Homological Mirror Symmetry for Theta Divisors

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Mirror symmetry origins in string theory

How to unite theories of quantum mechanics (small particles) and gravity in general relativity (large objects)?

Conjecture: Quarks made of vibrating strings in 4 spacetime + 6 additional dimensions wrapped tightly in a compact Calabi-Yau (CY) manifold X. CY means curvature quantity $c_1 = 0$.

How to measure string states?

Mirror symmetry origins continued

String state = Sigma Model $\sigma : S^1 \times \mathbb{R} \to X$.
String at time t is $\sigma(S^1 \times \{t\})$.

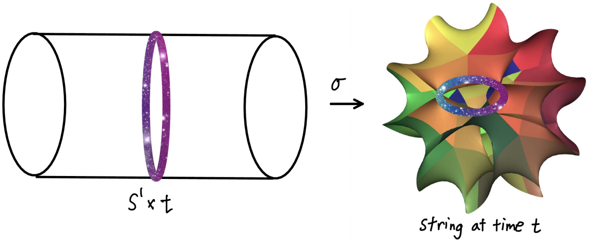
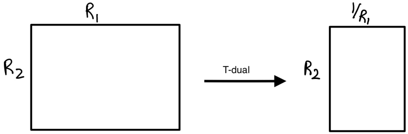


Figure: On right: 2D-slice of a 6D CY quintic from Wikipedia

Partition function = integrate over all states (sigma models) weighted by energy. T-duality finds pair X and Y with same partition function = foundation for mirror pair.

Mirror symmetry in math

Complex geometry on X (notion of $\cdot\sqrt{-1}$ in coordinates) = symplectic geometry on Y (closed, non-degen area 2-form ω)



- $X = T^2 = [0, R_1] \times [0, R_2]/\sim$. Area $R_1 R_2$, complex structure $= i\frac{R_2}{R_1}$.
- $Y = T^2 = [0, \frac{1}{R_1}] \times [0, R_2]/\sim$. Area $= R_2/R_1$, complex structure $= iR_1 R_2$.

[SYZ96] define SYZ mirror: same base, invert torus fiber radii.

Fano $(c_1 > 0)$ & general type (e.g. $c_1 < 0$) mirrors are non compact complex Y + holo $v : Y \to \mathbb{C}$ (Landau-Ginzburg)

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Homological mirror symmetry (HMS)

Kontsevich's HMS conjecture [Kon95]

$D^\pi Fuk(Y) = D^b Coh(X)$

Terms: symplectic "A-model" $\leftrightsquigarrow$ complex "B-model"

A-model: The Fukaya category has, roughly, Lagrangians L_i as objects, their intersection points are morphisms, denoted $H^*(CF(L_i, L_j) = \oplus_{p \in L_i \cap L_j} \mathbb{C} \cdot p, \partial)$, ∂ counts pseudo-holomorphic bigons between intersection points, and composition counts pseudo-holomorphic triangles.

B-model: D^bCoh has, roughly, line bundles as objects with morphisms of line bundles. Will use: $\mathscr{H}om(\mathcal{L}^{\otimes i}, \mathcal{L}^{\otimes j}) \cong \mathcal{L}^{-i} \otimes \mathcal{L}^j \cong \mathcal{L}^{j-i}$ therefore a morphism is a section of $\mathcal{L}^{j-i}$.

Some examples

HMS results are proven in many cases. Here are some.

- Abelian varieties $(\mathbb{C}^*)^n/\Gamma$: [PZ98], [Fuk02]
- B-model toric varieties (partial compactifications of $(\mathbb{C}^*)^n$): [Abo06], [Abo09], [Han19], [HH22]
- B-model hypersurfaces of $(\mathbb{C}^*)^n$: [AAK16], [AA]
- B-model hypersurfaces of abelian varieties: [Can20], [ACLLa], [ACLLb]
- Seidel (A-model genus 2 curve), Keating (A-model hypersurface cusp singularities), Hacking-Keating (B-model log CY surfaces), Ward (B-model elliptic Hopf surfaces), Sheridan (Fano & CY hypersurfaces of $\mathbb{CP}^n$), Sheridan-Smith (generalized Greene-Plesser mirrors), Lee (A-model open Riemann surfaces), Lekili-Ueda (Milnor fibers of simple singularities), Qin (Coisotropic Branes on Tori) . . .

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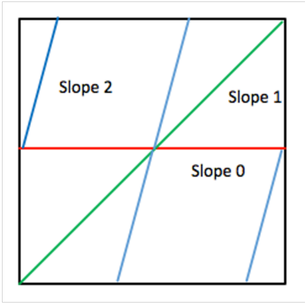
HMS for T^2



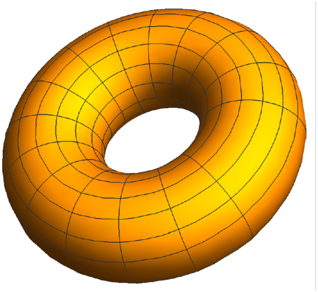


Figure: Slope j lines on torus of symplectic area $A \leftrightsquigarrow$ deg j line bundles on complex torus with lattice length A. Composition matches: counting triangles $\leftrightsquigarrow$ multiplying theta functions $=$ sections of line bundles. [PZ98]

Homological mirror symmetry for $\mathbb{T}^{2n}$

Definition (B-model principally polarized abelian variety)

–$V_\tau = (\mathbb{C}^*)^n/\Gamma_\tau$. Complex structure parametrized by $\tau = B + i\Omega$ where $(B, \Omega) \in S_n(\mathbb{R}) \times P_n(\mathbb{R})$ of symmetric and symmetric positive definite matrices respectively.

–Generator of $D^b_{\mathcal{L}_\tau}\mathrm{Coh}(V_\tau)$ is $\mathcal{L}_\tau = (\mathbb{C}^*)^n \times \mathbb{C}/\Gamma_\tau \to V_\tau$ ample line bundle of degree 1. $\mathbb{L}_{[z]} = \top^*_{[z]}\mathcal{L} \otimes \mathcal{L}^{-1}$ where $\top_{[z]}$ shifts torus coordinate by z. These are degree 0.

Definition (A-model)

$V_\tau^\vee = \{(\Omega^{-1}\xi =: r, \theta) \in \frac{\mathbb{R}^{2n}}{\mathbb{Z}^{2n}}\} \supset \ell_{j,b} = \{\theta = b - jr\}_{r \in \frac{\mathbb{R}^n}{\mathbb{Z}^n}}$

$\hat{\ell}_{j,[z]} = (\ell_{j,b}, \nabla_a := d - 2\pi i a dr$ on $\underline{\mathbb{C}})$, $F_{\nabla_a} = B$, $z := a + \tau b$

f.f. functor $D^b_{\mathcal{L}_\tau}\mathrm{Coh}(V_\tau) \hookrightarrow H^0\mathrm{Fuk}(V_\tau^\vee)$, $\mathcal{L}_\tau^j \otimes \mathbb{L}_{[z]} \mapsto \hat{\ell}_{j,[z]}$

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$n = 2$ theta divisor $\Theta_\tau \implies$ global HMS [ACLLa]

A theta divisor is the 0-set of a section of $\mathcal{L}_\tau \to V_\tau^{2n}$. For $n = 2$ this is a genus 2 curve.

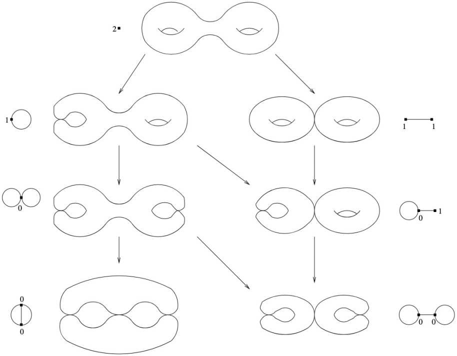


Figure: Global genus 2 curve moduli = theta divisor in abelian surface $\mathbb{T}^4$

Summary of [ACLLa]

In complex dimension 2, we can match up HMS for not just one genus 2 curve with its mirror (as in [Can20]), but for the moduli of all of them.

We first need to say which symplectic structure corresponds to which complex structure. This can be understood using the theory developed in [CMV13].

A Kähler form is a type of symplectic form. We use [KL19]'s.

Theorem ([ACLLa])

The Kähler cones in the Kähler space of the (generalized) SYZ mirror to the genus 2 curve are in one-to-one correspondence with the 3-dimensional cones in the *Voronoi decomposition* of $\mathcal{H}_2^{\text{trop}}$, the tropical Siegel space for genus 2.

Summary of [ACLLa], continued

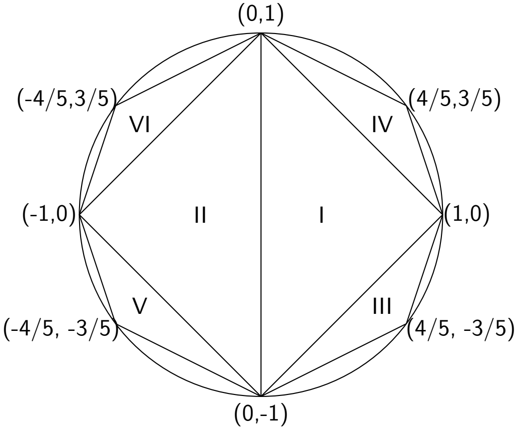


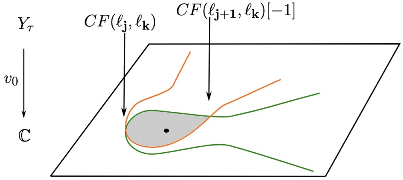
Figure: Pictorial depiction of the first six Kähler cones for the theta divisor mirror when $n = 2$

Generalized SYZ mirror to $\Theta_\tau \subset V_\tau$

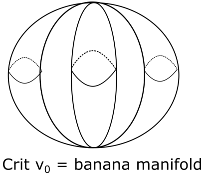
Complex side: restrict from V_τ to Θ_τ.

$$\iota^* : D^b_{\mathcal{L}_\tau}\mathsf{Coh}(V_\tau) \to D^b_{\mathcal{L}_\tau}\mathsf{Coh}(\Theta_\tau)$$

Symplectic side: $\mathsf{SYZ}(V_\tau)$ is now a fiber, get bigger by parallel transporting symplectic geometry in a fiber around arc in base.



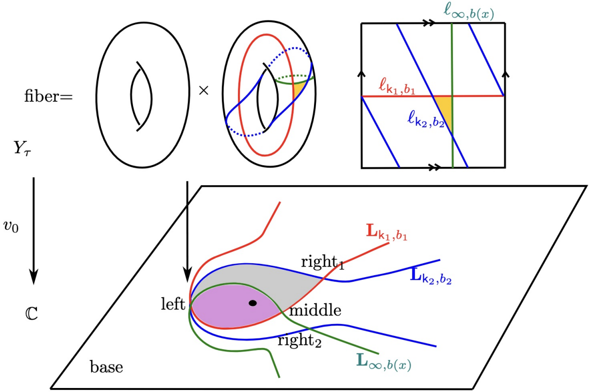
(a) (Mirror of Θ_τ) $= (Y_\tau, {v_0}_\tau)$ has fibers that are SYZ mirror of V_τ.



(b) $\mathsf{Crit}(v_0)$ = banana manifold when $n = 2$

Composition in the Fukaya-Seidel category

The Fukaya category is so named after the work of [FOOO09]. Seidel adapted it to certain symplectic fibrations [Sei08], so the corresponding category is called the Fukaya-Seidel category.



HMS for Theta Divisors

Theorem ([ACLLa], [ACLLb])

Globally for all τ when $n = 2$: [ACLLa]

Locally in τ for all dimensions n: [ACLLb]

$$\begin{array}{ccc} D^b_{\mathcal{L}_\tau}\mathsf{Coh}(V_\tau) & \xrightarrow{\quad \iota^* \quad} & D^b_{\mathcal{L}_\tau}\mathsf{Coh}(\Theta_\tau) \\ \text{HMS on } V_\tau \ \Big\downarrow \ \text{[Fuk02]} & & [ACLL2] \ \Big\downarrow \ \text{HMS on } \Theta_\tau \\ H^0\mathsf{Fuk}(V_\tau^\vee) & \xrightarrow{\quad \cup \quad} & H^0 FS(Y_\tau, v_0) \end{array}$$

$$\mathrm{Ext}_{\Theta_\tau}(\mathcal{L}|_{\Theta_\tau}^{\otimes k_1} \otimes L_{a_1+\tau b_1}, \mathcal{L}|_{\Theta_\tau}^{\otimes k_2} \otimes L_{a_2+\tau b_2}) \cong$$
$$HF_{(Y,v_0)}((L_{k_1,b_1}, \mathcal{E}_{k_1}), (L_{k_2,b_2}, \mathcal{E}_{k_2})).$$

Thank you!

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