Properties of Dirac ensembles and the double scaling limit

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Based on joint work with H.Hessam and M. Khalkhali.

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Dirac ensembles

- Barrett and Glaser in 2015 proposed studying integrals over the moduli space of Dirac operators

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where $S(D)$ is some polynomial function.

- Part of the motivation of this construction was that in theories of quantum gravity one typically tries to integrate over metrics/topologies (as well as matter fields). In the setting of spectral triples, Dirac operators take the place of metrics via Connes' distance formula.

Dirac ensembles

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Dirac ensembles

- To make sense of these integrals they considered Dirac operators of fuzzy geometries, thus making such integrals multi-trace multi-matrix models.
- We refer to fuzzy geometries equipped with a probability distribution on D as *Dirac ensembles*.
- Ideally we would like to relate these models to known physics through some sort of limit(s).
- Additionally, as mathematical objects they are inherently interesting.

Fuzzy spectral triples

- The Dirac operators of type (p, q) fuzzy geometries can be written in term of gamma matrices and the commutators or anti-commutators with Hermitian matrices H and skew-Hermitian matrices L.

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  - type $(0, 1)$, $D = -i[L, \cdot]$

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- For example
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- Example, type $(0, 2)$ geometry:

$$\gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then,

$$D = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot],$$

where L_1, L_2 are both skew-Hermitian.

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A type $(1,0)$ ensemble

- The simple example is a type (1,0) Dirac ensemble where

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$$Z = \int_{\mathcal{D}} e^{-g \text{ Tr } D^2 - \text{Tr } D^4} dD.$$

- The measure becomes the Lebesgue measure on the space of $N \times N$ Hermitian matrices:

$$dD = dH = \prod_{i=1}^{N} dH_{ii} \prod_{1 \le i < j \le N} d(\text{Re}(H_{ij}))\, d(\text{Im}(H_{ij})).$$

A type $(1, 0)$ ensemble

- The integral

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD$$

then becomes a bi-tracial matrix integral

$$= \int_{\mathcal{H}_n} \exp(-(2Ng\mathrm{Tr}H^2 + 2g(\mathrm{Tr}H)^2 + 2N\mathrm{Tr}(H^4)$$

$$+ 8\mathrm{Tr}H\mathrm{Tr}H^3 + 6(\mathrm{Tr}H^2)^2))dH.$$

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- In general Dirac ensembles with polynomial potentials are bi-tracial multi-matrix ensembles.

A quartic type $(2,0)$ ensemble

- The integral

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD,$$

where

$$\operatorname{Tr} D^2 = 4N\left(\operatorname{Tr} H_1^2 + \operatorname{Tr} H_2^2\right) + 4\left((\operatorname{Tr} H_1)^2 + (\operatorname{Tr} H_2)^2\right)$$

$$\begin{aligned}\operatorname{Tr} D^4 = {} & 4N\left(\operatorname{Tr} H_1^4 + \operatorname{Tr} H_2^4 + 4\operatorname{Tr} H_1^2 H_2^2 - 2\operatorname{Tr} H_1 H_2 H_1 H_2\right) \\ & + 16\left(\operatorname{Tr} H_1 \left(\operatorname{Tr} H_1^3 + \operatorname{Tr} H_2^2 H_1\right)\right. \\ & \left. + \operatorname{Tr} H_2 \left(\operatorname{Tr} H_1^2 H_2 + \operatorname{Tr} H_2^3\right) + \left(\operatorname{Tr} H_1 H_2\right)^2\right) \\ & + 12\left(\left(\operatorname{Tr} H_1^2\right)^2 + \left(\operatorname{Tr} H_2^2\right)^2\right) + 8\operatorname{Tr} H_1^2 \operatorname{Tr} H_2^2.\end{aligned}$$

The distributions of eigenvalues

- Often in RMT one studies the distribution of eigenvalues of random matrices in the large N limit. With Dirac ensembles one can study the eigenvalue distributions of the H's, L's, and D's.

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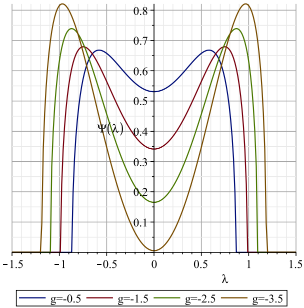
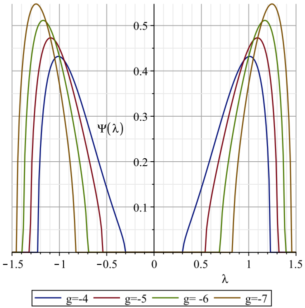
- Often in RMT one studies the distribution of eigenvalues of random matrices in the large N limit. With Dirac ensembles one can study the eigenvalue distributions of the H's, L's, and D's.
- For general Dirac ensembles this is difficult to do analytically but possible numerically with techniques such as Bootstrapping (subject of a later talk by H. Hessam) and Monte Carlo simulations (earlier talks and work by J. Barrett and L. Glaser).

Spectral phase transitions

- In the type $(1, 0)$ Dirac ensemble mentioned earlier, the distribution of eigenvalues of H, can be found analytically in the large N limit using Coulomb gas techniques.
- A precise critical value is found to be $g_c = -\frac{5\sqrt{2}}{2} \approx -3.5$ which is the value found by J. Barrett and L. Glaser 2016.





Spectrum of the Dirac operator

- Recall that for our model of interest

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- Partial answer: Sometimes the spectral density function of D is the integral convolution of the spectral density function of H with itself!

Spectrum of the Dirac operator

Theorem (M. and M. Khalkhali 2021)

With type (1,0) geometry, for any polynomial action such that

$$Z = \int_{\mathcal{G}} e^{-\operatorname{Tr} S(D)} dD$$

is convergent, the limiting eigenvalue distribution of the Dirac operator is

$$\rho_D(x) = \int_{\mathbb{R}} \rho_H(x-t)\rho_H(x)dt,$$

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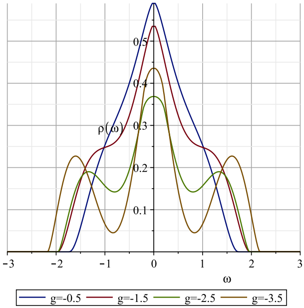
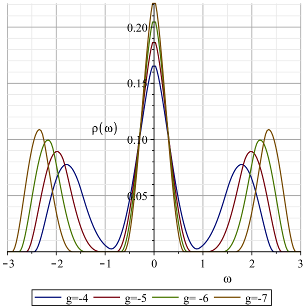
- This theorem generalizes further to certain classes of other models with different Dirac operators and even when the partition function is treated as a formal series.

Example: quartic type $(1, 0)$





Gaussian Dirac ensembles

- More generally,

Theorem (M. and M. Khalkhali 2021)

For type (p,q) Dirac ensemble with a Gaussian action i.e,

$$Z = \int_{\mathcal{G}} e^{-\frac{1}{4k^2}\operatorname{Tr} D^2} dD. \tag{1}$$

Then limit

$$\sqrt{\lim_{N\to\infty} \langle \operatorname{Tr} e^{xD} \rangle} = \frac{I_1(2x)}{x}, \tag{2}$$

holds, where I_1 denotes the modified Bessel's function of the first kind.

- Note that the right hand side of the above limit is precisely the moment generating function of the Wigner Semicircle Distribution, this hints at the following.

Gaussian Dirac ensembles

Corollary

For any such space, the limiting spectral density function of the transformed Dirac operators is

$$\rho_D(x) = \int_{\mathbb{R}} \rho_W(x-t)\rho_W(x)dt,$$

where

$$\rho_W(x) = \frac{1}{2\pi}\sqrt{4-x^2}_{[-2,2]},$$

is the Wigner Semicircle Distribution

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2-D quantum gravity from Dirac ensembles

- Random matrix theory has been known to have connections to 2D gravity:
  - The Kontsevich model and Witten's conjecture.
  - Liouville quantum gravity (LQG) i.e. 2 D conformal field theories coupled to gravity.
  - More recently Jackiw-Teitelboim (JT) gravity.

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  - Liouville quantum gravity (LQG) i.e. 2 D conformal field theories coupled to gravity.
  - More recently Jackiw-Teitelboim (JT) gravity.
- Of particular interest is LVQ. Physicists in the late 80's and 90's knew heuristically that asymptotics of random matrix models contained artifacts of LQG.

The double scaling limit

Rough idea:

- Formal Hermitian matrix integrals count maps which can be thought of as discretized Riemann surfaces.

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Rough idea:

- Formal Hermitian matrix integrals count maps which can be thought of as discretized Riemann surfaces.
- If the coupling constants of the models were tweaked such that the number of polygons that form maps goes to infinity, one would in essence be counting Riemannian surfaces.
- These critical points of matrix integrals exist in many models, in particular they exist in Dirac ensembles!

The double scaling limit

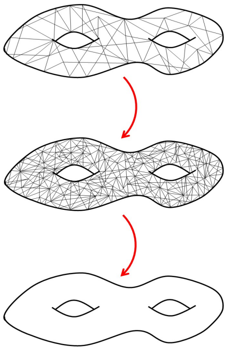

Figure: Intuitively if one fine tunes coupling constants of matrix models such that the number of polygons in maps goes to infinity, maps are replaced by smooth surfaces.

Liouville quantum gravity

- Consider the genus expansion of the partition function for some Dirac ensemble

$$\log Z := \sum_{g=0}^{\infty} N^{2-2g} F_g,$$

where

$$F_g = \sum_{v=1}^{\infty} t^v \sum_{\Sigma \in \mathbb{SM}_0^g(v)} \prod_{i,j=1}^{d} t_{i,j}^{n_{i,j}(\Sigma)} \frac{1}{|\mathrm{Aut}(\Sigma)|}.$$

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- In general, F_g is a function of the coupling constants with some algebraic or logarithmic singularities and these F_g's can be computed from the W_k^0's found using topological recursion.

Liouville quantum gravity

- Consider now the quartic type $(1,0)$ Dirac ensemble from earlier

$$Z = \int_{\mathcal{D}} e^{-t_2 \operatorname{Tr} D^2 - t_4 \operatorname{Tr} D^4} dD$$

then becomes a bi-tracial matrix integral

$$= \int_{\mathcal{H}_n} \exp(-(2Nt_2 \mathrm{Tr} H^2 + 2t_2(\mathrm{Tr} H)^2 + 2t_4 N \mathrm{Tr}(H^4)$$
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- One can show that near critical points the F_g's have an asymptotic expansion of the form:
$$\operatorname{sing}(F_g) = C_g\,(t_4 - t_c)^{5(1-g)/2}$$
except when $g = 1$,
$$\operatorname{sing}(F_1) = C_1 \log(t_4 - t_c).$$

Liouville quantum gravity

- We can define a new formal series

$$u(y) = \sum_{g=0}^{\infty} \text{sing}(F_g) y^{5(1-g)/2},$$

then $u''(y)$ satisfies the Painlevé I equation to all orders

$$y = (u''(y))^2 - \frac{1}{3} u^{(4)}(y).$$

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- The Liouville minimal model of conformal field theory coupled to gravity predicts that its "generating function of surfaces" should satisfy this equation!

Liouville quantum gravity

- Different matrix models are associated to different so called minimal models whose F_g's satisfy their own differential equation.

Liouville quantum gravity

- Different matrix models are associated to different so called minimal models whose F_g's satisfy their own differential equation.
- One can find such models by examining how the spectral density function scales near the critical point(s).

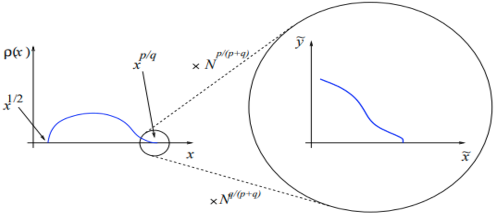

Figure: Borrowed from "Universal scaling limits of matrix models, and (p, q) Liouville gravity" by M. Bergère and B. Eynard.

The phases of the quartic type $(1,0)$ Dirac ensemble

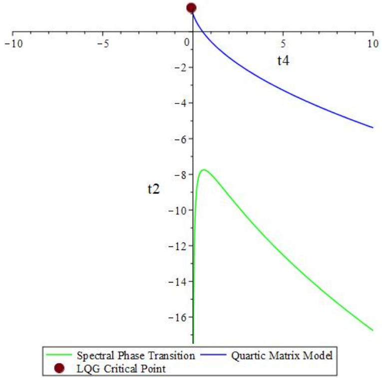

Figure: The phase diagram of the quartic Dirac ensemble.

The phases of the quartic type $(1,0)$ Dirac ensemble

- Curve of the spectral phase transition:

$$t_2 = -\frac{5t_4 + 3}{\sqrt{t4}}.$$

- The quartic Hermitian matrix model's curve:

$$t_2 = -\frac{(1+12t_4)^{3/2} - 4 - 144t_4 + (36t4 + 3)\sqrt{1+12t_4}}{72t_4}.$$

Liouville quantum gravity summary

- These minimal models and critical exponents correspond to representations of the conformal group in two dimensions classified by two integers (p, q) with critical exponents p/q. For example:
  - $(3, 2)$ is called pure gravity and corresponds to the cubic and quartic type $(1, 0)$ Dirac ensembles
  - $(5, 2)$ is called Lee-Yang edge singularity and corresponds to the hexic type $(1, 0)$ Dirac ensemble.

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- In general single trace single matrix model correspond to type $(p, 2)$ minimal models and general (p, q) can be found in multi-matrix models. What about other Dirac ensembles?

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A brief summary

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- Dirac ensembles are multi-tracial matrix integrals.
- They display many mathematical interesting properties such as: blobbed topological recursion, spectral phase transitions and universal properties.
- In certain models one can recover Liouville quantum gravity in the double scaling limit.

Open problems

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- Investigate the limiting eigenvalue distribution and critical points of Dirac ensembles with more complicated potentials.
- Are there minimal models associated with higher dimensional Dirac ensembles?
- Can one extra geometric data from Dirac ensembles?
- Apply the BV formalism to other Dirac ensembles.
- What is the general relationship between D and H's and L's? Are there recursive relationships strictly between moments of D.

References for our work

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Thank you for listening!
Questions?