Topological Expansion of BGW + HCIZ

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Fourier in RMT

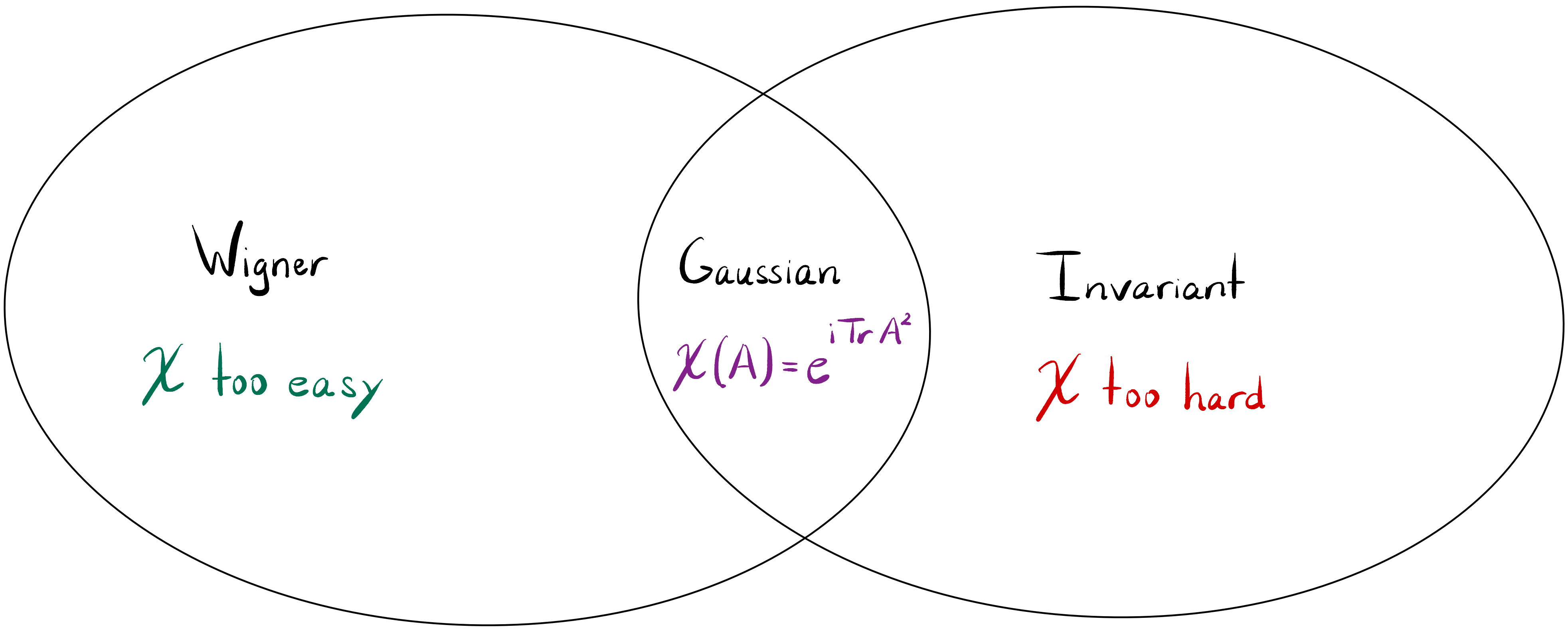
- Random Hermitian matrix: $X_N = [X_N(i,j)]_{1 \le i,j \le N}$.

- Characteristic function: $A^* = A \mapsto \mathbb{E}[e^{i \operatorname{Tr} A X_N}]$.

- Characterizes distribution of X_N – absent from RMT.

- Why?

$$\chi(A) = \mathbb{E}\left[e^{i \operatorname{Tr} A X_N}\right]$$



Invariant Ensembles

- Spectral theorem: $X_N = U_N B_N U_N^{-1}$, with independent eigenvectors U_N and eigenvalues $B_N = \mathrm{diag}(b_1 \geq \dots \geq b_N)$.

- Characteristic function:

$$\chi(A) = \mathbb{E}_{(B_N, U_N)}\left[e^{i\,\mathrm{Tr}\, A U_N B_N U_N^{-1}}\right]$$

$$= \mathbb{E}_{B_N} \int_{U(N)} e^{i\,\mathrm{Tr}\, A U B_N U^{-1}}\, dU.$$

New Kernel, New Problems

- RMT analogue of the scalar (and vector) Fourier kernel e^{iax} is the unitary matrix integral

$$I_N(A,B) = \int_{\mathcal{U}(N)} e^{i\,\mathrm{Tr}\,AUBU^{-1}}\,dU.$$

- Oscillatory integral over a compact real manifold of dimension N^2.

- No existing tools for $N \to \infty$ asymptotics.

Stationary Phase

- Rescale to get order N^2 action,

$$I_N := \int_{U(N)} e^{iN\,\mathrm{Tr}\,AUBU^{-1}}\,dU.$$

- Find stationary points of the action

$$N\,\mathrm{Tr}\,AUBU^{-1} = N\sum_{k,l=1}^{N} a_k b_l |U_{kl}|^2.$$

- Linear functional on Birkhoff polytope – extreme points are permutation matrices.

- Stationary phase approx $\rightsquigarrow$ determinant.

Theorem (Harish-Chandra, Itzykson-Zuber)

This is how complicated the kernel $I_N = I_N(A,B)$ is:

$$I_N = \mathrm{const}_N \, \frac{\det[e^{i a_k b_\ell}]}{\Delta(A)\,\Delta(B)}.$$

Useless for $N \longrightarrow \infty$ asymptotics; have to find another approach.

Analytic Continuation

- Make everything complex:

$$I_N = \int_{\mathcal{U}(N)} e^{zN \operatorname{Tr} AUBU^{-1}} dU.$$

- Entire function of $2N+1$ complex variables: z and eigenvals $a_1,\dots,a_N, b_1,\dots,b_N$ of $A,B \in \mathfrak{gl}_N(\mathbb{C})$.

- Reverts to RMT Fourier kernel on $i\mathbb{R} \times \mathbb{R}^{2N}$; becomes random matrix partition function on $\mathbb{R} \times \mathbb{R}^{2N}$.

Gibbs Measure

- I_N restricts to partition function of Gibbs measure on $U(N)$: inverse coupling $z \in \mathbb{R}$, Hamiltonian

$$H = N \sum_{i,j=1}^{N} a_i b_j |U_{ij}|^2,$$

external field parameters $a_1, \dots, a_N, b_1, \dots, b_N \in \mathbb{R}$.

- Large N behavior of $F_N = \log I_N$ anticipated by analogy with Hermitian matrix models.

Theorem (Ercolani-McLaughlin)

There exists $\varepsilon > 0$ such that free energy $F_N = \log Z_N$ of Hermitian one-matrix model

$$Z_N = \int_{H(N)} e^{tN \operatorname{Tr} X^4} \mu_N(dX), \quad \operatorname{Re}(t) < 0,$$

satisfies

$$F_N = \sum_{g=0}^{k} N^{2-2g} F_g + o(N^{2-2k})$$

as $N \to \infty$, for each $k \in \mathbb{N}_0$, with error uniform in $t \in [-\varepsilon, 0]$, and $F_g = F_g(t)$ generating fcn for genus g quadrangulations, which converges uniformly absolutely for $|t| \leq \varepsilon$.

- Existence of asymptotic expansion in Hermitian matrix models: Coulomb gas spectrum, orthogonal polynomials.

- Topological interpretation: Wick calculus.

- Alternative approach (Guionnet): Schwinger-Dyson equations plus concentration inequalities.

- HCIZ matrix model not an eigenvalue model; use second approach.

Theorem (Guionnet-Novak)

For each $k \in \mathbb{N}_0$ there exists $\varepsilon_k > 0$ such that

$$F_N = \sum_{g=0}^{k} N^{2-2g} F_{Ng} + o(N^{2-2k})$$

as $N \to \infty$, where the error term is uniform on compact box

$$B_N(\varepsilon_k) = [-\varepsilon_k, \varepsilon_k] \times \mathbb{R}^{2N} \subset \mathbb{R}^{2N+1}$$

inverse coupling

external field

and

$$\sup_{N \in \mathbb{N}} \| F_{Ng} \|_\varepsilon < \infty.$$

Demerits

- Argument does not extend to complex parameters.

- Box thickness $\varepsilon_k \to 0$ exponentially in k.

- No topological description of F_{Ng}.

Further Motivations

- HCIZ in Hermitian multimatrix models:

$$Z_N = \int_{H(N)^2} e^{N\mathrm{Tr}(V_1(X_1) + V_2(X_2) + z X_1 X_2)} \mu_N^{\otimes 2}(dX_1, dX_2)$$

- HCIZ in Hermitian matrix models with external source:

$$Z_N = \int_{H(N)} e^{N\mathrm{Tr}(V(X) + AX)} \mu_N(dX).$$

- HCIZ in representation theory:

$$Z_N = \frac{\chi_{(b_1,\dots,b_N)}(e^A)}{\chi_{(b_1,\dots,b_N)}(e^0)}.$$

Topological Expansion Conjecture (QFT 1980)

There exists $\varepsilon > 0$ such that, for each $k \in \mathbb{N}_0$, we have

$$\log \int_{U(N)} e^{zN \operatorname{Tr} AUBU^{-1}} \, dU = \sum_{g=0}^{k} N^{2-2g} F_{Ng} + o(N^{2-2k})$$

as $N \to \infty$, where the error term is uniform over $|z| \le \varepsilon$ and $|a_i|, |b_i| \le 1$, and the free energies are analytic functions of $z, a_1, \dots, a_N, b_1, \dots, b_N$ whose modulus is uniformly bounded in N, and which are generating functions for combinatorial invariants of compact connected genus g Riemann surfaces.

Theorem (Novak): The topological expansion conjecture is true: there exists $\varepsilon > 0$ such that, for each $k \in \mathbb{N}_0$,

$$\log \int_{U(N)} e^{zN \operatorname{Tr} AUBU^{-1}} dU = \sum_{g=0}^{k} N^{2-2g} F_{Ng} + o(N^{2-2k})$$

as $N \to \infty$, where the error term is uniform on $\mathbb{D}_N(\varepsilon) \subset \mathbb{C}^{2N+1}$ and

$$F_{Ng} = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta \vdash d} \frac{p_\alpha(a_1,\dots,a_N)}{N^{\ell(\alpha)}} \frac{p_\beta(b_1,\dots,b_N)}{N^{\ell(\beta)}} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_g(\alpha,\beta)$$

converges uniformly absolutely on $\mathbb{D}_N(\varepsilon)$; the topological invariants $\vec{H}_g(\alpha,\beta)$ are the monotone double Hurwitz numbers.

Theorem (Novak): Analogous statement for the BGW integral:

There exists $\varepsilon > 0$ such that, for each $k \in \mathbb{N}_0$,

$$\log \int_{\mathcal{U}(N)} e^{zN\operatorname{Tr}(AU + BU^{-1})} dU = \sum_{g=0}^{k} N^{2-2g} F_{Ng} + o(N^{2-2k})$$

as $N \to \infty$, where error term is uniform on $\mathbb{D}_N(\varepsilon) \subset \mathbb{C}^{N+1}$ and

$$F_{Ng} = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha \vdash d} \frac{p_\alpha(c_1, \ldots, c_N)}{N^{\ell(\alpha)}} (-1)^{\ell(\alpha)+d} \vec{H}_g(\alpha)$$

converges uniformly absolutely on $\mathbb{D}_N(\varepsilon)$ with $c_1, \ldots, c_N$ eigenvalues of $C = AB$, and $\vec{H}_g(\alpha) = \vec{H}_g(\alpha, 1^d)$ are the monotone single Hurwitz numbers.

Proof: The strong coupling expansion (Wilson 1974) and the large N expansion ('t Hooft 1974) are analytically compatible for sufficiently small complex parameters.

— Q.E.D.

~~Probability~~ Several Complex Variables

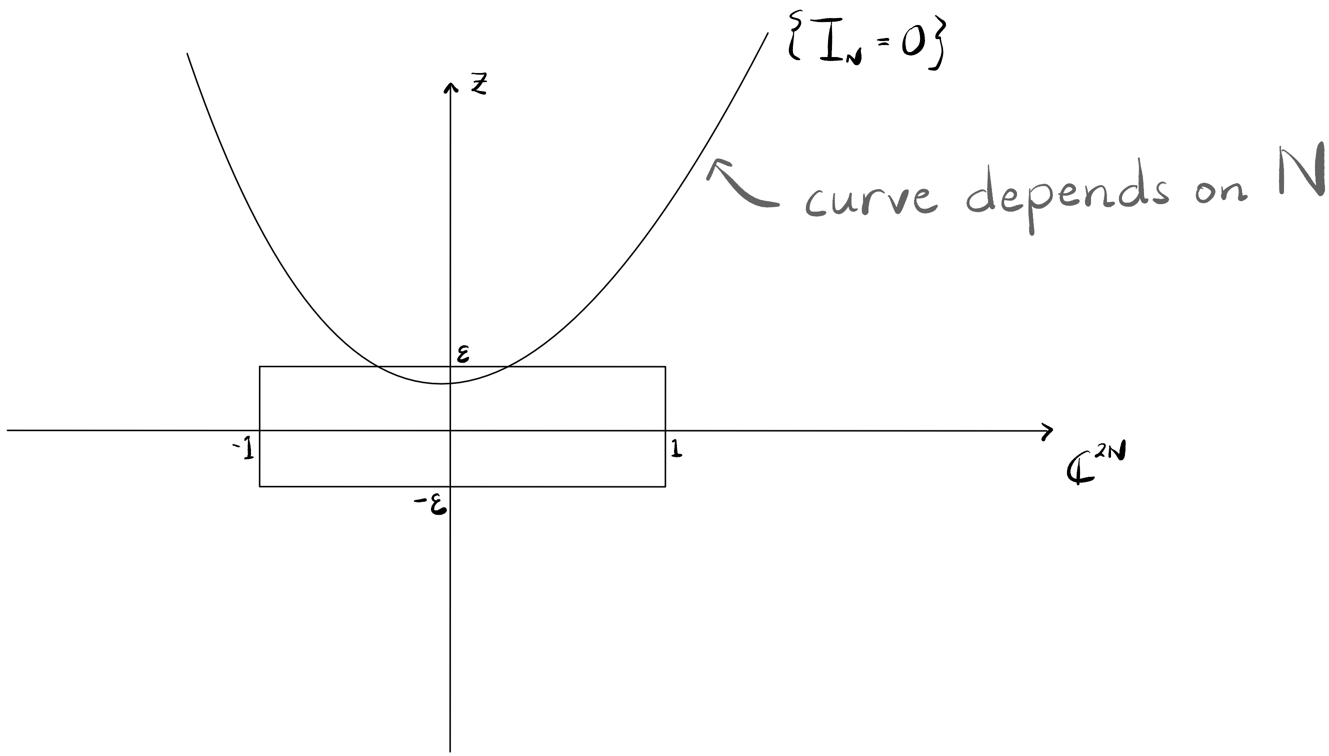
$$I_N = \int_{U(N)} e^{zN\,\mathrm{Tr}\,AUBU^{-1}}\,dU = I_N\left(z \,\middle|\, \begin{matrix} a_1,\dots,a_N \\ b_1,\dots,b_N \end{matrix}\right)$$

- <u>Have</u>: $I_N = 1$ on infinite coupling hyperplane $\{z=0\} \subset \mathbb{C}^{2N+1}$.

- <u>Make</u>: Cut out closed unit polydisc $\mathbb{D}_N \subset \mathbb{C}^{2N}$, embed in $\{z=0\} \subset \mathbb{C}^{2N+1}$, thicken out to closed polydisc $\mathbb{D}_N(\varepsilon)$ of polyradius $(\varepsilon,1,\dots,1)$ in $\mathbb{C}^{2N+1}$.

- <u>Want</u>: Asymptotics of $F_N = \log I_N$ on $\mathbb{D}_N(\varepsilon)$ as $N \to \infty$ with $\varepsilon > 0$ fixed.

- **Obstruction:** Could be that, for any $\varepsilon > 0$ we choose, hypersurface $\{I_N = 0\}$ intersects polydisc $\mathbb{D}_N(\varepsilon)$ non-trivially in $\mathbb{C}^{2N+1}$ for infinitely many $N \in \mathbb{N}$.



Strong Coupling Expansion

- Non-vanishing constant: if $\exists \delta > 0$ such thats $I_N \neq 0$ on $\mathbb{D}_N(\delta)$ for all $N \in \mathbb{N}$, then conditional proof?

- $F_N = \log I_N$ belongs to Banach algebra $(\mathcal{O}_N(\delta), \|\cdot\|_\delta)$ for all $N \in \mathbb{N}$.

- Strong coupling expansion = Maclaurin series,

$$F_N = \sum_{d=1}^{\infty} \frac{z^d}{d!} F_N^d, \qquad F_N^d : \mathbb{C}^{2N} \longrightarrow \mathbb{C}.$$

Large N Expansion

- QFT ansatz: strong coupling coefficients stratify topologically as $N \to \infty$,

$$F_N^d \sim \sum_{g=0}^{\infty} N^{2-2g} F_{Ng}^d .$$

- For each fixed $d \in \mathbb{N}$ and $k \in \mathbb{N}_0$, have

$$\lim_{N \longrightarrow \infty} N^{2k-2} \left\| F_N^d - \sum_{g=0}^{k} N^{2-2g} F_{Ng}^d \right\| = 0,$$

with $F_{Ng}^d : \mathbb{C}^{2N} \longrightarrow \mathbb{C}$ hom. deg. d polynomials such that

$$\sup_{N \in \mathbb{N}} \| F_{Ng}^d \| < \infty .$$

- Large N ansatz unknown in this context (lattice QCD).

- We can give it a topological meaning:

$$F_N^d \sim \sum_{g=0}^{\infty} N^{2-2g} \, \# \left\{ \begin{array}{c} \text{branched covers of } \mathbb{P}^1(\mathbb{C}), \text{ degree } d, \\ \text{genus } g, \text{ specified branch locus} \end{array} \right\}.$$

- Second obstruction: $\sum_{d=1}^{\infty} F_{Ng}^d$ is $\|\cdot\|$-absolutely convergent iff $d \leq N$ ("stable range").

NONCONVERGENCE OF THE $1/N$ EXPANSION FOR SU(N) GAUGE FIELDS ON A LATTICE

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We present specific examples that demonstrate the non-convergence of the $1/N$ expansion for the lattice theory of SU(N) gauge fields.

$1/N$ expansions in field theories with N or N^2 field components are a useful device for simplification and/or bookkeeping purposes of Feynman diagrams [1]. In the conventional perturbation expansion of SU(N) gauge theories one may consider the limit $N \to \infty$ keeping g^2N fixed, g being the coupling constant. One then finds at each order of g^2N a finite polynomial in $1/N$ with coefficients that are related in a precise manner to the topology of the corresponding diagrams as twodimensional surfaces [2]. In particular the leading term consists of planar Feynman diagrams only, which suggests that in the limit $N \to \infty$ one obtains hadrons that are essentially non-interacting. The $1/N$ expansion then corresponds to an expansion with respect to the coupling strength between the hadrons. Our general experience with coupling-constant expansions in field theories then suggests that the $1/N$ expansion will diverge at a fixed *value* for g^2N, even though the series is finite and therefore converges at fixed *order* in g^2N. We think that the probable formal divergence of the $1/N$ expansion is not a sufficient argument to reject $1/N$ expansions altogether, first because in the physically interesting case of SU(3) the effective coupling strength of $1/3$ may be small enough so that the spectrum obtained in the $N \to \infty$ limit will still resemble the physical spectrum, and secondly because fundamental problems such as the quark-confinement mechanism are likely to be independent of N, and understanding of such mechanisms in the $N \to \infty$ limit could be of great significance.

Thus we were motivated to study the $1/N$ expansion further, but now in the SU(N) gauge theory on a lattice. Here the usual expansion is made with respect to $1/g^2$ and $1/m_q$ where m_q are the masses of the quarks [3]. Alternatively, one may expand with respect to $1/g^2N$ and $1/N$, keeping m_q fixed and arbitrary [4]. Again we look at fixed order in $1/g^2N$ and this time we find that the series in $1/N$ does not only continue up to infinity as an essentially geometric series, but, more annoyingly, fails to produce the correct answer at finite N when summed. To be precise: we find for N larger than a few units pure rational functions of N, but when $N = 1, 2$ or 3 is substituted in here we find incorrect or even infinite answers. The critical value of N above which the rational function is valid and below which it fails depends on the order of $1/g^2N$ considered. We interpret this result as an aspect of the formal divergence of the $1/N$ expansion, but it must be kept in mind that also in this case we are unable to interchange the limits $g^2N \to \infty$ and $N \to \infty$.

To demonstrate the aforementioned properties of the $1/N$ expansion is the purpose of this note. The action for gauge fields and quarks on an infinite Euclidean lattice is given by [3]

$$S[\bar{\psi}_q, \psi_q, U^\dagger, U]$$

$$= \sum_{x,q} \bar{\psi}_q(x) \{\tfrac{1}{2} \sum_{\mu} (1 + \gamma_\mu) U(x, \hat{\mu}) \psi_q(x + \hat{\mu}) +$$

NON-PLANAR DIAGRAMS IN THE LARGE N LIMIT OF U(N) AND SU(N) LATTICE GAUGE THEORIES

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It is shown that the limit as $N \to \infty$ with g^2N fixed of the strong coupling expansion for the vacuum expectation values of a U(N) or SU(N) lattice gauge theory is not given by a sum of planar diagrams. This contradicts a result claimed by De Wit and 't Hooft.

Some time ago 't Hooft [1] showed that as $N \to \infty$ with g^2N fixed the leading Feynman diagrams for the Green's functions of a U(N) or SU(N) gauge theory are planar. This result suggested a possible way the relation between QCD and the string model might be made precise [1] and has been exploited in a wide variety of subsequent work [2]. Thus an interesting question is what happens to Wilson's strong coupling expansion for lattice gauge theories [3] in the limit $N \to \infty$, g^2N fixed. De Wit and 't Hooft [4] have claimed that once again planar diagrams dominate. This result has been applied by Eguchi [5] to relate a lattice U(N) gauge theory to a theory of non-interacting Nambu–Goto strings and has been reconsidered more recently by Bars and Green [6].

In the present article, however, we will show that planar diagrams do not dominate the large N limit of the strong coupling expansion of either U(N) or SU(N) lattice gauge theories. Thus the discussions of this question in refs. [4–6] are incorrect. Our proof will consist of exhibiting a non-planar term, in the expansion of a vacuum expectation, which has the same dependence on N as $N \to \infty$, g^2N fixed, as do the planar contributions.

Let us begin by briefly reviewing the euclidean lattice formulation of a gauge theory with gauge group G and the strong coupling expansion of vacuum expectation values [3]. For simplicity we will consider only a pure gauge theory without fermions. Let $\Lambda \subset \mathbf{Z}^4$ be a finite hypercubic lattice. For each oriented pair of nearest neighbor sites (x, y) in Λ let $U(x, y) = U(y, x)^\dagger$ be a matrix in G. For each oriented plaquette (square) p of four nearest-neighbor sites in Λ, let $U(p)$ be the ordered product of $U(x, y)$ around p starting at some arbitrarily chosen site of p. The action S is

$$S = (2g^2)^{-1} \sum_p \mathrm{Tr}\, U(p)\,, \tag{1}$$

and the vacuum expectation of any polynomial $\mathcal{F}$ of $U(x, y)$ is

$$\langle \mathcal{F} \rangle = \int \mathrm{d}\mu\, \mathcal{F} \exp(S) \Big/ \int \mathrm{d}\mu \exp(S)\,, \tag{2}$$

where μ is the product of one copy of Haar measure on G for each independent $U(x, y)$.

The strong coupling expansion for $\langle \mathcal{F} \rangle$ is obtained by expressing the exponentials in numerator and denominator of eq. (2) as power series in $(2g^2)^{-1}$. The terms which appear can be associated with surfaces of plaquettes. After the integrals over $\mathrm{d}\mu$ are carried out, the denominator yields a sum of closed surfaces and the numerator yields a sum of surfaces with boundary determined by $\mathcal{F}$. The effect of dividing the numerator by the denominator is equivalent to modifying the rules for calculating the numerator: Each integral over Haar measure of a product of $U(x, y)$ is replaced by some combination of connected vertices (contractions) [‡1] obtained from a cluster

[‡1] The definition of contractions is discussed by Wilson [3b, Appendix B].

Full Claim

<u>Stable Nonvanishing Constant:</u> There exist $\delta > 0$ such that

$$I_N = \int_{U(N)} e^{zN\,\mathrm{Tr}\,AUBU^{-1}}\,dU \quad \text{is nonvanishing on } \mathbb{D}_N(\delta) \text{ for all } N \in \mathbb{N}.$$

<u>Stable Topological Constant:</u> There exists $\gamma > 0$ such that

$$F_{Ng} = \sum_{d=1}^{\infty} \frac{z^d}{d!} F_{Ng}^d \quad \text{converges uniformly absolutely on } \mathbb{D}_N(\gamma) \text{ for all } N \in \mathbb{N},\ g \in \mathbb{N}_0.$$

<u>Asymptotic Interchange Constant:</u> There exists $0 < \varepsilon \leq \min(\delta, \gamma)$ such that

$$\lim_{N \to \infty} N^{2k-2} \left\| F_N - \sum_{g=0}^{k} N^{2-2g} F_{Ng} \right\|_{\varepsilon} = 0 \qquad \forall k \in \mathbb{N}_0.$$

FINITE N

Feynman Diagrams for Haar Measure

- Coupling expansion:

$$I_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} I_N^d .$$

- Coupling coefficients are polynomial Haar integrals.

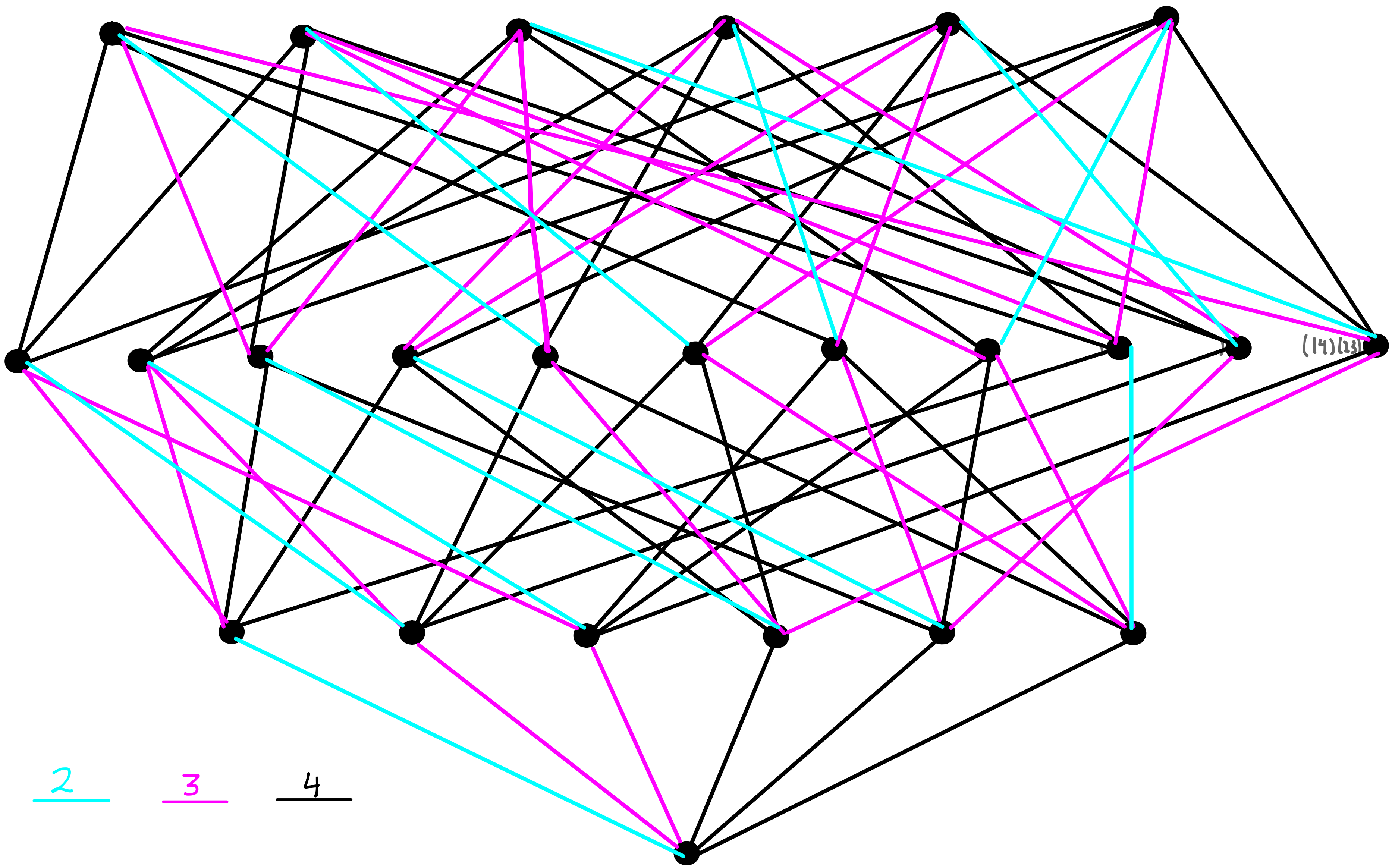
- Actually, BGW integral is generating fcn for Haar correlators.

$$\int_{U(N)} e^{zN\,\mathrm{Tr}(AU+BU^{-1})} = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!d!} N^{2d} \times$$

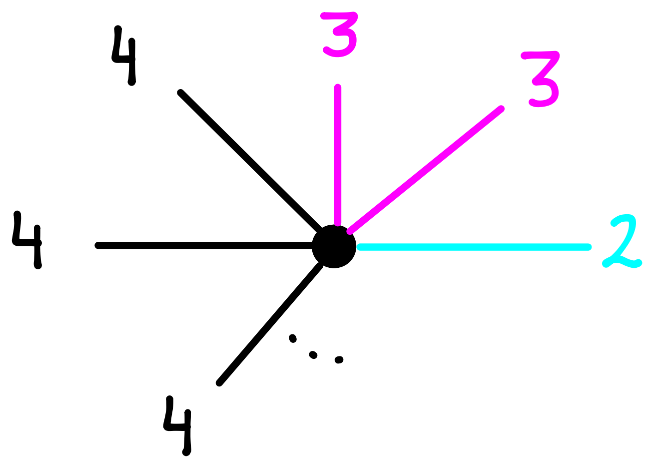
$$\sum_{i,j,i',j' \in \mathrm{Fun}(d,N)} A_{i(1)j(1)} \cdots A_{i(d)j(d)} B_{i'(1)j'(1)} \cdots B_{i'(d)j'(d)} \int_{U(N)} U_{i(1)j(1)} \cdots U_{i(d)j(d)} \overline{U_{i'(1)j'(1)} \cdots U_{i'(d)j'(d)}}\, dU.$$

- For $1 \leq d \leq N$, reduce to permutation correlators:

$$\mathrm{Wg}_N(\pi) = \int_{U(N)} U_{11} \cdots U_{dd} \overline{U_{1\pi(1)} \cdots U_{d\pi(d)}}\, dU \stackrel{?}{=} \sum_{\{\text{diagrams}\}}$$


(14)(23)
2
3
4

A fragment of S(d)



$$T = \begin{array}{c} \begin{bmatrix} (12) & (13) & (14) \\ & (23) & (24) \\ & & (34) \end{bmatrix} \\ \begin{matrix} \quad J_2 & \quad J_3 & \quad J_4 \end{matrix} \end{array}$$

Theorem (Novak)

For any $1 \leq d \leq N$, any $\alpha \vdash d$, and any $\pi \in S(d)$,

$$\int_{U(N)} U_{11} \cdots U_{dd} \overline{U_{1\pi(1)} \cdots U_{d\pi(d)}} \, dU = \frac{1}{N^d} \sum_{r=0}^{\infty} (-1)^r \frac{\vec{W}^r(\alpha)}{N^r},$$

where $\vec{W}^r(\alpha)$ is number of r-step monotone walks from id to π on $S(d)$.

(This is a disconnected monotone Hurwitz number).

Corollary: We have

$$\int_{U(N)} e^{zN\,\mathrm{Tr}(AU+BU^{-1})}\,dU$$

$$= 1 + \sum_{d=1}^{N} \frac{z^{2d}}{d!} N^d \sum_{\alpha \vdash d} p_\alpha(AB) \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \vec{W}^r(\alpha) + O(z^{2N+2})$$

$$\int_{U(N)} e^{zN\,\mathrm{Tr}\,AUBU^{-1}}\,dU$$

$$= 1 + \sum_{d=1}^{N} \frac{z^{d}}{d!} \sum_{\alpha \vdash d} p_\alpha(A)\, p_\beta(B) \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \vec{W}^r(\alpha,\beta) + O(z^{N+1})$$

Infinite N

Bott Periodicity

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}
$U(1)$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
$U(2)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
$U(3)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$	?	?	?	?
$U(4)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?	?	?
$U(5)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?

← Unstable

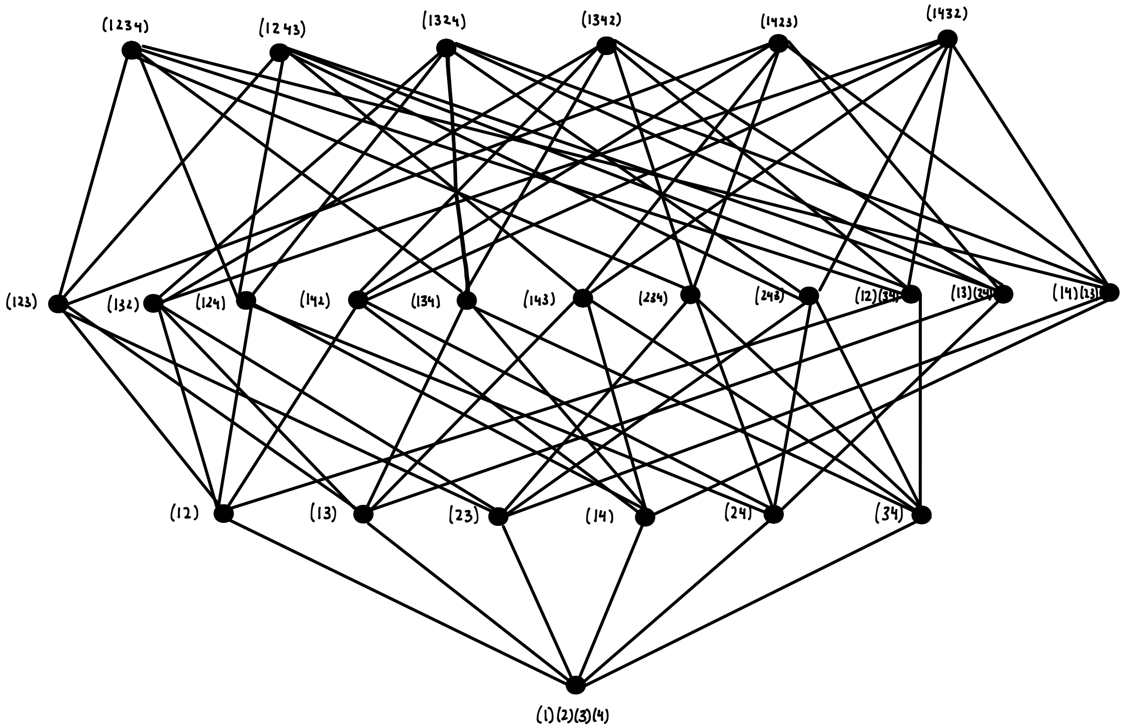
↑ Stable

$$U(\infty) = \varinjlim U(N)$$

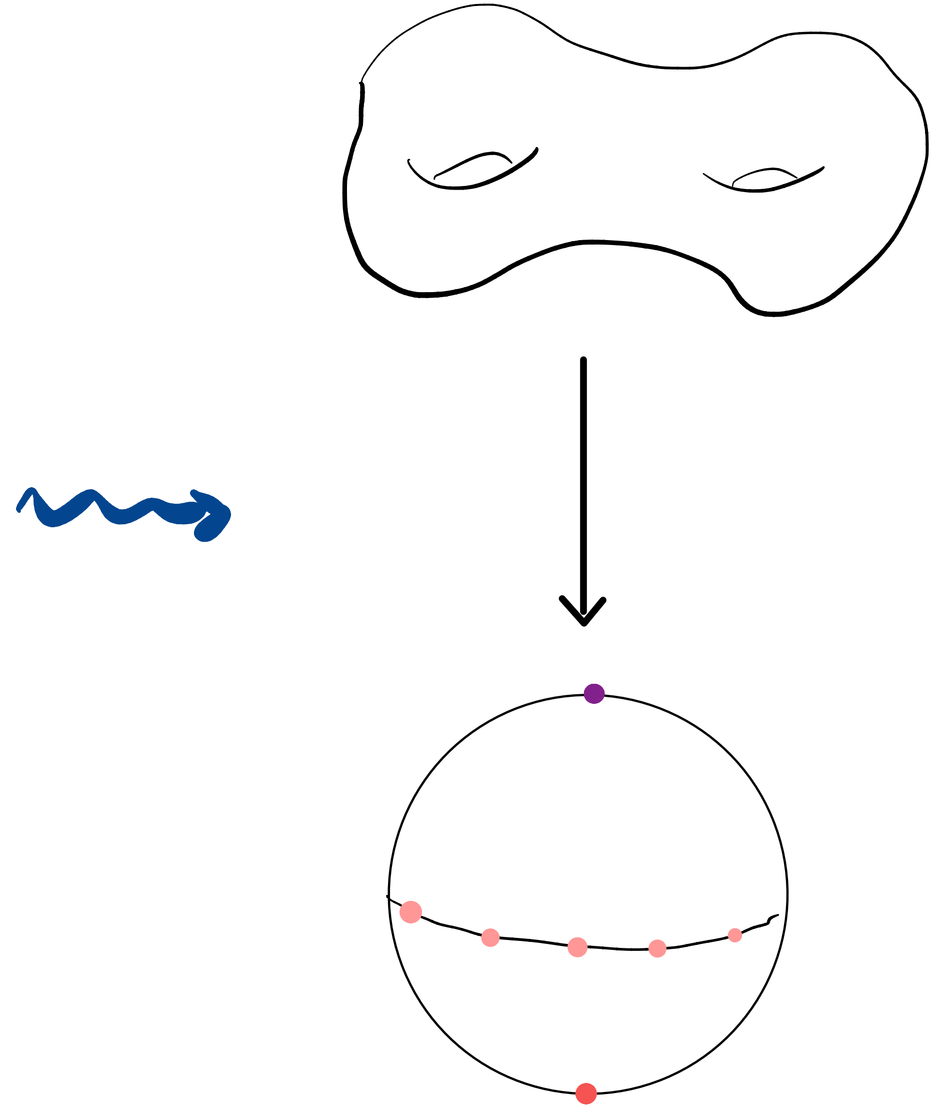
Stable BGW and HCIZ

$$\int_{U(\infty)} e^{z\hbar^{-1}\operatorname{Tr}(AU+BU^{-1})}\,dU = 1+\sum_{d=1}^{\infty}\frac{z^{2d}}{d!}\hbar^{-d}\sum_{\alpha\vdash d}p_\alpha(A)\sum_{r=0}^{\infty}(-\hbar)^r\,\vec{W}^r(\alpha)$$

$$\int_{U(\infty)} e^{z\hbar^{-1}\operatorname{Tr}AUBU^{-1}}\,dU = 1+\sum_{d=1}^{\infty}\frac{z^{d}}{d!}\sum_{\substack{\alpha\vdash d\\ \beta\vdash d}}p_\alpha(A)p_\beta(B)\sum_{r=0}^{\infty}(-\hbar)^r\,\vec{W}^r(\alpha,\beta)$$



Counting
Walks



Counting
Functions

Topological Interpretation

- Monotone Hurwitz numbers: $\vec{H}^{\bullet}_g(\alpha,\beta) = \vec{W}^{2g-2+\ell(\alpha)+\ell(\beta)}(\alpha,\beta)$.

$$\int_{U(\infty)} e^{z\hbar^{-1}\mathrm{Tr}(AU+BU^{-1})}\,dU = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha \vdash d} \frac{p_\alpha(A)}{\hbar^{-\ell(\alpha)}} (-1)^{\ell(\alpha)+d} \sum_{g=-\infty}^{\infty} \hbar^{2g-2} \vec{H}^{\bullet}_g(\alpha)$$

$$\int_{U(\infty)} e^{z\hbar^{-1}\mathrm{Tr}\,AUBU^{-1}}\,dU = 1 + \sum_{d=1}^{\infty} \frac{z^{d}}{d!} \sum_{\substack{\alpha \vdash d \\ \beta \vdash d}} \frac{p_\alpha(A)}{\hbar^{-\ell(\alpha)}} \frac{p_\beta(B)}{\hbar^{-\ell(\beta)}} (-1)^{\ell(\alpha)+\ell(\beta)} \sum_{g=-\infty}^{\infty} \hbar^{2g-2} \vec{H}^{\bullet}_g(\alpha,\beta)$$

Stable Free Energies

$$\log \int_{U(\infty)} e^{z\hbar^{-1}\operatorname{Tr}(AU+BU^{-1})}\, dU$$

$$= \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha \vdash d} \frac{p_\alpha(A)}{\hbar^{-\ell(\alpha)}} (-1)^{\ell(\alpha)+d} \sum_{g=0}^{\infty} \hbar^{2g-2} \vec{H}_g(\alpha)$$

$$\log \int_{U(\infty)} e^{z\hbar^{-1}\operatorname{Tr} AUBU^{-1}}\, dU$$

$$= \sum_{d=1}^{\infty} \frac{z^{d}}{d!} \sum_{\substack{\alpha \vdash d \\ \beta \vdash d}} \frac{p_\alpha(A)}{\hbar^{-\ell(\alpha)}} \frac{p_\beta(B)}{\hbar^{-\ell(\beta)}} (-1)^{\ell(\alpha)+\ell(\beta)} \sum_{g=0}^{\infty} \hbar^{2g-2} \vec{H}_g(\alpha,\beta)$$

Stable Topological Expansion

$$\log \int_{U(\infty)} e^{z\hbar^{-1}\,\mathrm{Tr}(AU+BU^{-1})}\, dU = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g,$$

$$F_g = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha \vdash d} \frac{p_\alpha(A)}{\hbar^{-\ell(\alpha)}} (-1)^{\ell(\alpha)+d}\, \vec{H}_g(\alpha)$$

$$\log \int_{U(\infty)} e^{z\hbar^{-1}\,\mathrm{Tr}\, AUBU^{-1}}\, dU = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g,$$

$$F_g = \sum_{d=1}^{\infty} \frac{z^{d}}{d!} \sum_{\substack{\alpha \vdash d \\ \beta \vdash d}} \frac{p_\alpha(A)}{\hbar^{-\ell(\alpha)}} \frac{p_\beta(B)}{\hbar^{-\ell(\beta)}} (-1)^{\ell(\alpha)+\ell(\beta)}\, \vec{H}_g(\alpha,\beta)$$

Large N

Holomorphic Candidates

Theorem: There exists $\gamma > 0$ such that

$$F_{Ng} = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta \vdash d} \frac{p_\alpha(a_1,\dots,a_N)}{N^{\ell(\alpha)}} \frac{p_\beta(b_1,\dots,b_N)}{N^{\ell(\beta)}} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_g(\alpha,\beta)$$

converges uniformly absolutely on $\mathbb{D}_N(\gamma)$ for all $N \in \mathbb{N}$ and $g \in \mathbb{N}_0$.

— stable topological constant exists —

Mystery

- Univariate power series $F_g = \sum_{d=1}^{\infty} \frac{z^d}{d!} \vec{H}_g(1^d, 1^d)$ has radius of convergence $z_c = \frac{2}{27}$.

- Based on parameterization of F_g by ${}_2F_1\left(\frac{2}{3}, \frac{4}{3}, \frac{3}{2}; \frac{27}{2} z\right)$.

$$\lim_{N \to \infty} \frac{1}{N^3} \log \left| \{\text{finite groups of order } p^N\} \right| = \frac{2}{27}.$$

Disconnected by Necessity

- Impossible to compare $F_N = \log I_N$ to $\sum_{g=0}^{k} N^{2-2g} F_{Ng}$ in $(\mathcal{O}_N(\gamma), \|\cdot\|_\gamma)$ because of complex zeros.

- Have to work with disconnected topologies: topologically normalized partition function

$$\Phi_{Nk} = e^{-\sum_{g=0}^{k} N^{2-2g} F_{Ng}} \int_{U(N)} e^{zN \operatorname{Tr} AUBU^{-1}} dU \in \mathcal{O}_N(\gamma).$$

Theorem: Topological expansion of $F_N = \log I_N$ is equivalent to topological concentration of I_N: there exists $0 < \varepsilon \leq \gamma$ such that for each $k \in \mathbb{N}_0$

$$\left\| e^{-\sum_{g=0}^{k} N^{2-2g} F_{N,g}} I_N - 1 \right\|_\varepsilon = O_k(N^{2-2k})$$

as $N \to \infty$.

- Can see topological concentration easily at $N=\infty$: it's topological cancellation.

$$\int_{U(\infty)} e^{z\hbar^{-1}\,\mathrm{Tr}\,AUBU^{-1}}\,dU = e^{\sum_{g=0}^{k}\hbar^{2g-2}F_g}\; e^{\sum_{g=0}^{k}\hbar^{2g-2}F_g}$$

$$\Rightarrow \quad e^{-\sum_{g=0}^{k}\hbar^{2g-2}F_g}\int_{U(\infty)} e^{z\hbar^{-1}\,\mathrm{Tr}\,AUBU^{-1}}\,dU = e^{\sum_{g=0}^{k}\hbar^{2g-2}F_g}$$

surfaces of genus $\geq k+1$

$$\Rightarrow \quad e^{-\sum_{g=0}^{k}\hbar^{2g-2}F_g}\int_{U(\infty)} e^{z\hbar^{-1}\,\mathrm{Tr}\,AUBU^{-1}}\,dU = O(\hbar^{2k})$$

A Possible Program

- How to reconcile formal and convergent topological expansions?

<u>Step 1</u>: Find holomorphic approximation of partition function.

<u>Step 2</u>: Determine holomorphic candidates using topological recursion.

<u>Step 3</u>: Prove topological expansion of free energy by establishing topological concentration of partition function.

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