Free Random Variables - Tutorial

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Workshop on Non-commutative Geometry, Free Probability Theory and Random Matrix Theory

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Recall

- $(\mathcal{A}, \varphi)$: non-commutative probability space (ncps) if

  $\mathcal{A}$ is a unital algebra over $\mathbb{C}$

  $\varphi : \mathcal{A} \to \mathbb{C}$ is linear such that $\varphi(1) = 1$.

- Given $a \in \mathcal{A}$. $G_a(z) = \sum_{n \geq 0} \frac{\varphi(a^n)}{z^{n+1}}$: the Cauchy transform of a

- Let $G_a^{\langle -1 \rangle}$ be the inverse composition of G_a

  $R_a(z) := G_a^{\langle -1 \rangle}(z) - \frac{1}{z}$: the R-transform of a.

  $\left(= \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots + \kappa_n z^{n-1} + \cdots \right)$

Example

Given a ncps $(\mathcal{A}, \varphi)$ $a \in \mathcal{A}$ is a semi-circular element

if $\varphi(a^n) = \int_{-2}^{2} t^n \underbrace{\frac{1}{2\pi}\sqrt{4-t^2}\, dt}_{\text{standard semi-circle law}}$.

When n is odd, $\varphi(a^n) = 0$

$n = 2k$,

$$\int_{-2}^{2} t^{2k}\sqrt{4-t^2}\, dt \overset{}{\underset{t=2\cos\theta}{=\!=\!=}} \int_0^{\pi} 2^{2k+2}\cos^{2k}\theta \cdot \sin^2\theta \, d\theta = 4^{k+1}\left(I_k - I_{k+1}\right)$$

where $I_k = \int_0^{\pi} \cos^{2k}\theta \, d\theta$

Example (Continue)

Note that $2^{2k}\cos^{2k}\theta = (e^{i\theta}+e^{-i\theta})^{2k} = \sum_{j=0}^{2k}\binom{2k}{j}(e^{i\theta})^{j}(e^{-i\theta})^{2k-j}$

$$= \sum_{j=0}^{2k}\binom{2k}{j} e^{(2j-2k)i\theta}$$

$$\int_0^{\pi} e^{i(2l)\theta}\,d\theta = \begin{cases} \pi & \text{if } l=0 \\ 0 & \text{if } l\neq 0 \end{cases} \implies I_k = \int_0^{\pi}\cos^{2k}\theta\,d\theta = \frac{\pi}{2^{2k}}\binom{2k}{k}$$

Thus, $\int_{-2}^{2} t^{2k}\sqrt{4-t}\,dt = 4^{k+1}(I_k - I_{k+1}) = \frac{2\pi}{k+1}\binom{2k}{k}$

$\implies \varphi(a^{2k}) = \underline{\frac{1}{k+1}\binom{2k}{k}} = C_k$: k-th Catalan number.

Example (Continue)

$$G_a(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(a^n)}{z^{n+1}}$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{z^{2k+1}} C_k = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{z^{2k+1}} \left(\sum_{j=1}^{k} C_{j-1} C_{k-j} \right)$$

$$= \frac{1}{z} + \frac{1}{z} \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{C_{j-1}}{z^{2j-1}} \frac{C_{k-j}}{z^{2(k-j)+1}} = \frac{1}{z} + \frac{1}{z} \sum_{j=1}^{\infty} \frac{C_{j-1}}{z^{2j-1}} \left(\sum_{k=j}^{\infty} \frac{C_{k-j}}{z^{2(k-j)+1}} \right)$$

$$= \frac{1}{z} + \frac{1}{z} \sum_{j=1}^{\infty} \frac{C_{j-1}}{z^{2j-1}} G_a(z) = \frac{1}{z} + \frac{1}{z} G_a(z)^2 .$$

$$\Longrightarrow \quad G_a(z)^2 - z\, G_a(z) + 1 = 0$$

Example (Continue)

$$\Longrightarrow \; G_a(z) = \frac{z \pm \sqrt{z^2-4}}{2} \;\Longrightarrow\; G_a(z) = \frac{z-\sqrt{z^2-4}}{2}$$

$$\lim_{|z|\to\infty} z\,G_a(z) = 1$$

Let $w = G_a(z) \Longrightarrow w = \dfrac{z-\sqrt{z^2-4}}{2} \Longrightarrow z = w + \dfrac{1}{w}$

Hence, $G_a^{\langle -1\rangle}(z) = z + \dfrac{1}{z}$, which implies

the $\mathcal{R}$-transform of a is $\mathcal{R}_a(z) = G_a^{\langle -1\rangle}(z) - \dfrac{1}{z} = z$.

$$\mathcal{R}_a(z) = z = 0 + 1\cdot z + 0\cdot z^2 + 0\cdot z^3 + \cdots \Longrightarrow \kappa_n^a = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{if } n \neq 2. \end{cases}$$

Recall

Given $n \in \mathbb{N}$, $[n] := \{1, 2, 3, \cdots, n\}$

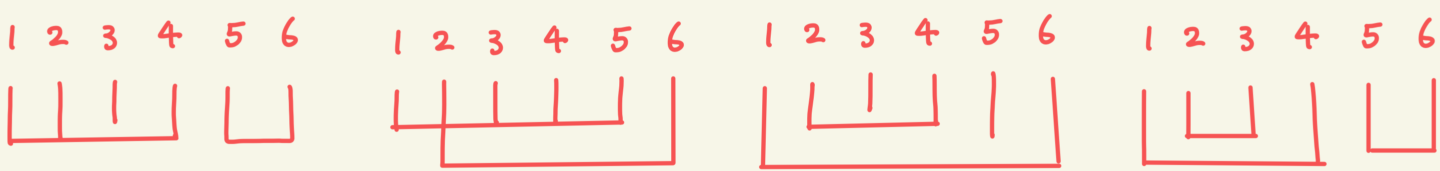
- $P(n)$ = the set of all partitions on $[n]$
- $NC(n)$ = the set of all non-crossing partitions on $[n]$

  $\nexists\ i<j<l<k$ s.t. $i, l \in V_\alpha$ and $j, k \in V_\beta$, $\alpha \neq \beta$
- $NC_2(n)$ = the set of all non-crossing pairings on $[n]$

  all blocks consist of two elements

- Fact: $|NC_2(n)| = \frac{1}{n/2+1}\binom{n}{n/2} = C_{n/2}$

e.g.



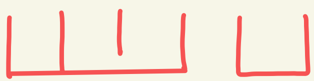
$\Uparrow$ $NC(6) \setminus NC_2(6)$ $\Uparrow$ $P(6) \setminus NC(6)$ $\Uparrow$ $NC(6) \setminus NC_2(6)$ $\Uparrow$ $NC_2(6) \subseteq NC(6)$

Recall

Given a ncps $(\mathcal{A}, \varphi)$ & a family of multilinear functionals $\{f_n : \mathcal{A}^n \to \mathbb{C}\}_{n=1}^{\infty}$. For $\pi \in P(n)$, $a_1, a_2, \cdots, a_n \in \mathcal{A}$

$$f_\pi(a_1, a_2, \cdots, a_n) := \prod_{j=1}^{s} f_{|V_j|}(a_{i_1}, \cdots, a_{i_k})$$ where $V_j = \{i_1 < \cdots < i_k\}$

e.g. $\pi = 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6$



$f_\pi(a_1, \cdots, a_6) = f_3(a_1, a_2, a_4)\, f_1(a_3)\, f_2(a_5, a_6)$

- $\{\kappa_n\}_{n=1}^{\infty}$: free cumulants if $\forall n \geq 0$, $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ is multilinear that defined recursively by $\varphi(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \cdots, a_n)$

Notations

Given ncps $(\mathcal{A}, \varphi)$ & $a \in \mathcal{A}$

- n-th moment of a is $m_n^a := \varphi(a^n)$;
- n-th free cumulant of a is $\kappa_n^a := \kappa_n(a, a, \cdots, a)$.

Example

Let a be a semi-circular element ; i.e. $m_n^a = \begin{cases} 0 & \text{if } n \text{ is odd} \\ C_{n/2} & \text{if } n \text{ is even} \end{cases}$

claim: $\kappa_n^a = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{if } n \neq 2. \end{cases}$

proof: Note that $\kappa_1 = m_1 = 0$

$$m_2 = \kappa_2 + \kappa_1^2 \Rightarrow 1 = \kappa_2 + 0^2 \Rightarrow \kappa_2 = 1$$

For even n,
$$C_{n/2} = m_n = \sum_{\pi \in NC(n)} \kappa_\pi = \underbrace{\sum_{\pi \in NC_2(n)} \kappa_\pi}_{\sum_{\pi \in NC_2(n)} 1 \,=\, |NC_2(n)| \,=\, C_{n/2}} + \sum_{\pi \in NC(n) \setminus NC_2(n)} \kappa_\pi$$

$$\Rightarrow \sum_{\pi \in NC(n) \setminus NC_2(n)} \kappa_\pi = 0.$$

Observe that
$$\sum_{\pi \in NC(3) \setminus NC_2(3)} \kappa_\pi = 0 \Rightarrow \kappa_3 + \cancelto{0}{\kappa_{\sqcup\,|}} + \cancelto{0}{\kappa_{\sqcup\!\!\!\!\;|\,}} + \cancelto{0}{\kappa_{|\,\sqcup}} + \cancelto{0}{\kappa_{|\,|\,|}} = 0$$

$$\Rightarrow \kappa_3 = 0.$$

$$\vdots$$

Recall

$\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are free $\iff$ $\kappa_n(a_1, \dots, a_n) = 0$ whenever

$\exists\, s, l \in [n]$, $a_l \in \mathcal{A}_{i_l}$, $a_s \in \mathcal{A}_{i_s}$ $\quad i_l \neq i_s$

(freeness $\iff$ Vanishing of mixed Cumulants)

Note : $a, b \in \mathcal{A}$ are free

$$\mathcal{R}_{a+b}(z) = \sum_{n \geq 0} \kappa_{n+1}(a+b, \dots, a+b)\, z^n$$

$$= \sum_{n \geq 0} \kappa_{n+1}^{a} z^n + \sum_{n \geq 0} \kappa_{n+1}^{b} z^n + \xcancel{\text{mixed terms}} \to 0$$

$$\Longrightarrow \mathcal{R}_{a+b}(z) = \mathcal{R}_a(z) + \mathcal{R}_b(z).$$

Theorem (Free Central Limit Thm)

Let $(\mathcal{A}, \varphi)$ be a ncps and $a_1, a_2, \cdots, a_n, \cdots \in \mathcal{A}$ are free, identically distributed random variables with $\varphi(a_1) = 0$, $\varphi(a_1^2) = 1$
(i.e. $\varphi(a_j^n) = \varphi(a^n)$ for all n, j, for some a)

then $S_n := \frac{1}{\sqrt{n}}(a_1 + \cdots + a_n) \longrightarrow$ standard semi-circle law, as $n \to \infty$

in the sense that $\varphi(S_n^k) \longrightarrow \begin{cases} 0 & \text{if } k \text{ is odd} \\ C_{k/2} & \text{if } k \text{ is even} \end{cases}$

proof $R_{S_n}(z) = R_{\frac{a_1 + \cdots + a_n}{\sqrt{n}}}(z) = \frac{1}{\sqrt{n}} \cdot R_{a_1 + \cdots + a_n}\left(\frac{1}{\sqrt{n}} z\right) = \sqrt{n} \cdot R_a\left(\frac{1}{\sqrt{n}} z\right)$

$= \sqrt{n} \cdot \left(\kappa_1^a + \kappa_2^a \left(\frac{z}{\sqrt{n}}\right) + \kappa_3^a \left(\frac{z}{\sqrt{n}}\right)^2 + \cdots + \kappa_s^a \left(\frac{z}{\sqrt{n}}\right)^{s-1} + \cdots \right)$

$= \sqrt{n} \cdot \left(\cancelto{0}{m_1^a} + [m_2^a - \cancelto{0}{(m_1^a)^2}] \frac{z}{\sqrt{n}} + O\left(\frac{1}{n}\right) \right) = z + \sqrt{n} \cdot O\left(\frac{1}{n}\right) \longrightarrow z$ as $n \to \infty$.

Thank you for your attention

Ref. Nica & Speicher:
Lectures on the Combinatorics of free Probability.