Curvature Bounded Above I

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Plan

1. Definitions, properties, examples.

2. Properties, characterizations, tools.

3. Spaces with extendible geodesics.

Advertisment:
Books by Alexander–Kapovitch–Petrunin, papers by L.–Nagano.

Very short history

Precursors: Gauss, Riemann, Hadamard, Cartan, Busemann, Wald ...

Foundation: Alexandrov, Reshetnyak, Nikolaev, Berestovskii ...

Modern period: Gromov, ...

Range of Applications: Algebraic Groups, GGT, PDE, Topology ....

Alexandrov's Lemma

$Q = xyzp$ Euclidean quadrangle, concave at p.

$\bar{Q} = \bar{x}\bar{y}\bar{z}\bar{p}$ quadrangle, same sidelengths and $\bar{p} \in [\bar{x}\bar{z}]$. Then

1. $|y - p| \leq |\bar{y} - \bar{p}|$.

2. Canonical comparison map $I : \bar{Q} \to Q$ is 1-Lipschitz.

3. There is 1-Lipschitz map $\hat{I}$ from Jordan domain $\bar{J}$ of $\bar{Q}$ onto Jordan domain J of Q, sending $\bar{x}$ to x .....

Some comments

On the last slide $3 \implies 2 \implies 1$.

Implications $1 \implies 2 \implies 3$ follow from Kirzsbraun's

Theorem

Any 1-Lipschitz map $f : A \to \mathbb{R}^n$ for $A \subset \mathbb{R}^n$ extends to 1-Lipschitz map $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$.

Statements 1-3 true for model spaces $\mathbb{H}^n(\kappa), \mathbb{S}^n(\kappa)$.

Definitions

X be a complete, geodesic metric space.
Triangle Δ in X are 3 geodesics $[xy], [yz], [zx]$.

Comparison triangle $\bar{\Delta}$ in M^2_κ has same sides.

Canonical *comparison map* $I_\Delta : \bar{\Delta} \to \Delta$.

Δ is κ-thin if I_Δ is 1-Lipschitz.

A complete, geodesic space X is CAT(κ) (*curvature globally bounded above by* κ) if all triangles are κ-thin.

Basic examples and comments

$\mathbb{H}^n$ is CAT(-1), CAT(0) and CAT(1).

$\mathbb{R}^n$ is CAT(0) and CAT(1) not CAT(-1)

$\mathbb{S}^n$ is CAT(1) not CAT(0).

CAT(κ) $\implies$ CAT(k) if $\kappa \leq k$.

X is CAT(κ) then $\lambda \cdot X$ is CAT($\frac{1}{\lambda^2} \cdot \kappa$).

After resclaing only $\kappa = \pm 1, 0$ matter.

Most theory does not depend on κ. Important exception: for $\kappa > 0$ the *globalization theorem* is not powerful. CAT(1) often plays an auxiliary role.

Basic properties I

We stick to $\kappa = 0$. Other cases similar.

Geodesics are unique. Depend continuously on endpoints.

The space is contractible. Balls are contractible.

Basic properties II: Angles

The κ-*comparison angle* $\bar{\angle}^\kappa xyz$ is the angle at $\bar{y}$ in the comparison triangle $\bar{x}\bar{y}\bar{z}$.

For geodesics γ_1, γ_2 starting at y the *angle* between γ_1 and γ_2

$$\angle_y(\gamma_1, \gamma_2) = \lim_{s,t \to 0} \bar{\angle}^\kappa \gamma_1(s) y \gamma_2(t) \ .$$

If X is CAT(κ) then κ-comparison angles monotone.

$\angle xyz$ exists and $\angle xyz \leq \angle^\kappa xyz$.

First variation formula for distances holds.

Basic properties III: convexity

CAT(0) spaces are as convex as Euclidean spaces.

Closed balls are convex. Hence CAT(0).

The distance function $X \times X \to \mathbb{R}$ is convex.

The distance function $dist_x^2$ is 2-convex (as convex as in $\mathbb{R}^2$).

Foot-point projection to convex set is 1-Lipschitz

Reshetnyak's Gluing Theorem

Lemma

Let $[xyz]$ be triangle in metric space X. Let $p \in [xz]$. If ypx and ypz are κ-thin then xyz is κ-thin.

Theorem

Let X_1, X_2 be CAT(κ), $A_i \subset X_i$ convex and $I : A_1 \to A_2$ isometry. Then the gluing $X_1 \cup_I X_2$ is CAT(κ).

Examples II

A complete Riemannian manifold X is CAT(κ) iff sectional curvature of (all tangent planes in) X is $\leq \kappa$ and injectivity radius not smaller than $\mathtt{diam}(M_\kappa^2)$.

For $\kappa \leq 0$ the condition on injectivity radius is equivalent to $\pi_1(X) = 1$ (Theorem of Cartan–Hadamard).

Closed convex subsets of CAT(κ) are CAT(κ).

$X_1 \times X_2$ is CAT(0) iff X_1, X_2 are CAT(0).

Cone(X) is CAT(0) iff X is CAT(1).

Examples III

Trees are CAT(κ) for every κ.

Gluing of spaces.

Jordan domains of polygons in $\mathbb{R}^2$.

A limit of CAT(κ) spaces is CAT(κ).

Arbitrary Jordan domains (Bishop).

Examples IV

Spaces of L^2-maps: $L^2(\Omega, X)$ if X CAT(0).

Space of Riemannian metrics, Teichmueller space, space of Kaehler potentials are CAT(0).

Ruled surfaces, minimal discs inherit CAT property.

Warped products, subsets with bounded second fundamental form, conformal changes, general subsets in dimension 2 ....

Thanks

Many thanks for your attention!