Ulam stability of lamplighters and Thompson groups

Fields Institute, Toronto

Francesco Fournier-Facio
joint with Bharatram Rangarajan

ETH Zürich

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Outline

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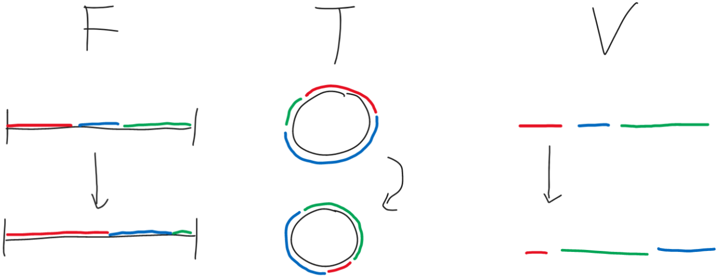

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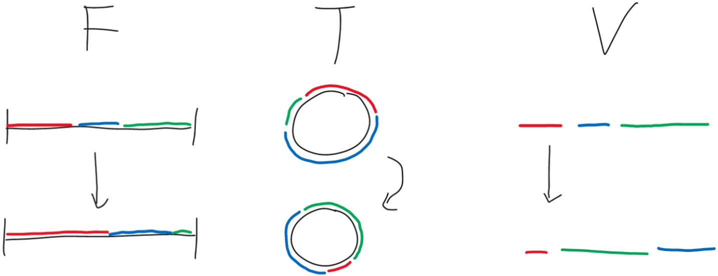

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- Brin–Squier: F has no free subgroups, does not satisfy a law

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  - Skipper–Witzel–Zaremsky: For every $n \geq 1$ there exists a simple group of type F_n but not of type F_{n+1}
  - Belk–Zaremsky: Every finitely generated group quasi-isometrically embeds into a finitely generated simple group

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- Ghys–Sergiescu: F', T and $\overline{T}$ have the homotopy type of loop spaces of S^3

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- Heuer–Löh, FF–Lodha: $\overline{T}$ and variants used to produce special values of simplicial volume
- Monod–Nariman, FF–Löh–Moraschini: T is the first finitely generated group whose bounded cohomology is fully computed in all degrees, and is non-trivial

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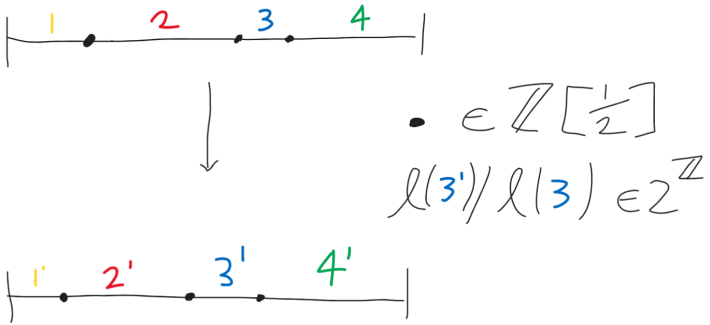

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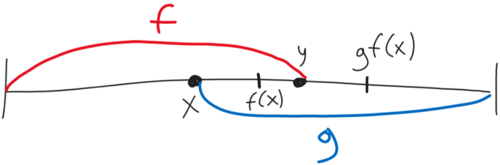

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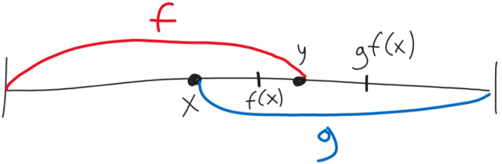

The main player: F'

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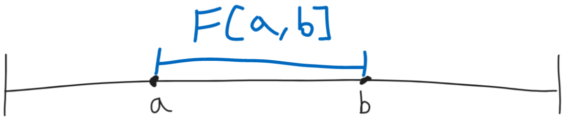

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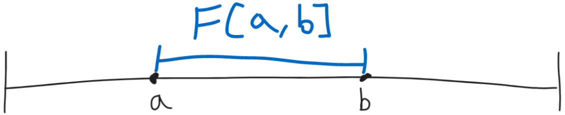

F' is the directed union of the $F[a, b]$, i.e. the subgroup of elements that act trivially near 0 and 1.

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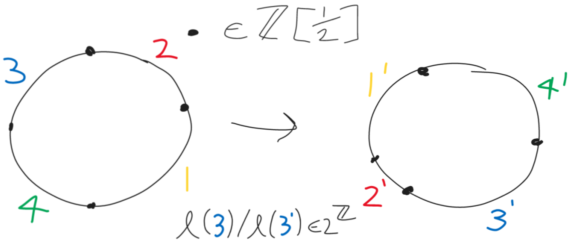

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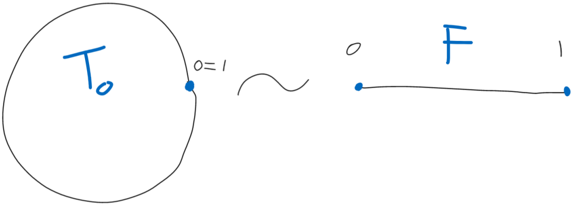

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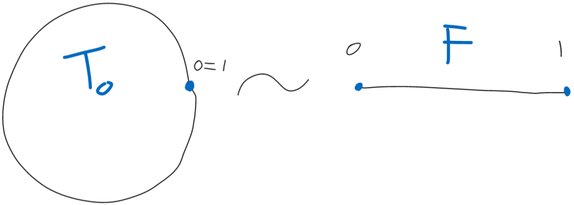

$T(x) := \{g \in T : g$ fixes a neighbourhood of x pointwise$\} \cong F'$.

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F' (so also F) acts transitively on ordered n-tuples in $\mathbb{Z}[1/2] \cap (0, 1)$.

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We will use these properties crucially!

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F' is simple and not linear. So the only homomorphism $F' \to U(n)$ is the trivial one. □

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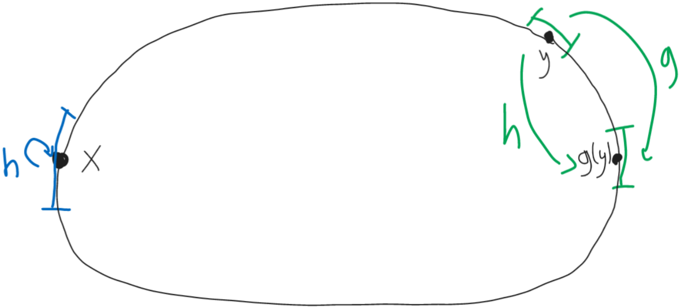

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$\mathcal{L}^\infty(({}^*\Gamma)^m, \mathcal{W}) := \prod_{n \to \omega} \ell^\infty(\Gamma^m, V_n)$.
Bounded / infinitesimal = $\tilde{\mathcal{L}}^\infty(({}^*\Gamma)^m, \mathcal{W})$.

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This defines a complex

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This property is called defect diminishing and is equivalent to Ulam stability with a linear estimate.

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One can think of Γ being amenable relative to Λ.
For instance, if Λ is amenable and coamenable, then Γ is amenable.

Examples

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Every finite subset of Γ fixes a point in Γ/Λ. Take an accumulation point of the corresponding Dirac masses. □

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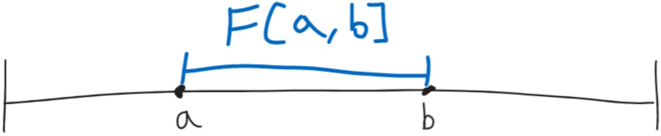

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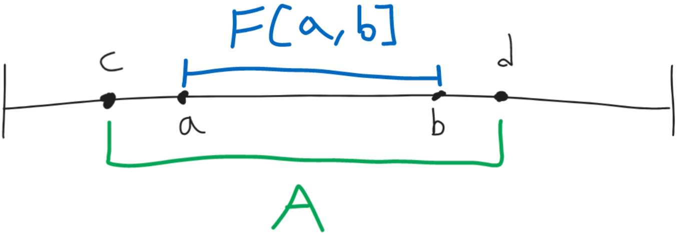

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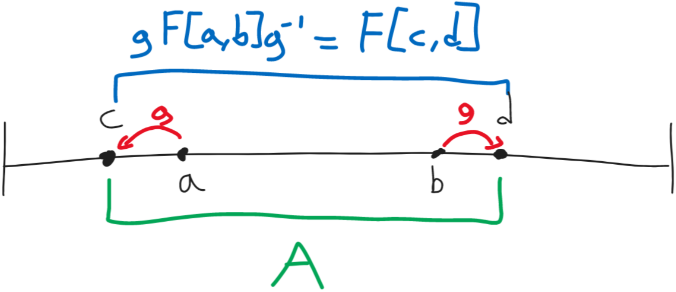

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If $\Lambda < \Gamma$ *is coamenable, then* $\mathrm{res}^2 : \mathrm{H}^2_a(\Gamma, \mathcal{W}) \to \mathrm{H}^2_a(\Lambda, \mathcal{W})$ *is injective,*

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This highlights the power of having **iff** statements for stability in terms of asymptotic cohomology.

Amenable kernels and stability

In a similar vein, one can prove:

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Suppose that $N \leq \Gamma$ is amenable and normal. If Γ is Ulam stable with a linear estimate, then so is Γ/N.

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It is possible that Γ is unstable, and contains an amenable normal subgroup N such that Γ/N is Ulam stable with a linear estimate (using T).

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Are there any nice subgroups of this form?

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where $\mathbb{Z}$ acts by shifting coordinates.

(Our results work more generally for wreath products of the form $\Gamma \wr \Lambda$ where Λ is infinite and amenable...)

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Example

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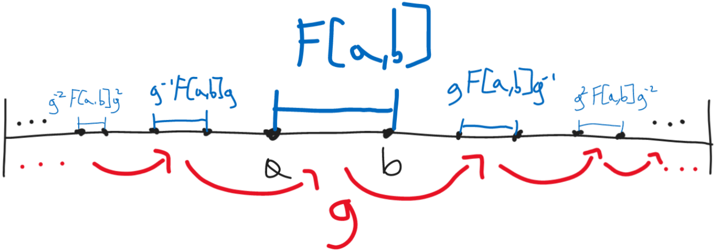

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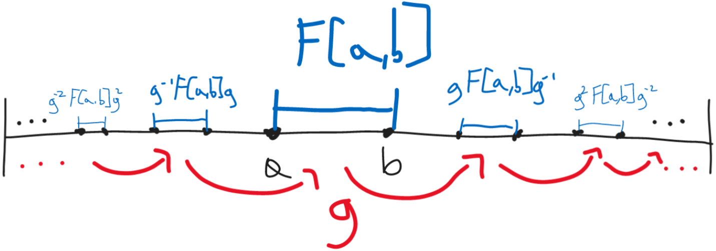

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So the following concludes the proof that F', F and T are Ulam stable with a linear estimate:

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*Let Γ be **any** countable group. Then the lamplighter $\Gamma \wr \mathbb{Z}$ is Ulam stable, with a linear estimate.*

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Corollary

Every countable group embeds into a 3-generated group which is Ulam stable, with a linear estimate.

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Lemma

If S is doubly ergodic, and E is a **separable** *Banach Γ-module, then every Γ-equivariant map $S \to E$ is essentially constant.*

Ergodicity and Zimmer-amenability

Corollary (Monod)

If there exists a highly ergodic Zimmer-amenable Γ*-space, then* $\mathrm{H}^n_b(\Gamma, E) = 0$ *for all dual* **separable** *Banach* Γ*-modules* E*.*

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Outline

Amenability of F

The biggest open question on Thompson's groups, and one of the biggest open questions in group theory, is the following:

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Theorem (Burger–Ozawa–Thom)

Groups with free subgroups are not strong Ulam stable.

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Theorem (Alpeev)

There exist (nonamenable) groups without free subgroups that are not strong Ulam stable.

More generally, Alpeev shows that $A \wr \Gamma$ is not strong Ulam stable, whenever A is abelian and Γ is nonamenable.

Stability and free subgroups

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In other words, if Γ is not Ulam stable, must Γ contain a free subgroup? This is not even known for quasimorphisms!

Thank you for your attention!