Homological methods in random noncommutative geometry

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Based on joint work w/ J. Gaunt and A. Schenkel
[arXiv:2203.04817]

Outline

Background

- Real spectral triples $\leftrightarrow$ NC Riemannian (spin) geometry
- Riemannian geometry in physics: gravity
- Quantum gravity: NC spacetime
- Recall: geometrical data/metrical data encoded in Dirac operator of spectral triple (Connes distance formula)

> Path integral over space of geometries $\leftrightarrow$ integration over space of Dirac operators

Fuzzy spectral triples

Consider $V = \mathbb{C}^k$ with $\mathrm{Cl}_{p,q}$-action ((p,q)-Clifford module).

- *Fuzzy spectral triples* [J. Barrett '15]: finite dimensional versions of real spectral triples $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ with
  - $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$
  - $\mathcal{H} = \mathcal{A} \otimes V$, with left $\mathcal{A}$-action $a(m \otimes v) = (a\, m) \otimes v$
  - $\Gamma(m \otimes v) = m \otimes \gamma(v)$
  - $J(m \otimes v) = m^* \otimes C(v)$
  - $+$ Properties
- Right action $Ja^*J^{-1}(m \otimes v) = (m\, a) \otimes v$.
- Terminology for $(\mathcal{A}, \mathcal{H}; \Gamma, J)$: (p,q)-*fermion space*

Dirac operator of fuzzy spectral triple

- Dirac operators fully classified: $L_i, H_i \in \mathrm{Mat}_N(\mathbb{C})$ (anti-)Hermitian

$$D(m \otimes v) = \sum_i [L_i, m] \otimes \alpha_i v + \sum_j \{H_j, m\} \otimes \tau_j v$$

- Recall: geodesic distance from Dirac operator of Riemannian spin manifold

> Space of geometries on fixed (p,q)-fermion space or *Dirac ensemble*: real finite dimensional vector space $\mathcal{D}$ of Dirac operators

Examples: $(0, 1)$ geometry

Simple example:

- $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$
- $\mathcal{H} = \mathcal{A} \otimes \mathbb{C}$
- $\Gamma(m \otimes v) = m \otimes v$
- $J(m \otimes v) = m^* \otimes \overline{v}$
- $D = -\mathrm{i}\,[L, \cdot\,]$

Examples: Fuzzy sphere

Irreducible spin $N/2$-representation $(W, \rho : \mathfrak{su}(2) \to \underline{\mathrm{end}}(W))$.

- Fuzzy sphere algebra: $\mathcal{A} = \underline{\mathrm{end}}(W) \cong \mathrm{Mat}_{N+1}(\mathbb{C})$
- Sphere? Consider basis of $\mathfrak{su}(2)$: $\{e_i\}$ with $[e_i, e_j] = \mathrm{i}\,\epsilon_{ijk}\, e_k$.

  Then $X_i := \lambda_N\, \rho(e_i)$ generate $\mathcal{A}$ and satisfy

$$[X_i, X_j] = \mathrm{i}\ \lambda_N\, \epsilon_{ijk}\, X_k\ , \quad X_1^2 + X_2^2 + X_3^2 = I_{N+1}$$

- $\mathcal{H} = \mathcal{A} \otimes \mathbb{C}^4$
- Dirac operator: $D_{\mathbb{S}_N^2} := I_{N+1} \otimes \widehat{\gamma}^0 + \frac{1}{2}\, \sum_{i,j=1}^{3} [L_{ij},\, \cdot\,] \otimes \widehat{\gamma}^0\, \widehat{\gamma}^i\, \widehat{\gamma}^j$

  with $L_{ij} = \frac{1}{\lambda_N^2}\, \left[X_i, X_j\right]$
- For more details, see [J. Barrett '15]

Random NCG and the path integral

Consider fermion space $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$, $\mathcal{H} = \mathcal{A} \otimes V$, Γ and J

- Finite dimensional space $\mathcal{D}$ of Dirac operators

  $\leftrightarrow$ Space of geometries/Dirac ensemble

- Partition function $Z = \int_{D \in \mathcal{D}} e^{-S(D)} \, \mathrm{d}D$, action $S : \mathcal{D} \to \mathbb{R}$
- Expectation value of $\mathcal{O} : \mathcal{D} \to \mathbb{C}$:

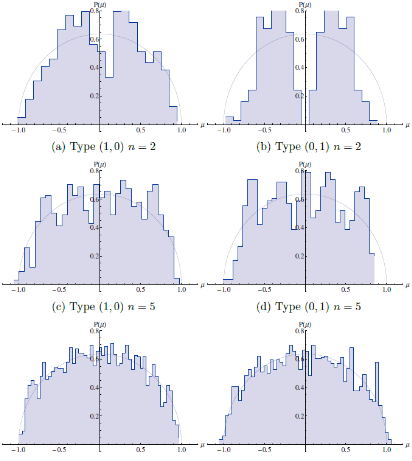
$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{D \in \mathcal{D}} \mathcal{O}(D) \, e^{-S(D)} \, \mathrm{d}D$$

Random NCG: previous works

- Numerical: J. Barrett, L Glaser, M. D'Arcangelo, P. Druce
- Analytical: M. Khalkhali, S. Azarfar, H. Hessam, N. Pagliaroli, C. Perez-Sanchez

The following is from [J. Barrett and L Glaser '16].

Comparison with results from random matrix theory: Wigner semicircle law



(a) Type $(1,0)$ $n=2$
(b) Type $(0,1)$ $n=2$
(c) Type $(1,0)$ $n=5$
(d) Type $(0,1)$ $n=5$
(e) Type $(1,0)$ $n=15$
(f) Type $(0,1)$ $n=15$

Figure 3: The semicircle law is compared with the density of states for H or L.

Comparison with results from random matrix theory: Phase transition

Consider action $S(g_2 \, D^2 + D^4)$ with $g_2 < 0$

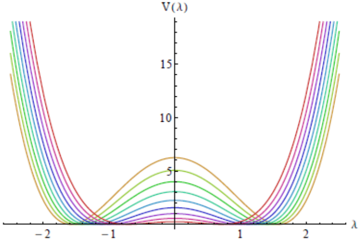


Figure 11: The potential $V = \lambda^4 + g_2\lambda^2$ for $g_2 = -1, -1.5, -2, -2.5, -3, -3.5, -4, -4.5, -5$. The lines are coloured from red ($g_2 = -1$) through to yellow ($g_2 = -5$).

Known: Random matrix model with this kind of potential leads to a phase transition

Numerical evidence of phase transitions

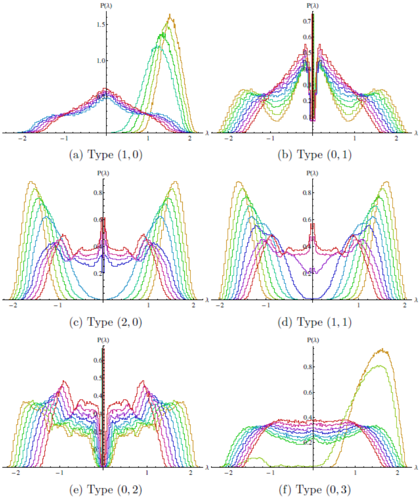


Figure 12: The eigenvalues of $S = \mathrm{Tr}\,(D^4 + g_2 D^2)$ for $n = 10$ and $g_2 = -1$, -1.5, -2, -2.5, -3, -3.5, -4, -4.5, -5. The lines are coloured from red ($g_2 = -1$) through to yellow ($g_2 = -5$).

Diffeomorphism symmetries?

- Overcounting problem: integration over $\mathcal{D}/$automorphisms
- So far ignored in literature
- Our attempt: [J. Gaunt, H. N. and A. Schenkel '22]
- Implementation: homological methods
  - Path integral determined by cohomology of a certain complex

    $\rightsquigarrow$ Batalin-Vilkovisky (BV) formalism
- (Toy-)model for quantum gravity

NC diffeomorphisms

Automorphism of (p, q)-fermion space $(\mathcal{A}, \mathcal{H}; \Gamma, J)$?

- Pair $(\varphi : \mathcal{A} \to \mathcal{A}, \Phi : \mathcal{H} \to \mathcal{H})$ with
  - $\varphi \in \mathrm{Aut}(\mathcal{A}) \cong PU(N) := U(N)/U(1)$ - NC diffeomorphism group
  - $\Phi(m \otimes v) = \varphi(m) \otimes T(v)$ where $T \in \mathrm{Aut}(V)$
    - $\langle T(v), T(v')\rangle_V = \langle v, v'\rangle_V$, $T\,\gamma = \gamma\, T$ and $T\,C = C\,T$
    - Group $K \subset \mathrm{Aut}(V)$ of such T - global transformations of spinors
- $\mathcal{G} := PU(N) \times K$ *gauge group* of (p, q)-fermion space

Gauge transformations

- Left $\mathcal{G}$-action on (p, q)-fermion space
  - $\rho_{\mathcal{A}} : \mathcal{G} \times \mathcal{A} \to \mathcal{A}, \quad \rho_{\mathcal{A}}(\varphi, T)(m) = \varphi(m)$
  - $\rho_{\mathcal{H}} : \mathcal{G} \times \mathcal{H} \to \mathcal{H}, \quad \rho_{\mathcal{H}}(\varphi, T)(m \otimes v) = \varphi(m) \otimes T(v)$

$\rightsquigarrow$ Induced left adjoint action on space of Dirac operators $\mathcal{D}$

$$\rho_{\mathcal{D}} : \mathcal{G} \times \mathcal{D} \to \mathcal{D}, \quad \rho_{\mathcal{D}}(\varphi, T)(D) = \rho_{\mathcal{H}}(\varphi, T) \circ D \circ \rho_{\mathcal{H}}(\varphi^{-1}, T^{-1})$$

- Infinitesimal gauge transformations: $\mathfrak{g} = \mathfrak{su}(N) \oplus \mathfrak{k}$
  - $\rho_{\mathcal{A}}(\epsilon \oplus k)(m) = [\epsilon, m]_{\mathcal{A}}$
  - $\rho_{\mathcal{H}}(\epsilon \oplus k)(m \otimes v) = [\epsilon, m]_{\mathcal{A}} \otimes v + m \otimes k(v)$
  - $\rho_{\mathcal{D}}(\epsilon \oplus k)(D) = [\rho_{\mathcal{H}}(\epsilon \oplus k), D]_{\mathrm{End}(\mathcal{H})}$

Perturbations

Goal: compute path integral $\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{D \in \mathcal{D}/\mathcal{G}} \mathcal{O}(D)\, e^{-S(D)}\, \mathrm{d}D$

with $\mathcal{G}$-invariant $(S : \mathcal{D} \to \mathbb{R}) \in \mathrm{Sym}\, D^{\vee}$ and polynomial $\mathcal{O}$.

- Formal perturbations $D = D_0 + \lambda\, \widetilde{D}$ where $\widetilde{D} \in \mathcal{D}$ and D_0 exact solution of Euler-Lagrange (EL) eqn of S
- Infinitesimal gauge transformation on perturbation:

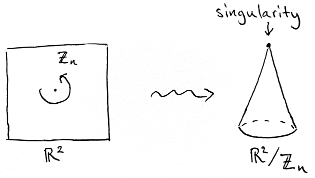
$$\widetilde{\rho}_{\mathcal{D}}(\epsilon \oplus k)(\widetilde{D}) = \boxed{[\rho_{\mathcal{H}}(\epsilon \oplus k), D_0]_{\mathrm{End}(\mathcal{H})}} + \lambda\, [\rho_{\mathcal{H}}(\epsilon \oplus k), \widetilde{D}]_{\mathrm{End}(\mathcal{H})}$$

- Induced action: $\widetilde{S}(\widetilde{D}) := \frac{1}{\lambda^2}\left(S(D_0 + \lambda\, \widetilde{D}) - S(D_0)\right)$

$\rightsquigarrow$ Compute path integral perturbatively around D_0 up to gauge transformations

BV-formalism

- Homological method to compute path integrals where the action S has gauge symmetries: $\frac{1}{Z}\int_{D\in\mathcal{D}/\mathcal{G}} \mathcal{O}(D)\, e^{-S(D)}\mathrm{d}D$
- However: **Cannot** take naive quotient $\mathcal{D}/\mathcal{G}$



- BV-formalism: assign dg-algebra to every gauge invariant S.
  Two parts: Classical BV-formalism and BV-quantisation
  Modern formulation by [K. Costello and O. Gwilliam '16]
- Computation: homological perturbation theory

Classical BV formalism

Input data:

- vector space $\mathcal{D}$ of perturbations $\widetilde{D}$ of $D_0 \leftarrow$ exact solution of EL-eqn of action S
- infinitesimal gauge symmetries by $\mathfrak{g} = \mathfrak{su}(N) \oplus \mathfrak{k}$, acting on $\mathcal{D}$
- $\mathfrak{g}$-invariant action $\widetilde{S} := \frac{1}{\lambda^2}\left(S(D_0 + \lambda\,\widetilde{D}) - S(D_0)\right)$

Output cochain complex:

- Classical observables $\mathrm{Obs}^{\mathrm{cl}} := \mathrm{Sym}(\mathcal{L})$ where

  $\mathcal{L} := \mathfrak{g}[2] \oplus \mathcal{D}[1] \oplus \mathcal{D}^\vee \oplus \mathfrak{g}^\vee[-1]$
- Differential $\mathrm{d} : \mathrm{Obs}^{\mathrm{cl}} \to \mathrm{Obs}^{\mathrm{cl}}$ determined by EL-eqns and infinitesimal gauge transformations

Shifted Poisson structure

- Consider dual pair of bases

$$\left\{e_a \in \mathcal{D}\right\}_{a=1}^{\dim \mathcal{D}} \quad , \quad \left\{f^a \in \mathcal{D}^\vee\right\}_{a=1}^{\dim \mathcal{D}} \quad ,$$
$$\left\{t_i \in \mathfrak{g}\right\}_{i=1}^{\dim \mathfrak{g}} \quad , \quad \left\{\theta^i \in \mathfrak{g}^\vee\right\}_{i=1}^{\dim \mathfrak{g}}$$

- Canonical (-1)-shifted symplectic structure

$$\omega \ = \ \mathrm{d}^{\mathrm{dR}} e_a \wedge \mathrm{d}^{\mathrm{dR}} f^a - \mathrm{d}^{\mathrm{dR}} t_i \wedge \mathrm{d}^{\mathrm{dR}} \theta^i$$

- Hamiltonian vector field ${}_aH$: $\mathrm{d}^{\mathrm{dR}} a = \iota_{{}_aH}\, \omega$

  $\rightsquigarrow$ *shifted Poisson bracket/antibracket*: $\{a, b\} := \iota_{{}_aH}\, \iota_{{}_bH}\, \omega$

  for all $a, b \in \mathrm{Obs}^{\mathrm{cl}}$

- Antibracket $\{\,\cdot\,,\,\cdot\,\}$ satisfies graded antisymmetry, graded Jacobi identity, derivation property and compatibility with d.

BV quantisation

- So far: from classical BV formalism, obtained
  $(\mathrm{Sym}(\mathcal{L}), \mathrm{d}, \{\,\cdot\,,\,\cdot\,\})$ with $\mathcal{L} := \mathfrak{g}[2] \oplus \mathcal{D}[1] \oplus \mathcal{D}^\vee \oplus \mathfrak{g}^\vee[-1]$
- Quantum BV formalism: Use antibracket to deform differential
  - *BV Laplacian*: $\Delta_{\mathrm{BV}} : \mathrm{Obs}^{\mathrm{cl}} \to \mathrm{Obs}^{\mathrm{cl}}$

$$\Delta_{\mathrm{BV}}(\varphi_1 \cdots \varphi_n) = \sum_{i<j} (-1)^{\sum_{k=1}^{i} |\varphi_k| + |\varphi_j| \sum_{k=i+1}^{j-1} |\varphi_k|} \times \{\varphi_i, \varphi_j\}\, \varphi_1 \cdots \check{\varphi}_i \cdots \check{\varphi}_j \cdots \varphi_n$$

  with $\varphi_i \in \mathrm{Sym}(\mathcal{L})$ homogeneous.
  - cochain complex of quantum observables

$$\mathrm{Obs}^{\mathrm{qu}} := \big(\mathrm{Sym}(\mathcal{L}), \mathrm{d}^{\mathrm{qu}} := \mathrm{d} + \hbar\, \Delta_{\mathrm{BV}}\big)$$

Computing the path integral

- Path integral $\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{D \in \mathcal{D}/\mathcal{G}} \mathcal{O}(D)\, e^{-S(D)}\, \mathrm{d}D$ determined

  by cohomology of $\mathrm{Obs}^{\mathrm{qu}} := \big(\mathrm{Sym}(\mathcal{L}), \mathrm{d}^{\mathrm{qu}} := \mathrm{d} + \hbar\, \Delta_{\mathrm{BV}}\big)$

- Computed perturbatively in λ using homological perturbation theory

  $\rightsquigarrow$ Split quantum differential $\mathrm{d}^{\mathrm{qu}} = \mathrm{d}^{\mathrm{free}} + \lambda\, \mathrm{d}^{\mathrm{int}} + \hbar\, \Delta_{\mathrm{BV}}$

- Starting point: $\mathrm{Obs}^{\mathrm{free}} := \big(\mathrm{Sym}(\mathcal{L}), \mathrm{d}^{\mathrm{free}}\big) = \mathrm{Sym}\big(\mathcal{L}, \mathrm{d}^{\mathrm{free}}\big)$ where

  $$(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) = \Big(\mathfrak{g}[2] \xrightarrow{\text{gauge}} \mathcal{D}[1] \xrightarrow{\text{EL}} \mathcal{D}^{\vee} \xrightarrow{\text{gauge}} \mathfrak{g}^{\vee}[-1] \Big)$$

Homological perturbation

Next: Choose *strong deformation retract* (SDR)

$$\left(\mathrm{H}^\bullet(\mathcal{L}, \mathrm{d}^{\mathrm{free}}), 0\right) \underset{\iota}{\overset{\pi}{\leftrightarrows}} (\mathcal{L}, \mathrm{d}^{\mathrm{free}}) \circlearrowright h$$

$\rightsquigarrow$ SDR

$$\left(\mathrm{Sym}\,\mathrm{H}^\bullet(\mathcal{L}, \mathrm{d}^{\mathrm{free}}), 0\right) \underset{I}{\overset{\Pi}{\leftrightarrows}} \mathrm{Obs}^{\mathrm{free}} \circlearrowright H \qquad (*)$$

Homological perturbation lemma: Let $\delta := \lambda\,\mathrm{d}^{\mathrm{int}} + \hbar\,\Delta_{\mathrm{BV}}$.
There exists a deformation of (*) into a SDR

$$\left(\mathrm{Sym}\,\mathrm{H}^\bullet(\mathcal{L}, \mathrm{d}^{\mathrm{free}}), \widetilde{\delta}\right) \underset{\widetilde{I}}{\overset{\widetilde{\Pi}}{\leftrightarrows}} \mathrm{Obs}^{\mathrm{qu}} \circlearrowright \widetilde{H}$$

Correlation functions

In particular:

n-point correlation function

$$\begin{aligned}\langle \varphi_1 \cdots \varphi_n \rangle &= \widetilde{\Pi}(\varphi_1 \cdots \varphi_n) \\ &= \sum_{k=0}^{\infty} \Pi\big((\delta H)^k (\varphi_1 \cdots \varphi_n)\big) \in \mathrm{Sym}\,\mathrm{H}^\bullet(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) \quad ,\end{aligned}$$

for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$ and $\delta = \lambda\, \mathrm{d}^{\mathrm{int}} + \hbar\, \Delta_{\mathrm{BV}}$

- Computes $\frac{1}{Z} \int_{D \in \mathcal{D}/\mathcal{G}} \varphi_1 \cdots \varphi_n e^{-S(D)} \mathrm{d}D$
- Feynman diagrams

Framework done. Next: Investigate if gauge symmetry contributes to correlation functions.

Two cases:

- $D_0 = 0$
- $D_0 \neq 0$

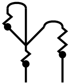
Case 1: $D_0 = 0$

- Action $S(D) = \frac{g_2}{2}\,\mathrm{Tr}_{\mathrm{End}(\mathcal{H})}(D^2) + S^{\mathrm{int}}(D)$,
  $S^{\mathrm{int}}(D)$ sum of monomials of degree ≥ 3.
- $(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) = \left(\mathfrak{g}[2] \xrightarrow{0} \mathcal{D}[1] \xrightarrow{\mathrm{EL}} \mathcal{D}^\vee \xrightarrow{0} \mathfrak{g}^\vee[-1] \right)$
- $\mathrm{H}^\bullet(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) = \mathfrak{g}[2] \oplus \mathfrak{g}^\vee[-1]$

Graphical calculus: denote elements $\varphi_1 \cdots \varphi_n \in \mathrm{Sym}(\mathcal{L})$ by n vertical lines with

Case 1: $D_0 = 0$, quartic interaction

- $S = \mathrm{Tr}_{\mathrm{End}(\mathcal{H})}(\frac{g_2}{2}\, D^2 + \frac{g_4}{4!}\, D^4)$
- 2-point correlation function: $\varphi_1, \varphi_2 \in \mathcal{L}^0 = \mathcal{D}^\vee$

$$\langle \varphi_1\, \varphi_2 \rangle \;=\; \hbar$$  $$+\;\; \frac{\hbar^2 \lambda^2}{2}$$  $$+\;\; \mathcal{O}(\lambda^3)$$

- Observe: no contribution from lines in $\mathfrak{g}[2]$ and $\mathfrak{g}^\vee[-1]$!

In fact:

> In the $D_0 = 0$ case, for $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^0 = \mathcal{D}^\vee$, $\langle \varphi_1 \cdots \varphi_n \rangle$ receive **no** contribution from lines in $\mathfrak{g}[2]$ and $\mathfrak{g}^\vee[-1]$.

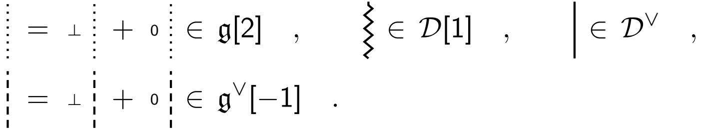
In other words: no contributions from gauge symmetry

in $\langle \varphi_1 \cdots \varphi_n \rangle$ in the $D_0 = 0$ case!

Case 2: $D_0 \neq 0$, quartic $(0,1)$-model

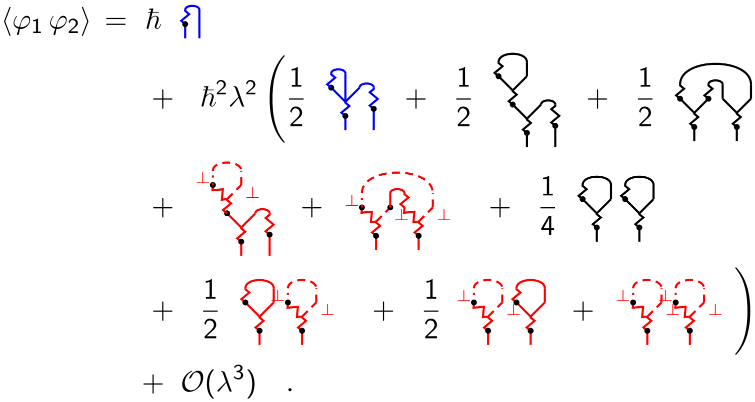
- $S = \mathrm{Tr}_{\mathrm{End}(\mathcal{H})}\big(-\frac{1}{2}\,D^2 + \frac{g_4}{4!}\,D^4\big)$ with $g_4 \geq 0$
  - $(0,1)$ geometry: $D = -i[L,\cdot\,]$ with L trace-free Hermitian $N \times N$-matrix
- $(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) = \Big(\mathfrak{g}[2] \xrightarrow{\text{gauge}} \mathcal{D}[1] \xrightarrow{\text{EL}} \mathcal{D}^{\vee} \xrightarrow{\text{gauge}} \mathfrak{g}^{\vee}[-1]\Big)$
- $\mathrm{H}^{\bullet}(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) = \mathfrak{g}_0[2] \oplus \mathfrak{g}_0[-1]$ with $\mathfrak{g}_0 \subset \mathfrak{g}$ Lie subalgebra stabilizing $D_0 \rightsquigarrow$ Symmetry broken!

Graphical calculus:



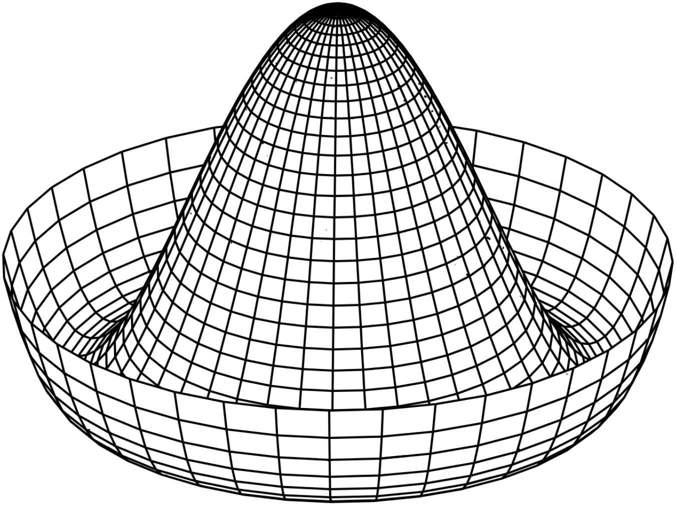
Case 2: $D_0 \neq 0$, quartic $(0,1)$-model, 2-point correlator

For $\varphi_1, \varphi_2 \in \mathcal{L}^0 = \mathcal{D}^\vee$,



Contributions from lines in $\mathfrak{g}[2]$ and $\mathfrak{g}^\vee[-1]$!

Analogy: Higgs mechanism



Summary

- Fuzzy spectral triples: $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$, $\mathcal{H} = \mathcal{A} \otimes \mathbb{C}^k$. Dirac operator encodes geometry.
- Dirac operators fully classified - Random NCG
- Numerical evidence of certain properties from random matrix theory also in randon NCG, e.g. phase transitions
- Overcounting problem: path integral $\mathcal{D}/$gauge symmetries?
- Homological methods: BV formalism and homological perturbation
- n-point correlation functions
    - $D_0 = 0$: gauge symmetry does **not** contribute!
    - $D_0 \neq 0$: gauge symmetry contributes (symmetry breaking)!

Thank you for your attention!