura-Zimmerer, Q. Gong, W. Kang, *Adaptive deep learning for high-dimensional HJB equations*, SIAM J. Scientific Computing, 2021



(*d*=6, 30 ~ 10^6, 10^{30})

- T. Nakamura-Zimmerer, Q. Gong, W. Kang, QRnet: optimal regulator with LQR-augmented neural networks, IEEE Control Systems Letters, 2021.

(*d*=64)

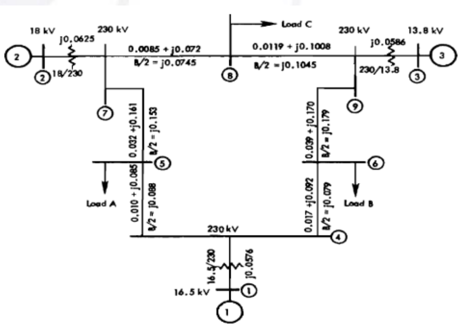
We combine the raw NN prediction (13) with the LQR value function (8) for the linearized dynamics (7) as

$$V^{\mathrm{NN}}(\boldsymbol{x}) = \frac{1}{c}\log\left[1 + cV^{\mathrm{LQR}}(\boldsymbol{x})\right] + W^{\mathrm{NN}}(\boldsymbol{x}), \qquad (14)$$

with a trainable parameter $c > 0$. Intuitively, LQR provides a

- W. Kang, Q. Gong, T. Nakamura-Zimmerer, F. Fahroo, Algorithms of data generation for deep learning and feedback design: A survey, Physica D: Nonlinear Phenomena, 2021

(*d*=30)



Background and literature

Why does deep learning work for so many high dimensional problems?

Background and literature

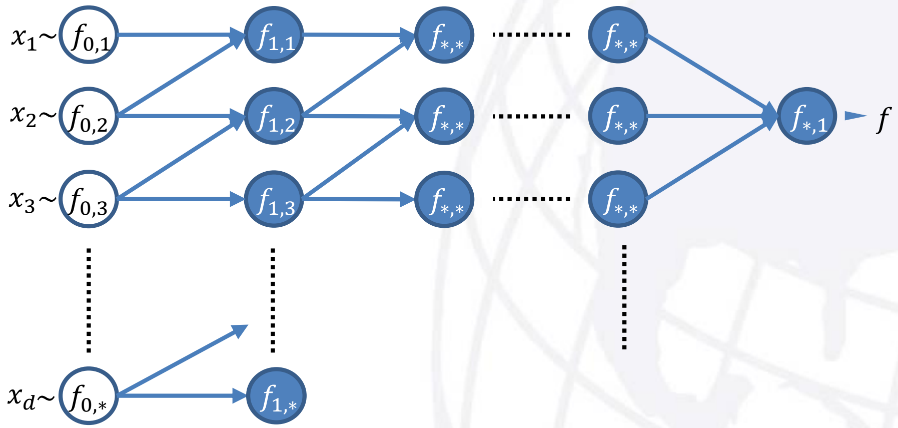
Approximation theory -

Rate of convergence, error upper bound, complexity, ……

- A. R. Barron, *Universal approximation bounds for superpositions of a sigmoidal function*, IEEE Trans. on Information Theory, 1993.
- W. E, C. Ma, S. Wojtowytsch and L. Wu, *Towards a mathematical understanding of neural network-based machine learning: what we know and what we don't*, arXiv 2020.
- P. C. Kainen, V. Kůrková, M. Sanguineti, *Approximating multivariable functions by feedforward neural nets*. In: Handbook on Neural Information Processing, Springer, 2013.
- T. Poggio, H. Mhaskar, L. Rosasco, B. Miranda, Q. Liao, *Why and when can deep - but not shallow - networks avoid the curse of dimensionality: a review*, arXiv 2017.
- H. N. Mhaskar and T. Poggio, *Deep vs. shallow networks: an approximation theory perspective*, arXiv 2016.
- W. Kang and Q. Gong, Feedforward *neural networks and compositional functions with applications to dynamical systems*, SIAM J. Control and Optim., 2022

Compositional function as a layered DAG

- W. Kang and Q. Gong, *Neural network approximations of compositional functions with applications to dynamical systems*, SIAM J. Control and Optimization, 2022.
- T. Poggio et al., *Why and when can deep – but not shallow – networks avoid the curse of dimensionality: A review*, arXiv 2017
- H. N. Mhaskar and T. Poggio, *Deep vs. shallow networks: an approximation theory perspective*, arXiv 2016.

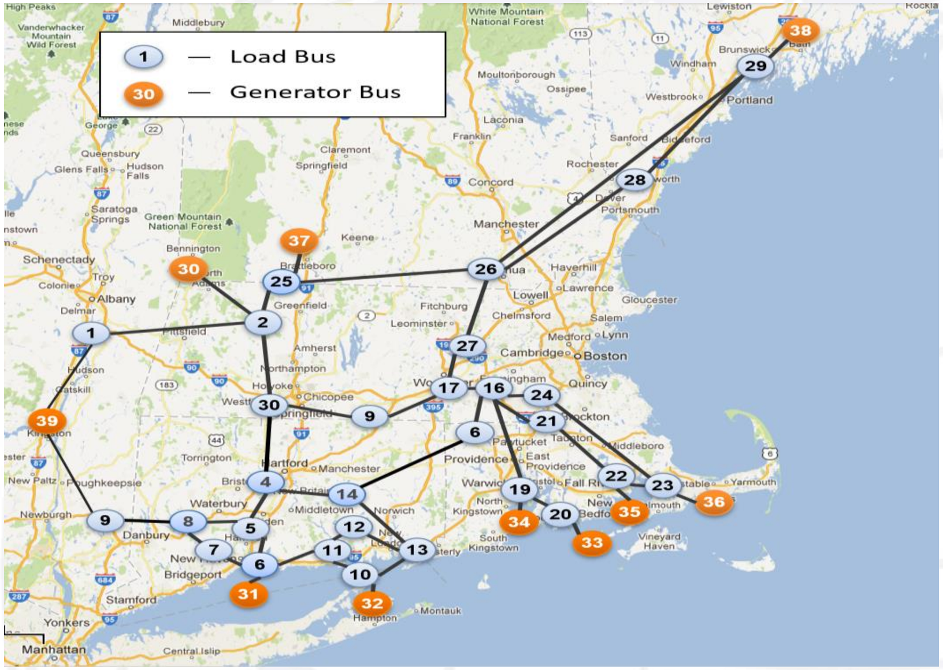


Layered directed acyclic graph (layered DAG)

Algebra of compositional functions

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

EI: A large electrical grid in North America. A simplified model has more than 25K buses, 28K lines, 8K transformers, and over 1,000 generators.

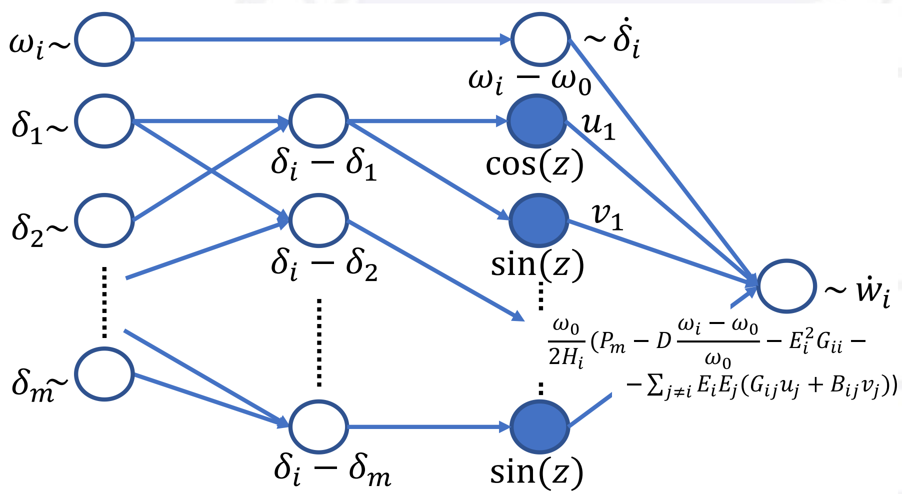


Algebra of compositional functions

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

The swing equation of power systems

$$\frac{d\omega_i}{dt} = \frac{\omega_0}{2H_i}\left(P_m - D\frac{\omega_i - \omega_0}{\omega_0} - E_i^2 G_{ii} \cdots \right.$$
$$\left. - \sum_{j=1, j\neq i}^{10} E_i E_j [B_{ij}\sin(\delta_i - \delta_j) + G_{ij}\cos(\delta_i - \delta_j)]\right)$$
$$\frac{d\delta_i}{dt} = \omega_i - \omega_0,$$



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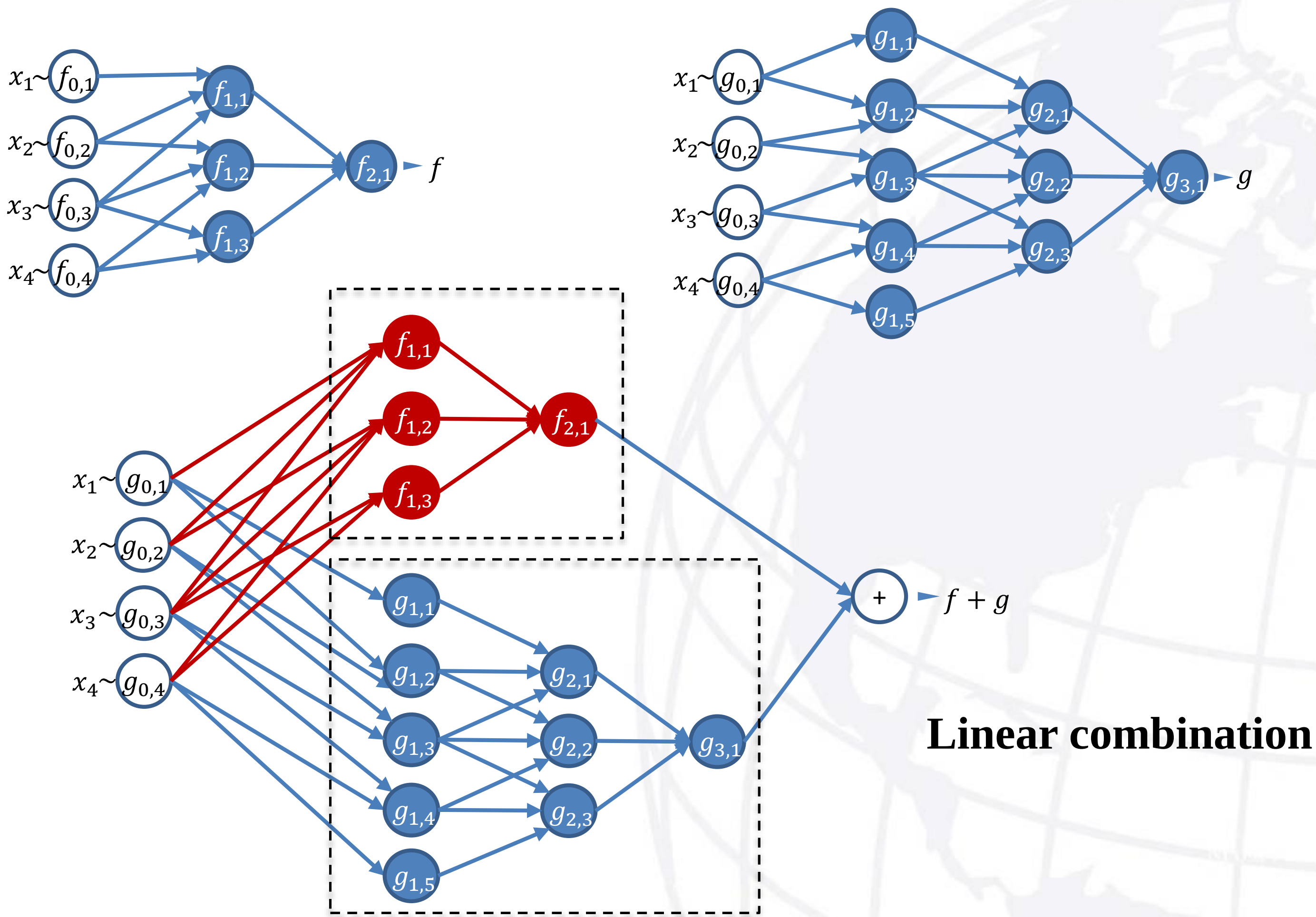

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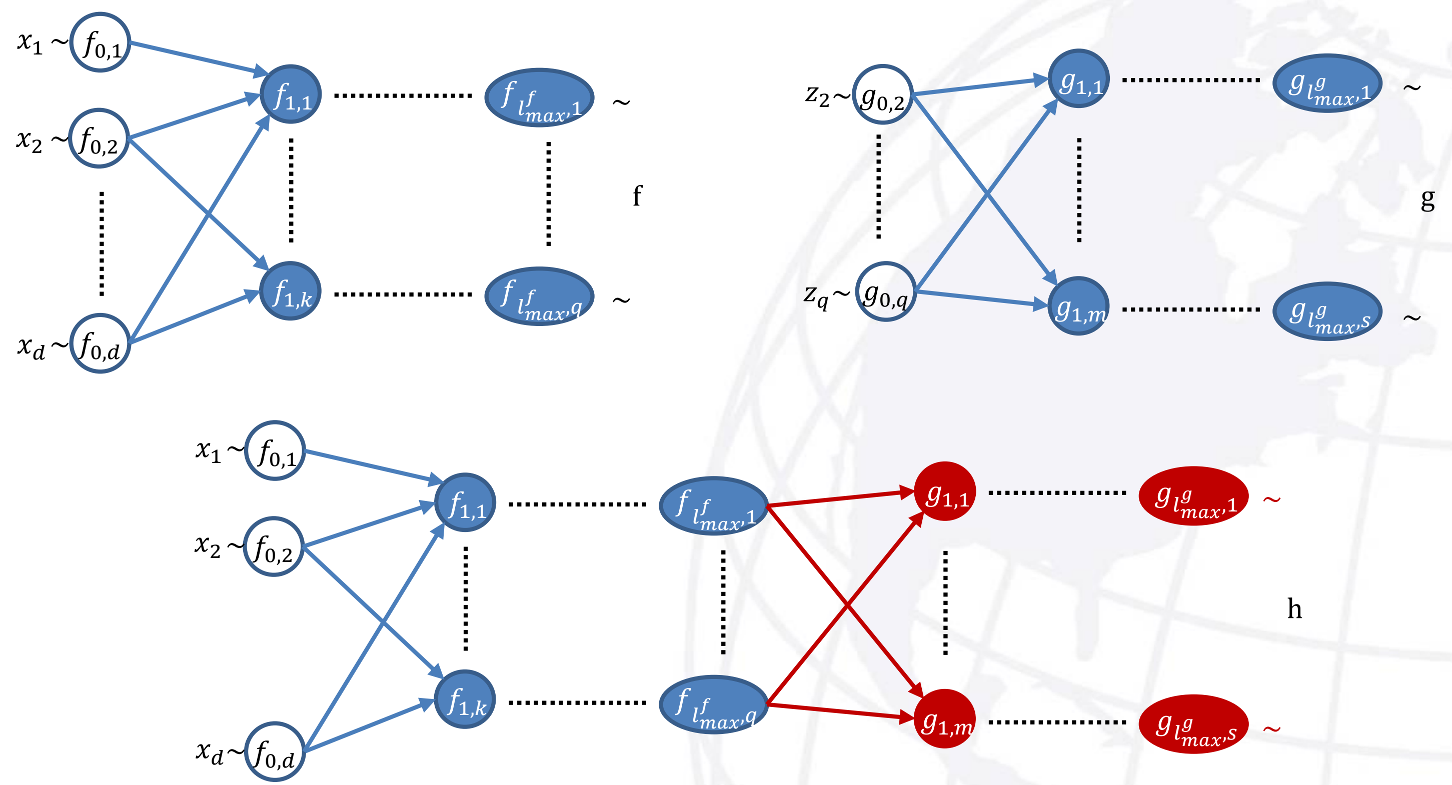

Composition of compositional functions

Algebra of compositional functions

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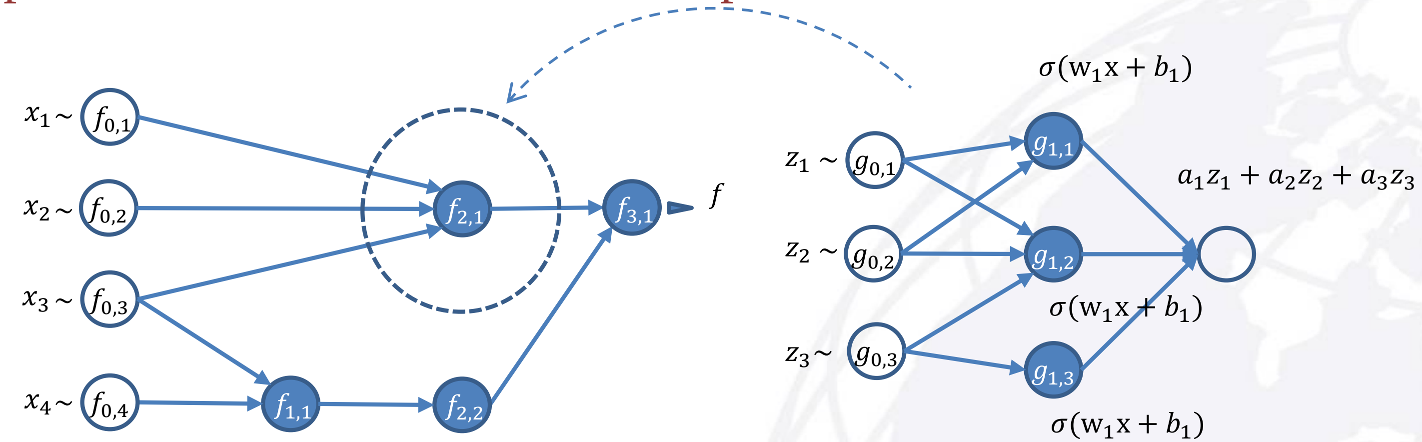
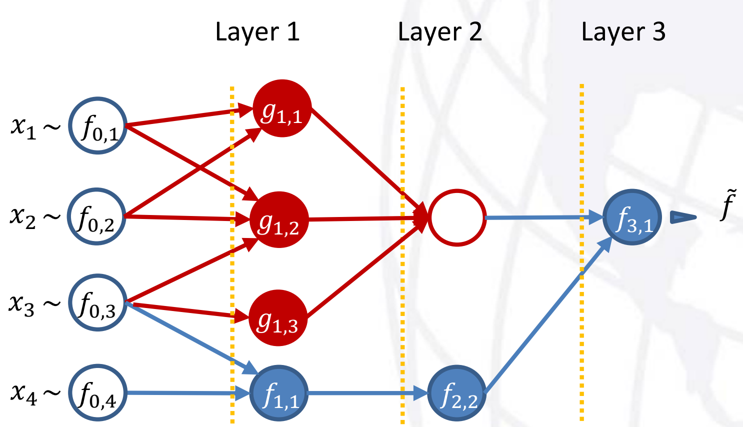


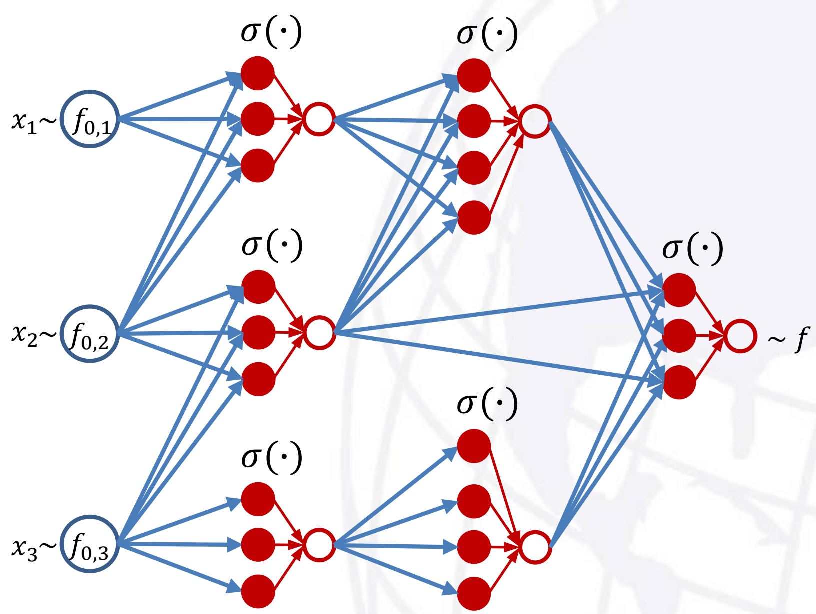


Substitution

Algebra of compositional functions

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

Neural networks are compositional functions



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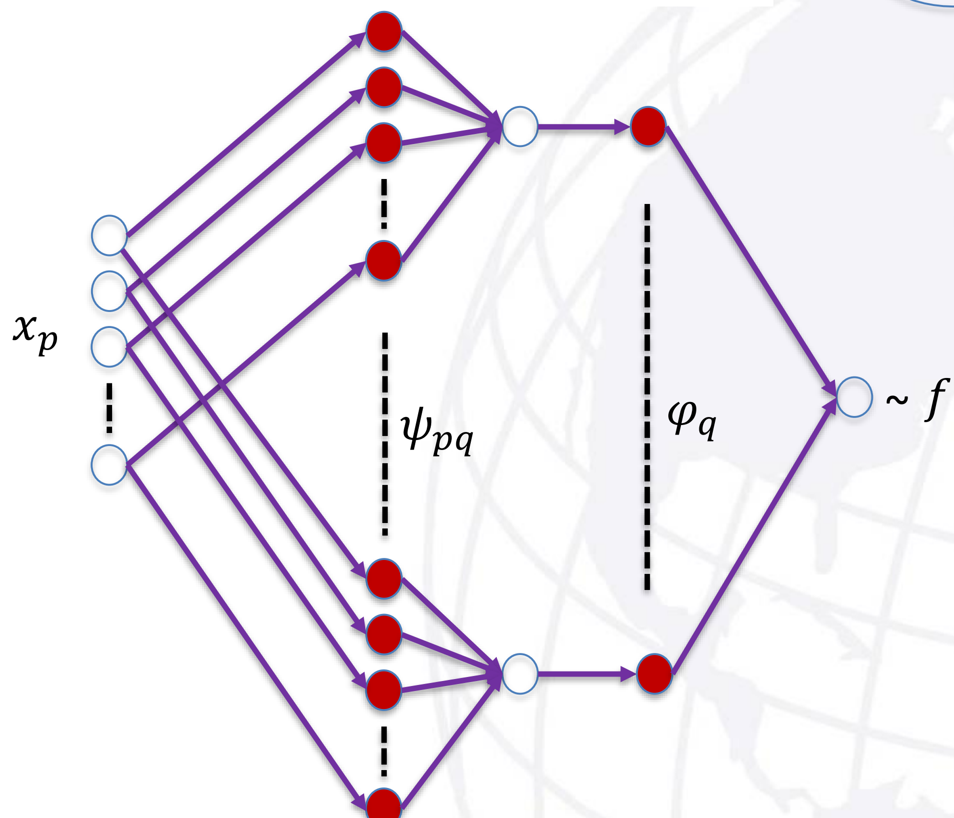
Algebra of compositional functions

- Iterative computational algorithms are (deep) compositional functions.
- Kolmogorov-Arnold representation theorem

$$f(x_1, \cdots, x_d) = \sum_{q=1}^{2d+1} \phi_q \left(\sum_{p=1}^{d} \psi_{pq}(x_p) \right)$$

Every continuous function is a layered DAG



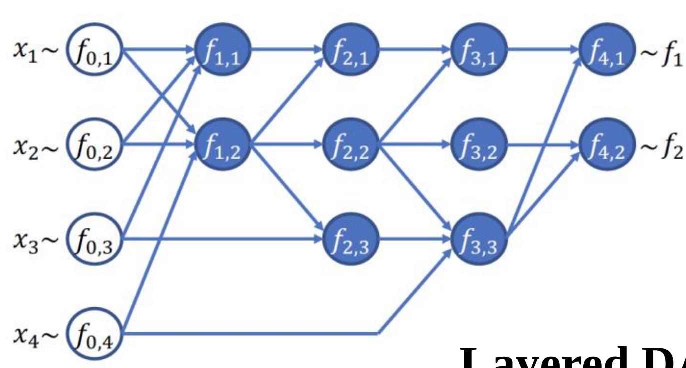
Compositional features

Dimension feature: $r^{\mathbf{f}}_{max}$ – the largest $d_{i,j}/m_{i,j}$

Volume feature: $\Lambda^{\mathbf{f}}$ – the largest $\max\{(R_{i,j})^{m_{i,j}}, 1\} \|f_{i,j}\|_{W^{\infty}_{m_{i,j},d_{i,j}}}$

Lipschitz constant feature: $L^{\mathbf{f}}_{max}$ – the largest $|L_{i,j}|$

Complexity feature: $|\mathcal{V}^{\mathbf{f}}_G|$– the total number of nodes



Layered DAG

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

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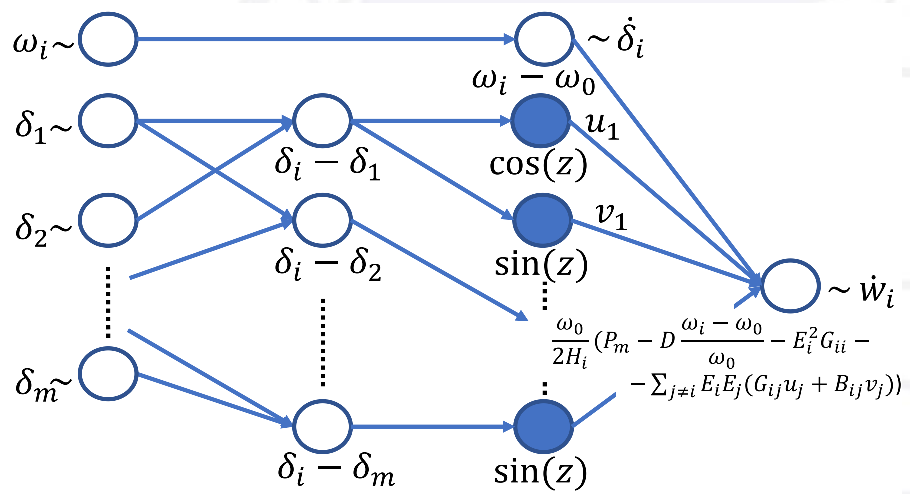
$$\frac{d\omega_i}{dt} = \frac{\omega_0}{2H_i}\left(P_m - D\frac{\omega_i - \omega_0}{\omega_0} - E_i^2 G_{ii} \cdots - \sum_{j=1, j\neq i}^{10} E_i E_j [B_{ij}\sin(\delta_i - \delta_j) + G_{ij}\cos(\delta_i - \delta_j)]\right)$$

$$\frac{d\delta_i}{dt} = \omega_i - \omega_0,$$

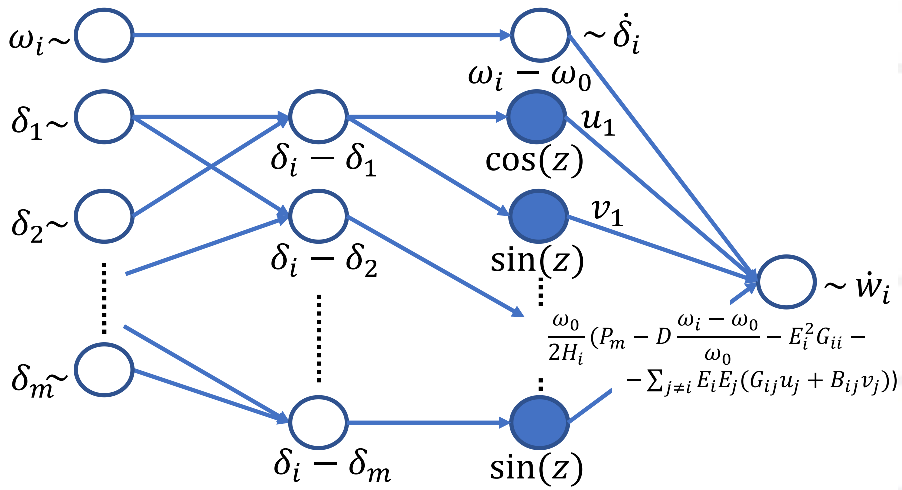


Examples

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

Compositional features

$$r_{max} = 1, \quad \Lambda = 4\pi, \quad |\mathcal{V}| = 2(N_g - 1)N_g,$$
$$L_{max} = \max_{1\leqslant i,j\leqslant N_g, i\neq j}\left\{\frac{\omega_0}{2H_i}E_iE_jG_{ij}, \frac{\omega_0}{2H_i}E_iE_jB_{ij}\right\}$$

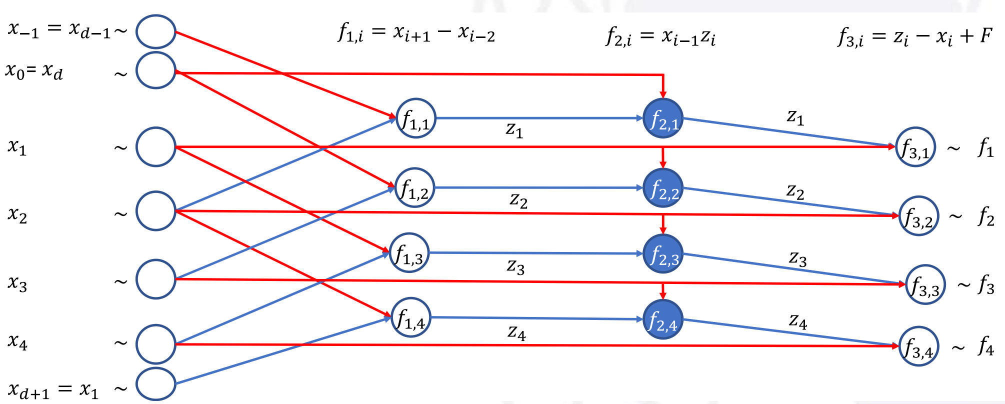


Examples

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

The Lorenz-96 model

$$\dot{\mathbf{x}} = \begin{bmatrix} x_0(x_2 - x_{-1}) - x_1 + F \\ x_1(x_3 - x_0) - x_2 + F \\ \vdots \\ x_{i-1}(x_{i+1} - x_{i-2}) - x_i + F \\ \vdots \\ x_{d-1}(x_{d+1} - x_{d-2}) - x_d + F \end{bmatrix}$$

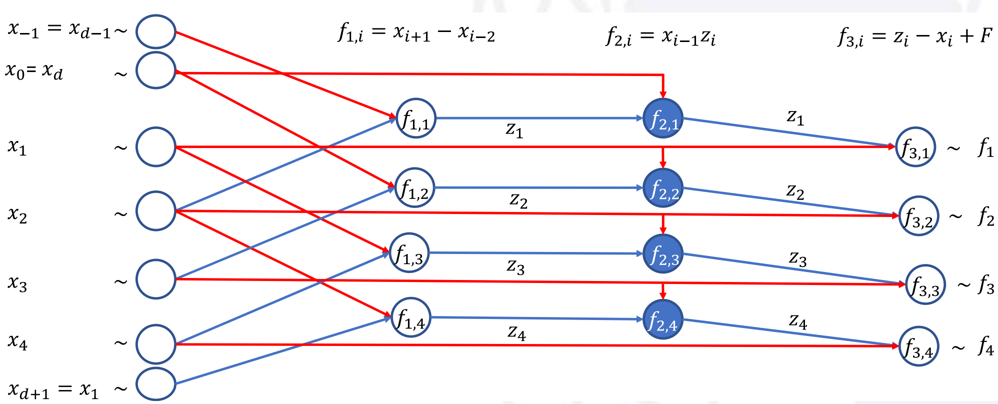


Examples

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

The Lorenz-96 model

$$\begin{array}{lllllll} r_{max} & = & 2, & \Lambda & = & \max\{(2R),1\}(2R+4R^2), \\ L_{max} & = & 1, & |\mathcal{V}| & = & d \end{array}$$



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Examples

Observation: It is widely observed in science and engineering that complicated and high dimensional input-output information relations can be represented as compositions of functions with low input dimensions.

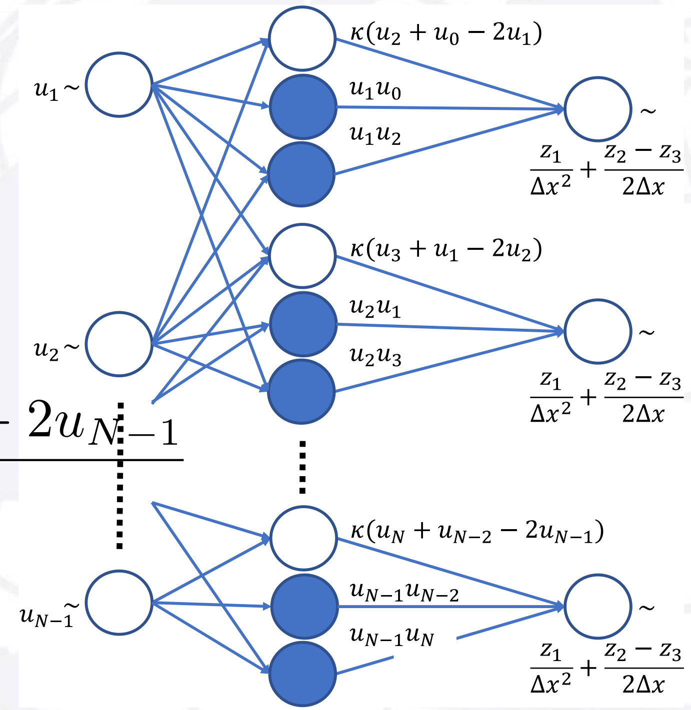
Discretization of PDEs

$$\dot{u}_1 = -u_1 \frac{u_2 - u_0}{2\Delta x} + \kappa \frac{u_2 + u_0 - 2u_1}{\Delta x^2}$$

$$\dot{u}_2 = -u_2 \frac{u_3 - u_1}{2\Delta x} + \kappa \frac{u_3 + u_1 - 2u_2}{\Delta x^2}$$

$$\vdots$$

$$\dot{u}_{N-1} = -u_{N-1} \frac{u_N - u_{N-2}}{2\Delta x} + \kappa \frac{u_N + u_{N-2} - 2u_{N-1}}{\Delta x^2}$$



Examples

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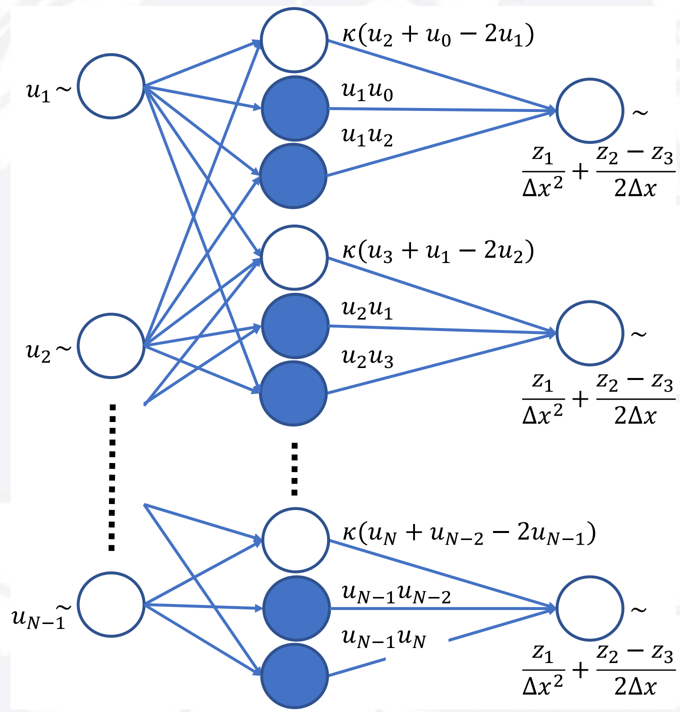
Discretization of PDEs

$$r_{max} = 2, \Lambda = \frac{1}{2}R^2 + R,$$
$$L_{max} = \frac{N}{L}, |\mathcal{V}_G| = 2N.$$

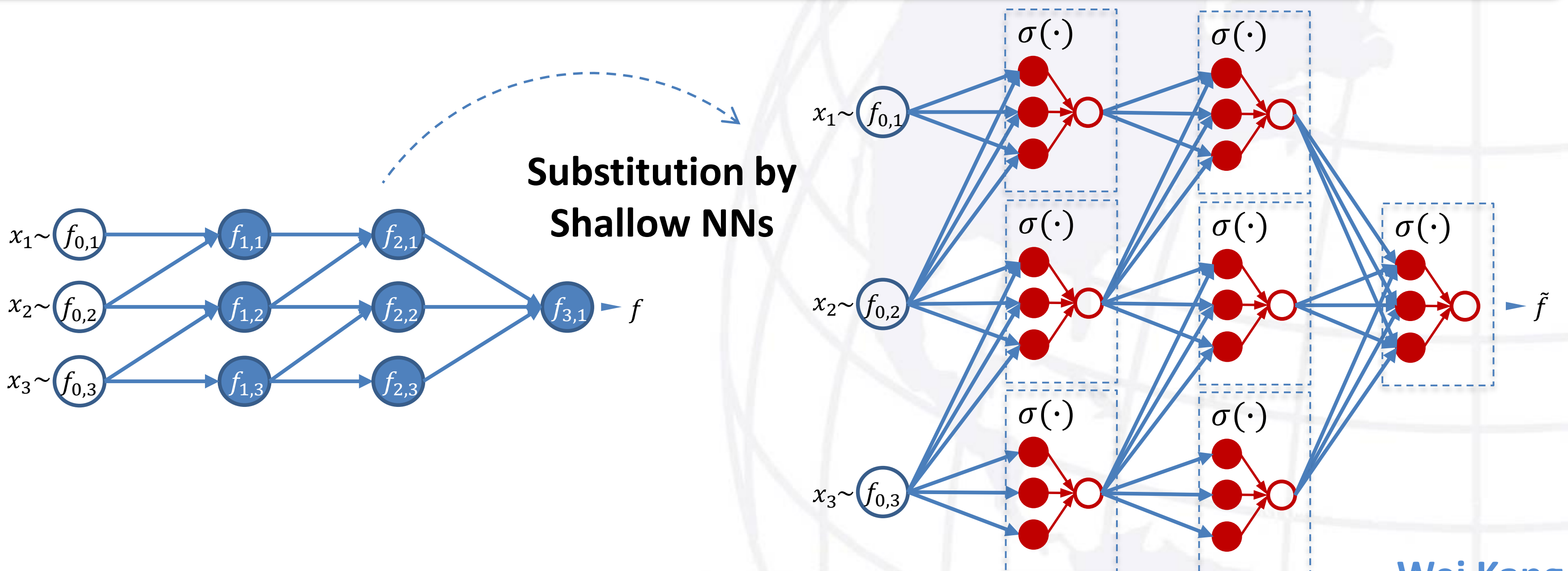


Error propagation - substitution

Theorem Under smoothness assumptions, the neural network approximation error is bounded by

$$\left\|\tilde{\mathbf{f}}(\boldsymbol{x}) - \mathbf{f}(\boldsymbol{x})\right\|_p \leq \sum_{i,j=1}^{K} L^{\mathbf{f}}_{i,j}\epsilon_{i,j}, \quad \text{for all } \boldsymbol{x} \text{ in the domain of } \mathbf{f}.$$

where $L_{i,j}$ is the Lipschitz constant associated with the node $f_{i,j}$, $\epsilon_{i,j}$ is the error of each node due to substitution.

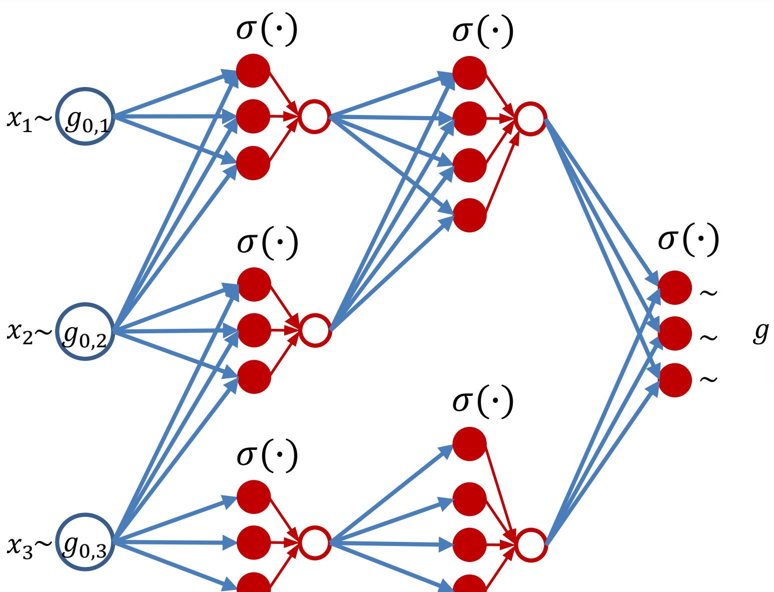


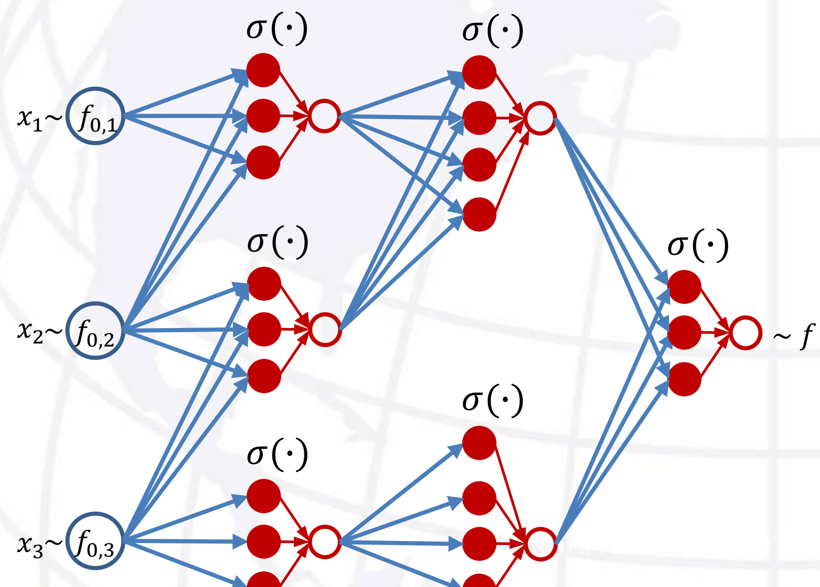
Error propagation - composition

Proposition Let $\tilde{\mathbf{f}}$, $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$ be approximations of $\tilde{\mathbf{f}}$, $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$ with errors bounded by ϵ_1, ϵ_2 and ϵ_3, respectively. Then

$$\left\|(\mathbf{f}(\cdot))^K \circ \mathbf{g}(\boldsymbol{x}) - (\tilde{\mathbf{f}}(\cdot))^K \circ \tilde{\mathbf{g}}(\boldsymbol{x})\right\|_p \leq \frac{(L^{\mathbf{f}})^K - 1}{L^{\mathbf{f}} - 1} e_1 + (L^{\mathbf{f}})^K e_2.$$

$$\left\|\mathbf{h} \circ (\mathbf{f}(\cdot))^K(\boldsymbol{x}) - \tilde{\mathbf{h}} \circ (\tilde{\mathbf{f}}(\cdot))^K(\boldsymbol{x})\right\|_p \leq L^{\mathbf{h}} \frac{(L^{\mathbf{f}})^K - 1}{L^{\mathbf{f}} - 1} e_1 + e_3.$$





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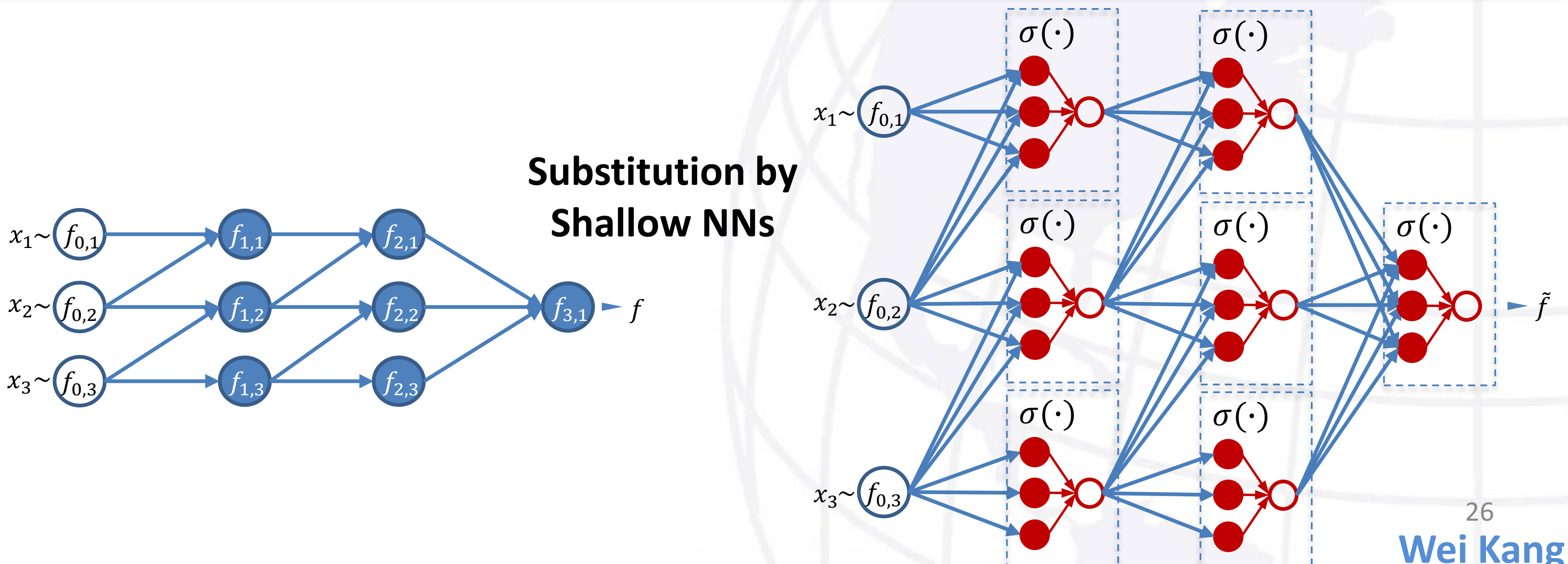
Approximation theory

Theorem Assume all nodes of a compositional function, $\mathbf{f}$, are C^1. For any integer $n_{width} > 0$, there exists a neural network, $\mathbf{f}^{NN}$, that has the following error upper bound,

$$\left\|\mathbf{f}(\boldsymbol{x}) - \mathbf{f}^{NN}(\boldsymbol{x})\right\|_p \leq C_1 L^{\mathbf{f}}_{max} \Lambda^{\mathbf{f}} \left|\mathcal{V}^{\mathbf{f}}_G\right| (n_{width})^{-1/r^{\mathbf{f}}_{max}}, \quad \text{for all } \boldsymbol{x} \in [-R, R]^d,$$

where C_1 is a constant determined by $\{d_{i,j}, m_{i,j}; \quad f_{i,j} \in \mathcal{V}^{\mathbf{f}}_G\}$. The complexity of $\mathbf{f}^{NN}$ is

$$n \leq \left|\mathcal{V}^{\mathbf{f}}_G\right| n_{width}.$$



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Trajectory of ODE

ODE

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, t \in [0, T]$$

Trajectory

$$\boldsymbol{x}(0) = \boldsymbol{x}$$
$$t \to \boldsymbol{\phi}(t; \boldsymbol{x})$$

Theorem Under C^1 assumption, for any $\epsilon > 0$, there exists a neural network approximation, $\boldsymbol{\phi}^{NN}$, satisfying

$$||\boldsymbol{\phi}(T; \boldsymbol{x}) - \boldsymbol{\phi}^{NN}(\boldsymbol{x}))||_p \leq \epsilon, \quad \boldsymbol{x} \in D^{\mathbf{f}}.$$

The complexity of $\boldsymbol{\phi}^{NN}$ is bounded by

$$n \leq C\epsilon^{-(r^{\mathbf{f}}+1)}$$

where C is a polynomial function of the compositional features of $\mathbf{f}$.

Optimal control

The problem of optimal control

$$\begin{cases} \min\limits_{U} J = \Psi(\boldsymbol{x}(T)), & \boldsymbol{x} \in \mathbb{R}^d \\ \dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, \boldsymbol{u}), & \boldsymbol{u} \in \mathbb{R}^q, \end{cases}$$

Zero-order hold control

$$U = \begin{bmatrix} \boldsymbol{u}(t_0)^T & \boldsymbol{u}(t_1)^T & \cdots & \boldsymbol{u}(t_{N_t}) \end{bmatrix}$$

Trajectory and terminal state

$$\boldsymbol{x}(T) = \boldsymbol{\phi}(T; \boldsymbol{x}_0, U)$$

Cost function

$$J = \Psi \circ \boldsymbol{\phi}(T; \boldsymbol{x}_0, U)$$

Theorem Under convexity and C^1 assumptions, for any $\epsilon > 0$ there exists a neural network approximation of the optimal control satisfying

$$||U^{*NN}(\boldsymbol{x}) - U^*(\boldsymbol{x})||_2 \leq 3\epsilon, \quad \boldsymbol{x} \in D.$$

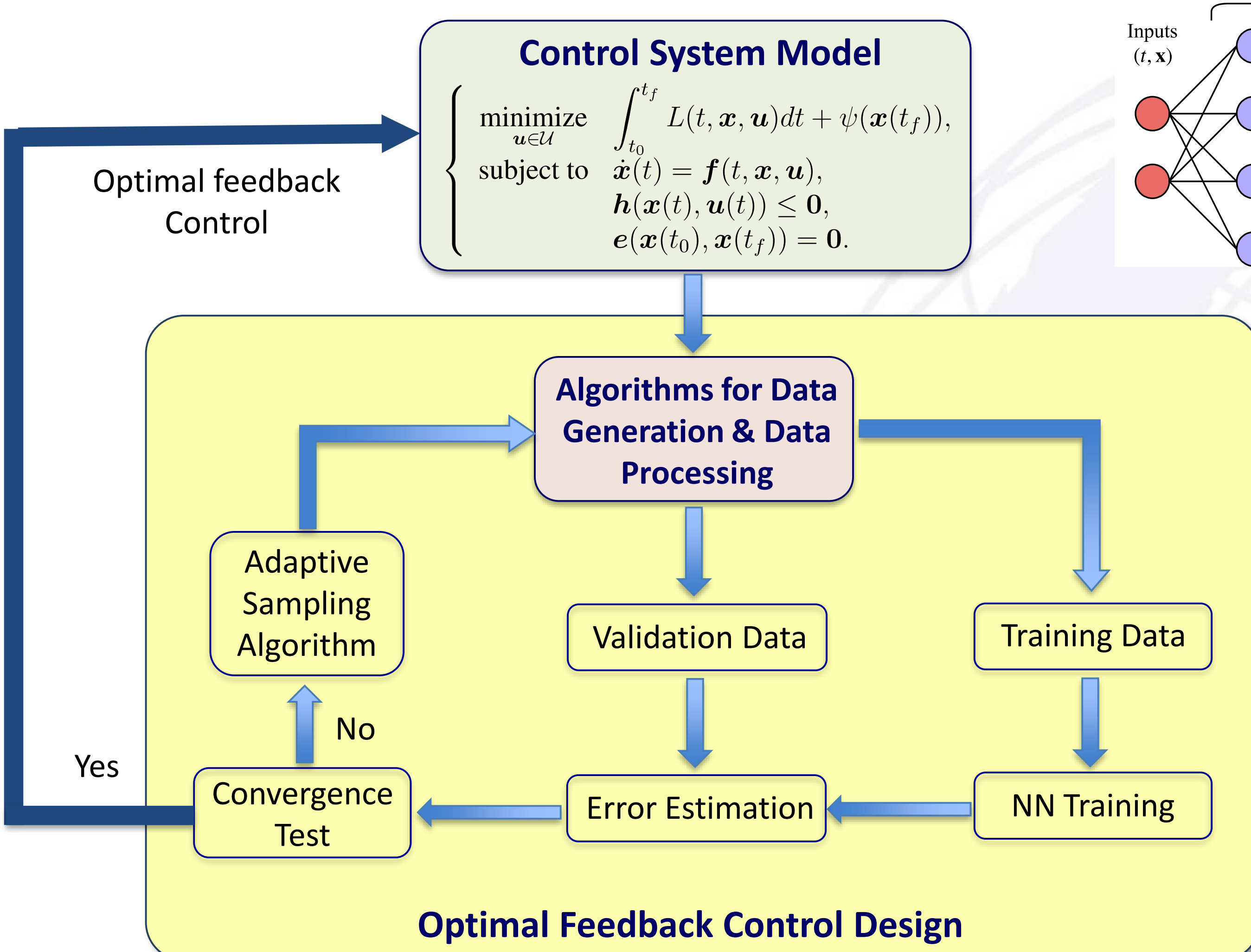
The complexity of U^{*NN} is bounded by

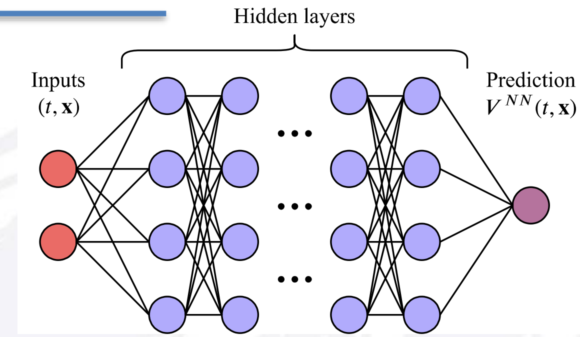
$$n \leq C\epsilon^{-(4r+1+\frac{4r}{r^{\mathbf{f}}_{max}})}$$

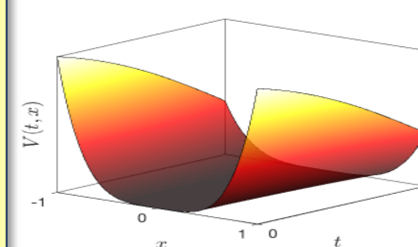
where $r = \max\{r^{\mathbf{f}}_{max}, r^{\Psi}_{max}\}$, C is a polynomial function of the compositional features of $\mathbf{f}$, Ψ, and the Hessian matrix of the cost function.

A model-based data-driven approach

❖ T. Nakamura-Zimmerer, Q. Gong, W. Kang, *Adaptive deep learning for high-dimensional HJB equations*, SIAM J. Scientific Computing, 2021.



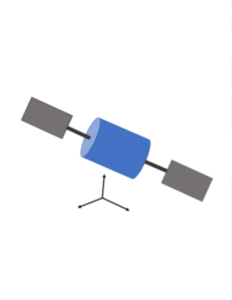




Example - rigid body optimal control

Problem formulation

$$\begin{cases} \min\limits_{\boldsymbol{u}(\cdot)} \int_0^{t_f} L(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u})d\tau + \frac{W_4}{2}\|\boldsymbol{v}(t_f)\|^2 + \frac{W_5}{2}\|\boldsymbol{\omega}(t_f)\|^2, \\ \dot{\boldsymbol{v}} = \boldsymbol{E}(\boldsymbol{v})\boldsymbol{\omega}, \\ \boldsymbol{J}\dot{\boldsymbol{\omega}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}(\boldsymbol{v})\boldsymbol{h} + \boldsymbol{B}\boldsymbol{u}. \end{cases}$$



State variables:

$\boldsymbol{v} = \begin{pmatrix}\phi & \theta & \psi\end{pmatrix}^T$ Euler angles

$\boldsymbol{\omega} = \begin{pmatrix}\omega_1 & \omega_2 & \omega_3\end{pmatrix}^T$ angular velocity

Running cost and weights:

$L(\boldsymbol{v},\boldsymbol{\omega},\boldsymbol{u}) = \frac{W_1}{2}\|\boldsymbol{v}\|^2 + \frac{W_2}{2}\|\boldsymbol{\omega}\|^2 + \frac{W_3}{2}\|\boldsymbol{u}\|^2,$

$W_1 = 1, W_2 = 10, W_3 = 0.5, W_4 = 1, W_5 = 1, t_f = 20.$

Matrices

$$\boldsymbol{E}(\boldsymbol{v}) := \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{pmatrix}, \qquad \boldsymbol{S}(\boldsymbol{\omega}) := \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix},$$

$$\boldsymbol{R}(\boldsymbol{v}) := \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\theta\sin\phi \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\theta\cos\phi \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1/20 & 1/10 \\ 1/15 & 1 & 1/10 \\ 1/10 & 1/15 & 1 \end{pmatrix}, \qquad \boldsymbol{J} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad \boldsymbol{h} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Example – rigid body optimal control

HJB equation and feedback law

Define the Hamiltonian

$$H(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}) = L(t, \boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{\lambda}^T \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u})$$

Solve the HJB equation to find the optimal feedback law

$$\begin{cases} V_t(t, \boldsymbol{x}) + \min\limits_{\boldsymbol{u} \in \mathcal{U}} H(t, \boldsymbol{x}, V_{\boldsymbol{x}}, \boldsymbol{u}) = 0, \\ V(t_f, \boldsymbol{x}) = \psi(\boldsymbol{x}), \end{cases}$$

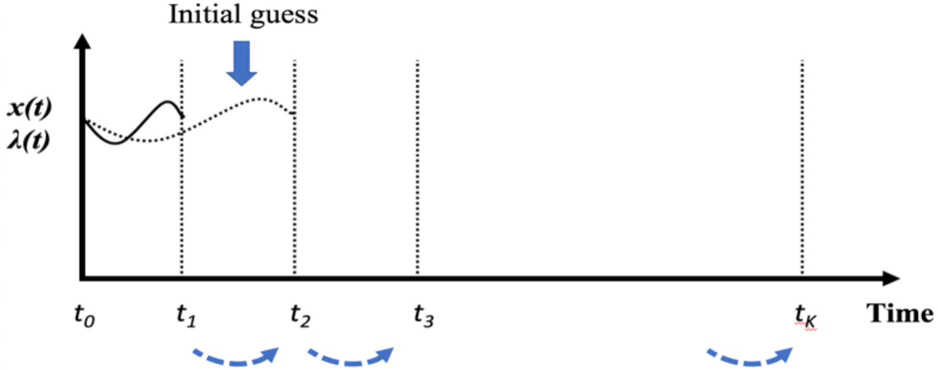
$$\boldsymbol{u}^*(t, \boldsymbol{x}) = \arg \min_{\boldsymbol{u} \in \mathcal{U}} H(t, \boldsymbol{x}, V_{\boldsymbol{x}}, \boldsymbol{u}).$$

Time-marching

Pontryagin's maximum principle

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}^*), & \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \dot{\boldsymbol{\lambda}}(t) = -H_{\boldsymbol{x}}(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}^*), & \boldsymbol{\lambda}(t_f) = F_{\boldsymbol{x}}(\boldsymbol{x}(t_f)), \\ \dot{v}(t) = -L(t, \boldsymbol{x}, \boldsymbol{u}^*), & v(t_f) = F(\boldsymbol{x}(t_f)). \end{cases}$$

where $\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} H(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u})$



1. Choose a partition of $[t_0, t_f]$,
$$t_0 < t_1 < t_2 < \cdots < t_K = t_f.$$
2. In $[t_0, t_1]$, solve the TPBVP,
$$(\boldsymbol{x}^1(t), \boldsymbol{\lambda}^1(t)).$$
3. Extending the trajectory to $[t_0, t_2]$,
$$\boldsymbol{x}_0^2(t) = \begin{cases} \boldsymbol{x}^1(t), & \text{if } t_0 \le t \le t_1, \\ \boldsymbol{x}^1(t_1), & \text{if } t_1 < t \le t_2, \end{cases}$$
$\boldsymbol{\lambda}_0^2(t)$ is similarly defined.
4. $(\boldsymbol{x}_0^2(t), \boldsymbol{\lambda}_0^2(t))$ is used as an initial guess to solve the TPBVP over $[0, t_2]$.
5. Repeating the process until $t_K = t_f$.

Neural network warm start

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}^*), & \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \dot{\boldsymbol{\lambda}}(t) = -H_{\boldsymbol{x}}(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}^*), & \boldsymbol{\lambda}(t_f) = F_{\boldsymbol{x}}(\boldsymbol{x}(t_f)), \\ \dot{v}(t) = -L(t, \boldsymbol{x}, \boldsymbol{u}^*), & v(t_f) = F(\boldsymbol{x}(t_f)). \end{cases}$$

where $\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} H(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u})$

1. Generate a first data set without initial guess (time-marching).
2. Train a neural network $V^{NN}(t, \boldsymbol{x})$.
3. Generate more data using warm start $\boldsymbol{\lambda}_0(t) = V^{NN}_{\boldsymbol{x}}(t, \boldsymbol{x})$.

Time-marching method

K	% BVP convergence	mean integration time
1	0.3%	0.37 s
2	38.7%	0.44 s
3	76.2%	0.40 s
4	92.9%	0.45 s
8	98.4%	0.53 s

Neural network warm start

μ	% BVP convergence	mean integration time
0	90%	0.44 s
10^{-3}	99.6%	0.41 s
10^{1}	100%	0.40 s

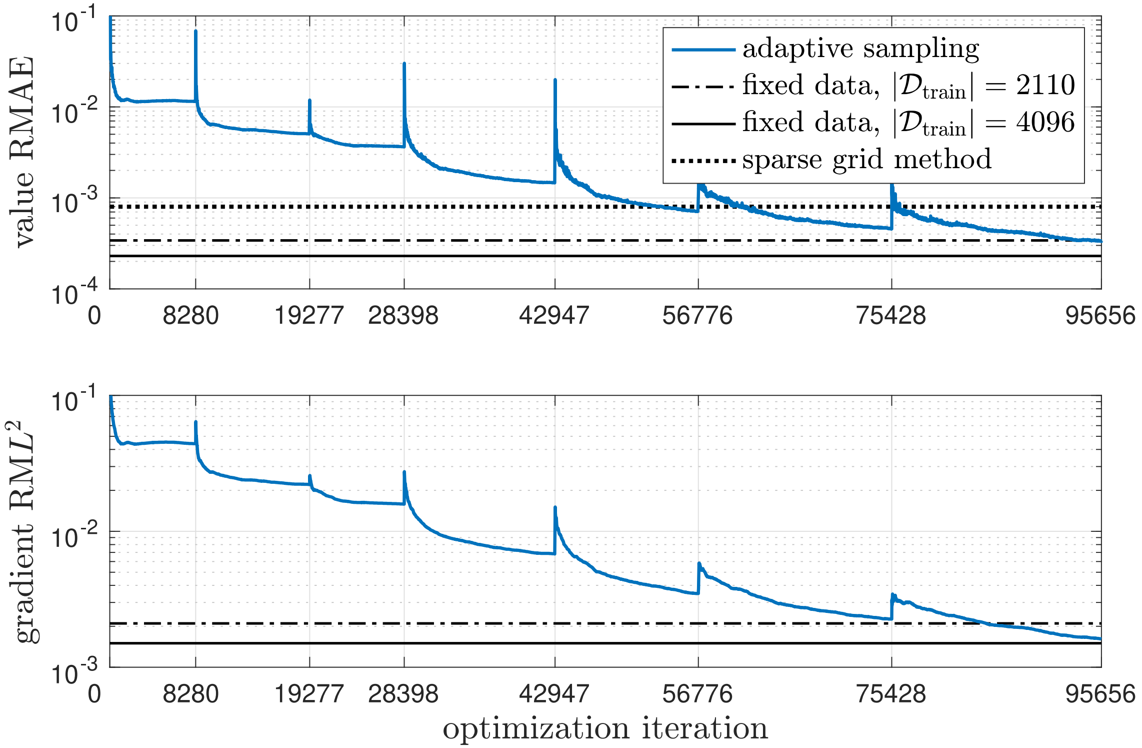
Rigid body optimal attitude control

NAVAL POSTGRADUATE SCHOOL

Example - rigid body optimal control

❖ T. Nakamura-Zimmerer, Q. Gong, W. Kang, *Adaptive deep learning for high-dimensional HJB equations*, SIAM J. Scientific Computing, 2021.

Neural network approximation of value function

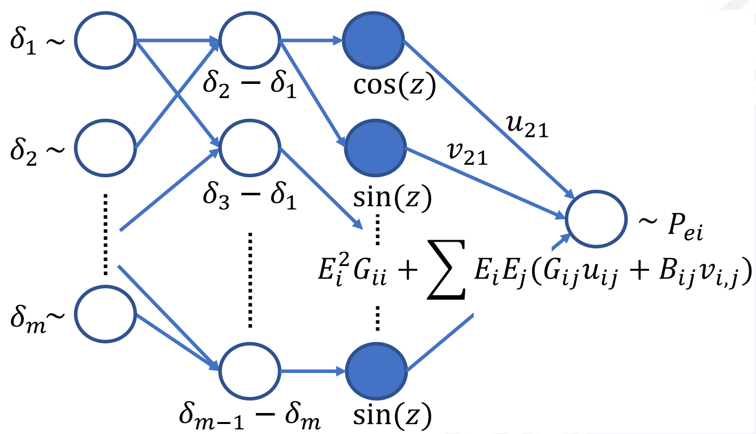


- Gradient loss weight $\mu = 10$; tolerance $\epsilon = 0.1$
- Training data set updated adaptively
- The final adaptive data set has 2110 points

Power system stability

Power system electric air-gap torque

$$(\mathbf{P}_e)_i = E_i^2 G_{ii} + \sum_{j=1, j\neq i}^{m} E_i E_j (G_{ij}\cos(\delta_i - \delta_j) + B_{ij}\sin(\delta_i - \delta_j)), \quad 1 \leq i \leq m,$$



$$\mathbf{r_{max}} = \mathbf{1},$$
$$\boldsymbol{\Lambda} = \mathbf{4}\pi,$$
$$|\mathcal{V}| = \mathbf{2}(\mathbf{N_g} - \mathbf{1})\mathbf{N_g},$$
$$\mathbf{L_{max}} = \max_{1\leq i,j\leq N_g, i\neq j}\left\{\frac{\omega_0}{\mathbf{2H_i}}\mathbf{E_iE_jG_{ij}}, \frac{\omega_0}{\mathbf{2H_i}}\mathbf{E_iE_jB_{ij}}\right\}$$

Power system domain of attraction

Theorem: Let $\mathcal{R} \subset \mathbb{R}^{2N_g}$ be a bounded set. Then, there exists a solution, $V(\mathbf{x})$, to Zubov's equation (a special Lyapunov function that characterizes the domain of attraction) and a neural network, $V^{NN}(\mathbf{x})$, that has n^{NN} hyperbolic tangent neurons. They satisfy

$$\left|V^{NN}(\mathbf{x}) - V(\mathbf{x})\right| < (C_1 N_g^2 + C_2)\frac{N_g}{\sqrt{n^{NN}}}$$

for $\mathbf{x} \in \mathcal{R}$, where C_1 and C_2 are constants independent of N_g.

Power system stability

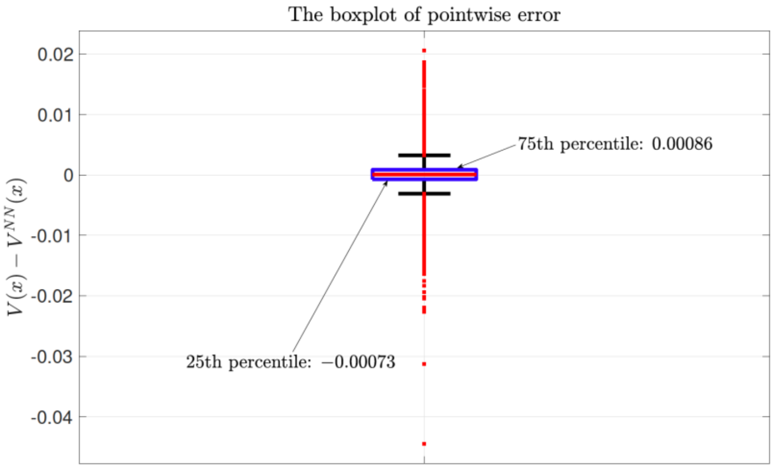
New England 10-generator 39-bus power system

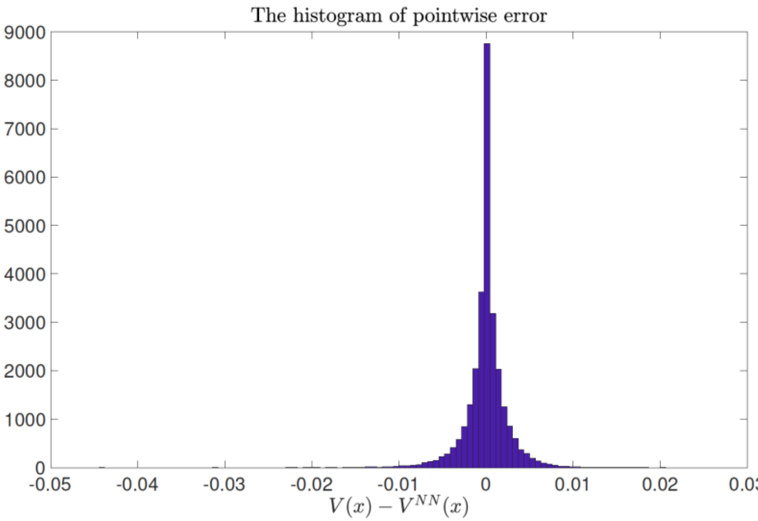
Characterization of the domain of attraction

--Neural network approximation of Zubov's equation

Activation:	hyperbolic tangent
Depth:	16
Width:	40

Error: $$RMSE = 2.0 \times 10^{-3}$$





Next step research

Future work

A lot more questions than answers

- The compositional features of nonsmooth problems, ReLU activation?
- How does compositional structure help to improve NN design and training?
- How does compositional structure help to validate a result, empirical risk vs population risk, L_2-norm vs infinity norm?
- Applications to dynamical systems (space dimension >5)
  - Deep filter and data assimilation (observability in unobservable systems)
  - Output regulation and FBI equation
  - Control Lyapunov function
  - Optimal control and viscosity solutions
  - Reachable set of control systems
  - The boundary of domain of attraction

THANK YOU

Problems appropriate for ML

What makes a good problem for machine learning?

- The overall problem does not have tractable solution.
  And
- You have access to lots of data.
  And
- The problem does not require highly accurate numerical solution (such as machine precision).

Give machine learning a try.

Deep learning for dynamics and control

Quantitative performance + **High dimension (not in normal form)** + **Real-time**

↓

Curse-of-dimensionality

Performance associated with PDEs

- HJB equation – optimal control
- HJI equation – differential game
- FBI equation – output regulation
- Zakai equation – optimal estimation
- Zubov's equation – the boundary of domain of attraction