Curvature Bounded Above II

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September 7, 2022

Majorization theorem of Reshetnyak

Theorem

A complete, geodesic space X is CAT(0) iff any rectifiable curve $\gamma : S^1 \to X$ is majorized by a convex subset in $\mathbb{R}^2$.

Characterizations of CAT

The following are **equivalent** for a complete, geodesic space X

X is CAT(0).

Sum of angles in any triangle is at most π.

The majorization theorem holds true.

Lipschitz curves $\gamma : S^1 \to X$ bound discs of area $\leq \frac{1}{4\pi} \mathit{length}^2(\gamma)$.

$|x_1 - x_3|^2 + |x_2 - x_4|^2 \leq \sum |x_i - x_{i+1}|^2$ for all $x_1, ..., x_4 \in X$.

For $A \subset \mathbb{R}^n$ and $f : A \to X$ 1-Lipschitz there is 1-Lipschitz extension $\hat{f} : \mathbb{R}^n \to X$.

Localization

X has curvature $\leq \kappa$ if X is locally CAT(κ).

A Riemannian manifold has curvature $\leq \kappa$ if all sectional curvatures are $\leq \kappa$ (Rauch).

In a geodesic space of curvature $\leq \kappa$ there are no "conjugate points" along local geodesics up to length $\mathrm{diam}(M^2_\kappa)$.

Globalization I

A complete, geodesic space X of curvature $\leq \kappa$ is CAT(κ) iff geodesics of length $< \mathrm{diam}(M_\kappa^2)$ unique and continuously depend on endpoints.

Finite Euclidean simplicial complex is non-positively curved iff injectivity radius is positive.

Globalization II

Let $\kappa \le 0$

There exist curve-shortening process $\gamma \to \gamma_t$ non-changing endpoints, such that $\gamma = \gamma_0$ and γ_1 is a local geodesic.

X be complete, geodesic of curvature $\le \kappa \le 0$. Then X is CAT(κ) iff $\pi_1(X) = 1$.

A geodesic, compact, locally CAT(1) space is CAT(1) iff there are no small isometrically embedded circles.

CAT(0) and groups

If X is locally CAT(0) then universal covering is contractible.

X is $K(\pi_1(X), 1)$.

Topology of X and coarse toology of $\tilde{X}$ are encoded in $\pi_1(X)$.

Isometries of $\tilde{X}$ are elliptic, hyperbolic or parabolic. They act on the boundary at infinity of $\tilde{X}$.

Rank-one and higher-rank dichotomy.

Tools I: Convexity

Controlled (semi)-convexity of distance functions.

Abundance of convex subsets.

Semiconcave functions have well-defined gradients and controlled gradient flows.

Tools II: Center of mass

There is a center-of-mass map $c : \mathcal{M}(X) \to X$ which assignes to a probability measure with small support a center of mass.

If $\kappa \leq 0$ then c is contracting.

For convex combinations of Dirac measures $\mu = \sum_{i=0}^{m} a_i \delta_{x_i}$ the map c defines *barycentric m-simplices* in X.

Tools III: Dimension theory

The (completed) space of direction Σ_x at any x is CAT(1).

The (geodesic) tangent cone $T_x X = C(\Sigma_x)$ is CAT(0), embedded in any blow up at x.

For a separable CAT-space the following are equal:
1. Homological dimension
2. Topological dimension
3. Maximal dimension of non-degenerated barycentric simplex
4. 1+ (Maximal dimension of Σ_x)
5. 1+(Maximal dimension of $\mathbb{S}^{m-1}$ in some Σ_x)

Tools IV: Minimizing discs

Discs in X can be used to verify CAT(κ) and to investigate CAT(κ) spaces.

Reshetnyak majorization, isoperimetric characterization.

CAT-metrics on discs are all known (uniformization theorem).

Ruled discs in CAT(κ) are CAT(κ).

Minimal discs in CAT(κ) are conformal and CAT(κ).

Distance-minimizing discs are CAT(κ).

Harmonic discs are CAT(κ).