Blobbed Topological Recursion for Dirac Ensembles

Shahab Azarfar

School of Data Science, University of Virginia

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- This talk is based on joint work with Masoud Khalkhali, *Random Finite Noncommutative Geometries and Topological Recursion*, arXiv:1906.09362

- The major ideas discussed through the talk are certainly due to the following mathematicians (just to name a few):

  - (Blobbed) Topological Recursion: Eynard, Chekhov, Borot, Orantin, ...
  - Noncommutative Geometry: Connes, Marcolli, Barrett, ...

Outline

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Classical 1-Hermitian Matrix Models

- One starts from a U_N -invariant measure $\mathrm{d}\mu(H)$ on $\mathcal{H}_N$

$$\mathrm{d}\mu(H) = \exp\left[-(N/t)\,\mathrm{Tr}(\mathcal{V}(H))\right]\,\mathrm{d}H$$

- Weyl integration formula: $\mathrm{d}\mu(H)$ induces a measure $\mathrm{d}\tilde{\mu}(\boldsymbol{\lambda})$ on $\mathbb{R}^N$

$$\mathrm{d}\tilde{\mu}(\boldsymbol{\lambda}) = \prod_{1\le i<j\le N} |\lambda_j - \lambda_i|^2 \prod_{i=1}^{N} e^{-(N/t)\mathcal{V}(\lambda_i)}\,\mathrm{d}\lambda_i$$

- Formal matrix models: all integrations are w.r.t. the normalized Gaussian measure

$$\mathrm{d}\mu^0(H) = c\,\exp\left(-\frac{N\,\mathrm{Tr}(H^2)}{2t}\right)\,\mathrm{d}H$$

Correlation Functions

- Partition function $Z_N = \int_{\mathcal{H}_N} \mathrm{d}\mu(H)$, and Free energy $F = \log Z_N$

- Disconnected correlators are the moments of the following form

$$\hat{W}_n(x_1, \cdots, x_n) = \mathbb{E}\left[\prod_{j=1}^{n}\left(\sum_{i=1}^{N}\frac{1}{x_j - \lambda_i}\right)\right], \quad x_j \in \mathbb{C}\backslash\mathbb{R},$$

where $\sum_{i=1}^{N}\frac{1}{x-\lambda_i}$ is the trace of the resolvent.

- Connected correlators are the joint cumulants of the following form

$$W_n(x_1, \cdots, x_n) = \sum_{K \vdash [\![1,n]\!]} (-1)^{[K]-1}\,([K]-1)!\,\prod_{i=1}^{[K]} \hat{W}_{|K_i|}\,(x_{K_i})$$

Counting Discretized Surfaces

- Wick's Theorem: computation of the moments of the Gaussian measure can be reduced to enumeration of the maps (ribbon graphs)

- A term $\tau_{\ell_i} = t_{\ell_i} \frac{\mathrm{Tr}(H^{\ell_i})}{\ell_i}$ in $\mathrm{Tr}(\mathcal{V}(H)) \rightsquigarrow$ an ℓ_i-gon of Boltzmann weight t_{ℓ_i}

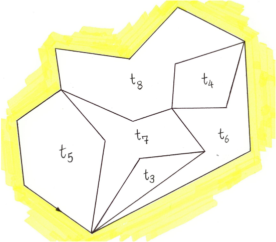


Figure: A planar map with one marked rooted face (the colored 9-gon containing the point ∞)

Topological Expansion

- Using Wick's theorem, one obtains a large-N topological expansion

$$F = \sum_{g \geq 0} (N/t)^{2-2g} F_g \,, \qquad F_g = \sum_{[M] \in \mathbb{M}_g} \mathfrak{Bw}([M]) \,,$$

$$W_n(x_1, \cdots, x_n) = \sum_{g \geq 0} (N/t)^{2-2g-n} \, W_{g,n}(x_1, \cdots, x_n) \,,$$

where $W_{g,n}(x_1, \cdots, x_n)$ enumerates the connected closed maps of genus g with n marked rooted polygonal faces

- Topological Recursion provides a machinery for computing the $W_{g,n}$'s recursively, given certain initial data

Outline

Real Spectral Triples

- A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ equipped with a real structure $J : \mathcal{H} \to \mathcal{H}$ and a chirality operator $\gamma : \mathcal{H} \to \mathcal{H}$
- The real spectral triple $\left(C^\infty(\mathcal{M})\,,\, L^2(\mathcal{M}, \not{S})\,,\, \not{D}\,,\, \gamma\,,\, J\right)$ associated to a spin Riemannian manifold $\mathcal{M}$
- Connes' distance formula: The Dirac operator $\not{D}$ encapsulates all the information about the Riemannian metric over $\mathcal{M}$

Dirac Ensembles

- Moduli space of Dirac operators D encodes all possible geometries ("metrics") over a fixed fermion space $(\mathcal{A}, \mathcal{H}, \gamma, J)$

- As a model for *Quantum Gravity* on a finite noncommutative space, one considers a distribution of the following form over the moduli space of Dirac operators

$$e^{-\mathcal{S}(D)}\,\mathrm{d}D$$

- The action functional is *spectral*

$$\mathcal{S}(D) = \mathrm{Tr}\,(f(D)) = \sum_{\lambda \in \mathrm{Spec}(D)} f(\lambda)$$

Matrix Geometries of type (p, q) [Barrett (2015)]

- A particular class of finite real spectral triples
  - $\mathcal{A} = M_N(\mathbb{C})$
  - $\mathcal{H} = V_{p,q} \otimes M_N(\mathbb{C})$
  - $\langle v \otimes A, u \otimes B \rangle = \langle v, u \rangle \operatorname{Tr}(AB^*)$, $\quad v, u \in V_{p,q}$, $A, B \in M_N(\mathbb{C})$
  - $\pi(A)(v \otimes B) = v \otimes (AB)$,

  where $V_{p,q}$ is an irreducible complex module over the Clifford algebra $C\ell_{p,q}$
- The Dirac operators are expressed in term of commutators or anticommutators with given Hermitian matrices H and anti-Hermitian matrices L

Random matrix geometries of type $(1,0)$

- The Dirac operator

$$D = \{H, \cdot\} \ , \quad H \in \mathcal{H}_N$$

- We consider a model with the following action functional

$$\mathcal{S}(D) = \mathcal{S}_{\text{unstable}}(D) + \mathcal{S}_{\text{stable}}(D) \, ,$$

where

$$\mathcal{S}_{\text{unstable}}(D) = \mathrm{Tr}\left(\mathcal{V}(D)\right) \ , \quad \mathcal{V}(x) = \frac{1}{2t}\left(\frac{x^2}{2} - \sum_{n=3}^{d} \alpha_n \frac{x^n}{n}\right) ,$$

and

$$\mathcal{S}_{\text{stable}}(D) = -\sum_{s=1}^{\mathfrak{g}} (N/t)^{-4s} \sum_{n_I \in \mathbb{N}_{\uparrow}^{s}} \hat{\alpha}_{n_I} \prod_{i=1}^{s} \mathrm{Tr}\left(D^{n_i}\right) \, .$$

Topological expansion of the action functional

- An elementary 2-cell of topology (g,n) with polygonal boundaries of perimeters $\{\ell_i\}_{i=1}^{n}$, $\ell_i \geqslant 1$, is a (equivalence class of) surface of genus g whose boundary has n connected components, and consists of the 1-skeleton of ℓ_i-gons

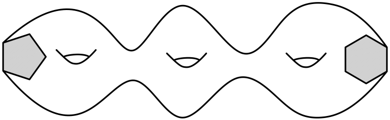


Figure: An elementary 2-cell of topology $(g,n)=(3,2)$ with polygonal boundaries of perimeters $(\ell_1,\ell_2)=(5,6)$

- Decomposition of $\mathcal{S}(D)$ in terms of underlying elementary 2-cells

$$\begin{aligned}\mathcal{S}(D) &= \mathcal{S}_0(H) + \mathcal{S}_{\text{int}}(H) \\ &= \frac{N}{2t}\operatorname{Tr}\left(H^2\right) - \sum_{[C]\in\mathcal{C}} \frac{(N/t)^{\chi(C)}}{(\beta_0\left(\partial C\right))!}\, T_{[C]}(H)\,,\end{aligned}$$

where

  - $\chi(C)\ =$ the Euler characteristic of an elementary 2-cell C
  - $\beta_0\left(\partial C\right)\ =$ the number of connected components of the boundary of C
  - Classifying the elementary 2-cells based on whether $\chi(C) \geqslant 0$ or $\chi(C) < 0$

$$\mathcal{C} = \mathcal{C}_{\text{unstable}} \cup \mathcal{C}_{\text{stable}}$$

  - For an elementary 2-cell C of topology (g, n) with polygonal boundaries of perimeters $\{\ell_i\}_{i=1}^n$

$$T_{[C]}(H) := \mathrm{t}_{\vec{\ell}}^{(g)} \prod_{i=1}^{n} \frac{\operatorname{Tr}\left(H_i^{\ell}\right)}{\ell_i}$$

Corresponding Hermitian matrix model

- A formal multi-trace 1-Hermitian matrix model
- The term $\exp\left(-\mathcal{S}_{\text{int}}(H)\right)$ is considered as a formal power series in the Boltzmann weights $\mathrm{t}_{\vec{\ell}}^{(g)}$
- Using Wick's theorem, we get the following large-N topological expansion

$$W_n(x_1,\cdots,x_n)=\sum_{g\geqslant 0}(N/t)^{2-2g-n}\;W_{g,n}(x_1,\cdots,x_n)\,,$$

where $W_{g,n}(x_1,\cdots,x_n)\in\mathbb{Q}[\mathbf{t}][[t]][[(x_j^{-1})_j]]$ enumerates the connected closed stuffed maps of genus g with n marked rooted polygonal faces

Stuffed maps

- Each term of the form $\frac{(N/t)^{\chi(C)}}{(\beta_0(\partial C))!}\, T_{[C]}(H)$ is represented by the corresponding elementary 2-cell C

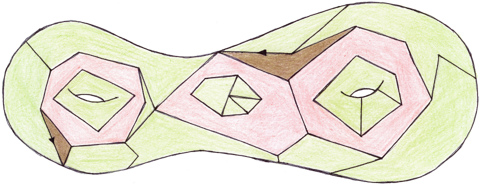


Figure: A closed stuffed map of genus two with two marked rooted polygonal faces (brown disk) obtained by gluing the unstable elementary 2-cells of topology $(g, n) = (0, 1)$ (green disk) and $(g, n) = (0, 2)$ (red cylinder)

Large-N expected spectral distribution ("equilibrium measure")

- *Tame* Boltzmann weights are those numerical values of $\mathrm{t}_{\vec{\ell}}^{(g)}$ for which the generating function of the rooted planar stuffed maps with topology of a disk and perimeter ℓ is finite for all $\ell \in \mathbb{N}$.

- The formal series $W_{0,1}(x)$ is a holomorphic function with discontinuity locus $\Gamma = [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R}$.

- The large-N expected spectral density

$$\varphi(s) = \frac{1}{2\pi\mathrm{i}} \lim_{\epsilon \to 0^+} \left(W_{0,1}(s - \mathrm{i}\epsilon) - W_{0,1}(s + \mathrm{i}\epsilon) \right) , \quad \forall s \in \Gamma_{\text{interior}}$$

is supported on Γ, and assumes positive values on Γ_{interior}.

Outline

Schwinger-Dyson Equations (SDEs)

- The Schwinger-Dyson equations for a matrix model are a tower of equations satisfied by the n-point correlation functions of the model.

- The root of SDEs is the invariance of the integral of a top-degree differential form under a 1-parameter family of orientation-preserving diffeomorphisms on a manifold, i.e.

$$\int_{\phi_t(\Omega)} \Psi\omega = \int_{\Omega} {\phi_t}^* (\Psi\omega) \ , \quad \forall t \in (-\epsilon, \epsilon) \, ,$$

where $\Psi : \mathcal{M} \to \mathbb{R}$ is a smooth function over a Riemannian n-manifold $\mathcal{M}$ with the canonical volume form ω , and $\phi_t : \Omega \to \Omega$ is a local flow over a compact n-dimensional submanifold $\Omega \subset \mathcal{M}$.

SDEs for Random Matrix Geometries of type $(1,0)$

The rank n SDE is given by:

$$
\begin{aligned}
& W_{n+1}(x,x,x_I) + \sum_{J\subseteq I} W_{|J|+1}(x,x_J)\, W_{n-|J|}(x,x_{I\setminus J}) \\
& + \sum_{i\in I} \oint_\Gamma \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \frac{W_{n-1}(\xi,x_{I\setminus\{i\}})}{(x-\xi)(x_i-\xi)^2} \\
& + \sum_{k=1}^{2\mathfrak{g}} \sum_{\substack{K\vdash[\![1,k]\!] \\ J_1\sqcup\cdots\sqcup J_{[K]}=I}} \oint_\Gamma \Big[\prod_{r=1}^{k} \frac{\mathrm{d}\xi_r}{2\pi\mathrm{i}}\Big] \frac{\partial_{\xi_1} T_k(\xi_1,\cdots,\xi_k)}{(k-1)!\,(x-\xi_1)} \prod_{i=1}^{[K]} W_{|K_i|+|J_i|}\,(\xi_{K_i},x_{J_i}) \\
& = 0\,,
\end{aligned}
$$

where the symmetric *k-point interactions* T_k are defined by

$$
\sum_\lambda T_k(\lambda_{i_1},\lambda_{i_2},\cdots,\lambda_{i_k}) = \sum_{\substack{[C]\in\mathcal{C} \\ \beta_0(\partial C)=k}} (N/t)^{\chi(C)}\, T_{[C]}(H)\,.
$$

By considering a large-N expansion of topological type for the correlation functions W_n and the k-point interactions T_k , the rank n SDE to order N^{3-2g-n} is given by:

$$W_{g-1,\,n+1}(x,x,x_I) + \sum_{J\subseteq I\,,\ 0\leqslant f\leqslant g} W_{f,\,|J|+1}(x,x_J)\,W_{g-f,\,n-|J|}(x,x_{I\setminus J})$$

$$+\sum_{i\in I}\oint_\Gamma \frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\,\frac{W_{g,\,n-1}(\xi,x_{I\setminus\{i\}})}{(x-\xi)(x_i-\xi)^2}$$

$$+\sum_{\substack{1\leqslant k\leqslant 2\mathfrak{g}\\ 0\leqslant h}}\ \sum_{\substack{K\vdash[\![1,k]\!]\\ J_1\sqcup\cdots\sqcup J_{[K]}=I}}\ \sum_{\substack{0\leqslant f_1,\cdots,f_{[K]}\\ h+k-[K]+\sum_i f_i=g}}$$

$$\oint_\Gamma\Big[\prod_{r=1}^{k}\frac{\mathrm{d}\xi_r}{2\pi\mathrm{i}}\Big]\,\frac{\partial_{\xi_1}T_{h,k}(\xi_1,\cdots,\xi_k)}{(k-1)!\,(x-\xi_1)}\,\prod_{i=1}^{[K]}W_{f_i,\,|K_i|+|J_i|}\,(\xi_{K_i},x_{J_i})$$

$$=0\,.$$

Spectral Curve

- The rank one SDE to leading order in N

$$(W_{0,1}(x))^2+\sum_{k=1}^{2}\oint_{\Gamma}\Big[\prod_{r=1}^{k}\frac{\mathrm{d}\xi_r}{2\pi\mathrm{i}}\Big]\frac{\partial_{\xi_1}T_{0,k}(\xi_1,\cdots,\xi_k)}{x-\xi_1}\prod_{r=1}^{k}W_{0,1}(\xi_r)=0\,.$$

- The Stieltjes transform $W_{0,1}(x)$ of the large-N spectral distribution $\mu=\varphi(s)\,\mathrm{d}s$ of the model satisfies a quadratic algebraic equation:

$$W_{0,1}(x)=Q(x)+M(x)\sqrt{(x-\mathfrak{a})(x-\mathfrak{b})}\,,$$

where the coefficients of the polynomials $Q(x)$ and $M(x)$ depend on the Boltzmann weights and the moments of μ.

Spectral Curve

- Using the Joukowski map $x : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, given by

$$x(z) = \frac{\mathfrak{a}+\mathfrak{b}}{2} + \frac{\mathfrak{b}-\mathfrak{a}}{4}\left(z + \frac{1}{z}\right),$$

the function $W_{0,1}(x(z))$ gets an *analytic continuation* over the spectral curve Σ of the model.

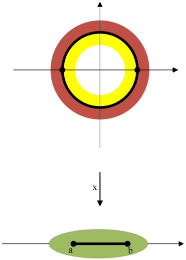


Figure: Illustration of the Joukowski map an the spectral curve Σ of the model

- From the coefficients $W_{g,n}$ of the correlation functions to meromorphic symmetric differentials $\omega_{g,n}$ of degree n, i.e. sections of the n-times external tensor product
  $K_\Sigma^{\boxtimes n} \to \Sigma^n$ of the canonical line bundle $K_\Sigma \to \Sigma$, given by

$$\omega_{g,n}(z_1,\cdots,z_n) = W_{g,n}\left(x(z_1),\cdots,x(z_n)\right) dx(z_1)\, dx(z_2)\cdots dx(z_n) + \delta_{n,2}\,\delta_{g,0}\,\frac{dx(z_1)\,dx(z_2)}{\left(x(z_1)-x(z_2)\right)^2}$$

- Input for the (Blobbed) Topological Recursion Formula
  - The Riemann surface Σ equipped with a local biholomorphic involution
  - The 1-form $\omega_1^0(z)$
  - The symmetric bidifferential $\omega_2^0(z,z_1)$

Blobbed Topological Recursion [Borot (2014)]

Main Result

For random matrix geometries of type $(1,0)$ with the distribution $\mathrm{d}\rho = e^{-\mathcal{S}(D)}\,\mathrm{d}D$, all the stable $\omega_{g,n}$, $2g+n-2>0$, can be computed recursively, using the blobbed topological recursion formula given by

$$\omega_{g,n}(z,z_I) = \sum_{p\in\mathfrak{R}} \operatorname*{Res}_{\zeta=p} K(z,\zeta)\,\mathcal{E}_{g,n}(\zeta,\iota(\zeta);z_I) - \frac{1}{2\pi\mathrm{i}}\oint_{\partial\Sigma}\omega_{0,2}(z,\zeta)\,\mathcal{V}_{g,n}(\zeta;z_I)\,,$$

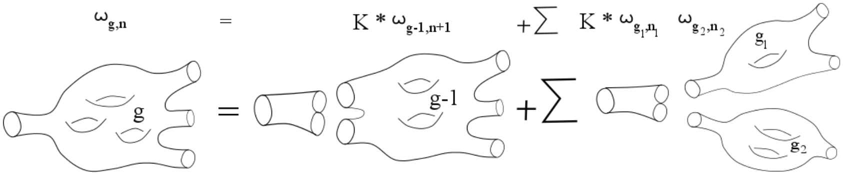
where

$$K(z,\zeta) = \frac{1}{2}\,\frac{\int_{\iota(\zeta)}^{\zeta}\omega_{0,2}(z,\tau)}{\omega_{0,1}(\zeta)-\omega_{0,1}(\iota(\zeta))}$$

$$\mathcal{E}_{g,n}(z,\iota(z);z_I) = \omega_{g-1,\,n+1}(z,\iota(z),z_I) + \sum_{\substack{J\subseteq I\,,\ 0\leqslant f\leqslant g\\ (J,f)\neq(\emptyset,0)\,,\,(I,g)}} \omega_{f,\,|J|+1}(z,z_J)\,\omega_{g-f,\,n-|J|}(\iota(z),z_{I\setminus J})$$

Schematic illustration of the Topological Recursion

- The operator $\sum_{p\in\mathfrak{R}} \operatorname*{Res}_{\zeta=p} K(z,\zeta)$ is represented by a pair of pants
- A differential $\omega_{g,n}$ of degree n is represented by a surface of genus g with n boundary components



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