Tutorial for "How topological recursion organises quantum fields on noncommutative geometries"

This tutorial aims to understand in more details the derivation of the (closed) Dyson-Schwinger equations related to Raimar Wulkenhaar's Mini-course "How topological recursion organises quantum fields on noncommutative geometries".

Let H_N be the space of hermitian $N\times N$ matrices, then we are considering the following partition function

$$\mathcal{Z}^k[J]=\int_{H_N} dM \exp\Big(-N\mathrm{Tr}\big(EM^2+\frac{\lambda}{k}M^k-JM\big)\Big),$$

where $E,J\in H_N$ and E has distinct positive eigenvalues $(E_i)_i$. Denote by $S^k_{int}(M)=N\mathrm{Tr}\big(\frac{\lambda}{k}M^k\big)$ the interaction. We are in particular interested in $\mathcal{Z}^3$ (Kontsevich model) and $\mathcal{Z}^4$ (Grosse-Wulkenhaar model).

The correlation functions (the so-called $(N_1+...+N_b)$-point function of genus g with b boundaries):

$$G^{(g)}_{|a_1^1a_2^1...a_{N_1}^1|...|a_1^b...a_{N_b}^b|}:=N^{2g+b-2}\frac{\partial^{N_1+...+N_b}}{\partial J_{a_1^1a_2^1}..\partial J_{a_{N_1}^1a_1^1}...\partial J_{a_{N_b}^ba_1^b}}\log\mathcal{Z}^k[J]\big|_{J=0},$$

where we assume for the defintion that all a_i^j are pairwise distinct.

Supporting Exercises

Exercise/Remark 1: Show that $\mathcal{Z}^3[0]$ is equivalent to

$$\mathcal{Z}^3[0]=C\int_{H_N} d\tilde{M}\exp\Big(-N\mathrm{Tr}\big(\tilde{E}\tilde{M}+\tilde{\lambda}\tilde{M}^3\big)\Big)$$

by determining $C,\tilde{E},\tilde{\lambda},\tilde{M}$. There is the notation of *generalised Kontsevich model* with the partition function

$$\mathcal{Z}^{gKont}=\int_{H_N} d\tilde{M}\exp\Big(-N\mathrm{Tr}\big(\tilde{E}\tilde{M}+V(\tilde{M})\big)\Big),$$

where $V(x)$ is a polynomial with real coefficients. Show that $\mathcal{Z}^4$ can not be transformed to $\mathcal{Z}^{gKont}$. From this point of view $\mathcal{Z}^k$ with $k>3$ is a different type of generalisation of the classical Kontsevich model.

Exercise 2: Show that the partition function can be represented as

$$\mathcal{Z}^k[J]=K\exp\Big(-S_{int}\Big(\frac{1}{N}\frac{\partial}{\partial J}\Big)\Big)\exp\Big(\frac{N}{2}\sum_{n,m}\frac{J_{nm}J_{mn}}{E_n+E_m}\Big),$$

where K is some constant depending on E.

Exercise 3: Show that the correlation function is the connected expectation values

$$\frac{1}{N^K}\frac{\partial^K}{\partial J_{p_1q_1}\partial J_{p_2q_2}..\partial J_{p_Kq_K}}\log\mathcal{Z}[J]\big|_{J=0}=\langle M_{q_1p_1}M_{q_2p_2}..M_{q_Kp_K}\rangle_c.$$

Exercise 4: Find an argument why any $(N_1+...+N_b)$-point function with $\sum_{i=1}^b N_b$ odd vanishes for $\mathcal{Z}^k$, whenever k is even.

Exercise 5: Prove the Leibniz rule

$$e^{f(\partial_x)}(x\cdot g(x))=f'(\partial_x)e^{f(\partial_x)}g(x)+xe^{f(\partial_x)}g(x).$$

Exercise 6: Compute the Ward identity

$$\frac{E_a-E_b}{N}\sum_m\frac{\partial}{\partial J_{am}}\frac{\partial}{\partial J_{mb}}\mathcal{Z}^k[J]=\sum_m\left(J_{ma}\frac{\partial}{\partial J_{mb}}-J_{bm}\frac{\partial}{\partial J_{am}}\right)\mathcal{Z}^k[J].$$

by considering invariance of the partition function under unitary transformation. Le $U=e^{\mathrm{i}A}\in U(N)$, choose an infinitesimal transformation of the form $M\mapsto M'=UMU^\dagger=M+\mathrm{i}AM-\mathrm{i}MA+\mathcal{O}(A^2)$.

Exercise 7: Prove the recursive algebraic relation between correlation functions for $N_i>k-2$ boundary length.

For $k=3$

$$G_{|a_1^1..a_{N_1}^1|\mathcal{J}|}=-\lambda\frac{G_{|a_2^1a_3^1..a_{N_1}^1|\mathcal{J}|}-G_{|a_1^1a_3^1..a_{N_1}^1|\mathcal{J}|}}{(E_{a_1^1}+E_{a_2^1})(E_{a_1^1}-E_{a_2^1})}. \tag{1}$$

For $k=4$

$$\begin{aligned}G_{|a_1^1a_2^1..a_{N_1}^1|\mathcal{J}|}=&-\frac{\lambda}{E_{a_2^1}-E_{a_{N_1}^1}}\Bigg\{\frac{1}{N^2}\sum_{k=2}^{N_1}\frac{G_{|a_2^1a_3^1..a_k^1|a_{k+1}^1..a_N^1a_1^1|\mathcal{J}|}-G_{|a_1^1a_2^1a_3^1..a_{k-1}^1|a_k^1..a_{N_1}^1|\mathcal{J}|}}{E_{a_k^1}-E_{a_1^1}}\\&+\sum_{\beta=2}^{b}\sum_{k=1}^{N_\beta}\frac{G_{|a_1^\beta a_2^\beta..a_k^\beta a_2^1a_3^1..a_{N_1}^1a_1^1a_{k+1}^\beta..a_{N_\beta}^\beta|\mathcal{J}\setminus\{J^\beta\}|}-G_{|a_1^\beta a_2^\beta..a_{k-1}^\beta a_1^1a_2^1..a_{N_1}^1a_k^\beta..a_{N_\beta}^\beta|\mathcal{J}\setminus\{J^\beta\}|}}{E_{a_k^\beta}-E_{a_1^1}}\\&+\sum_{k=2}^{N_1}\sum_{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J}}\frac{G_{|a_{k+1}^1..a_{N_1}^1a_1^1|\mathcal{I}|}G_{|a_2^1..a_k^1|\mathcal{I}'|}-G_{|a_k^1..a_{N_1}^1|\mathcal{I}|}G_{|a_1^1a_2^1..a_{k-1}^1|\mathcal{I}'|}}{E_{a_k^1}-E_{a_1^1}}\Bigg\}.\end{aligned} \tag{2}$$