Bootstrapping Dirac Ensembles

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Real Spectral Triples

- A finite real spectral triple is composed of
  - a finite dimensional Hilbert space $\mathcal{H}$
  - an algebra $\mathcal{A}$
  - a chirality operator Γ
  - an antilinear real structure J
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- We are interested in when $\mathcal{A}$ is the space of $\mathrm{M}_N(\mathbb{C})$ and $\mathcal{H} = \mathbb{C}^k \otimes \mathrm{M}_N(\mathbb{C})$.
- Let all of the above objects besides D be fixed and Dirac operator D can vary.

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  - Type $(0,1)$, $D = -i[L, \cdot]$ $\quad$ L is skew-Hermitian
- Signature two models
  - Type $(2,0)$, $D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$
  - Type $(1,1)$, $D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes [L_2, \cdot]$
  - Type $(0,2)$, $D = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot]$

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- We are interested in solving these models in the large N limit for different potential $S(D)$.

Type (1,0) Dirac Ensembles

Consider the type (1,0) Dirac Ensembles

$$D = \{H, \cdot\} = H \otimes I + I \otimes H^t,$$

$$\operatorname{Tr} D^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} \operatorname{Tr} H^{\ell-k} \operatorname{Tr} H^k.$$

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then

$$\begin{aligned} S(D) &= g(2N \operatorname{Tr} H^2 + 2 \operatorname{Tr} H \operatorname{Tr} H) \\ &\quad + (2N \operatorname{Tr} H^4 + 8 \operatorname{Tr} H \operatorname{Tr} H^3 + 6 \operatorname{Tr} H^2 \operatorname{Tr} H^2). \end{aligned}$$

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- The partition function of a Dirac ensemble can be written in terms of a random $N \times N$ Hermitian matrix variable H:

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- We are mainly concerned with finding the moments of these models in the large N limit.

$$m_k = \lim_{N\to\infty} \langle \frac{1}{N} \operatorname{Tr} H^k \rangle = \lim_{N\to\infty} \frac{1}{N} \frac{1}{Z} \int_{\mathcal{H}_N} \operatorname{Tr} H^k e^{-\tilde{S}(H)} dH.$$

$$d_k = \lim_{N\to\infty} \langle \frac{1}{N^2} \operatorname{Tr} D^k \rangle = \lim_{N\to\infty} \frac{1}{N^2} \frac{1}{Z} \int_{\mathcal{G}} \operatorname{Tr} D^k e^{-S(D)} dD.$$

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- **Step 3**: Apply positivity constraint with a cutoff.

Hamburger Moment Problem

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The Hamburger moment problem is solvable if and only if the corresponding Hankel matrix is positive semi-definite.

$$\begin{bmatrix} m_0 & m_1 & m_2 & m_3 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \\ m_2 & m_3 & m_4 & m_5 & \cdots \\ m_3 & m_4 & m_5 & m_6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \geq 0$$

Schwinger-Dyson Equations

- In general, the Schwinger-Dyson equations come from the following integral.

$$\sum_{i,j=1}^{N} \int_{\mathcal{H}_N^m} \frac{\partial}{\partial (H_q)_{ij}} \left(W_{ij} \, e^{-\tilde{S}(H_1, H_2, \ldots, H_m)} \right) dH_1 \ldots dH_m = 0.$$

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- The above equation generates the following SDEs in single matrix model.

$$\sum_{k=0}^{\ell-1} \langle \mathrm{Tr}\, H_1^{\ell-1-k}\, \mathrm{Tr}\, H_1^{k} \rangle = \langle \mathrm{Tr}\, H_1^{\ell} \tilde{S}'(H_1) \rangle.$$

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$$\begin{aligned}\tilde{S}(H) &= g(2N\,\mathrm{Tr}\,H^2 + 2\,\mathrm{Tr}\,H\,\mathrm{Tr}\,H) \\ &\quad + (2N\,\mathrm{Tr}\,H^4 + 8\,\mathrm{Tr}\,H\,\mathrm{Tr}\,H^3 + 6\,\mathrm{Tr}\,H^2\,\mathrm{Tr}\,H^2).\end{aligned}$$

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- **Step 1**: The SDEs
$$m_{2\ell+2} = \frac{1}{8} \sum_{k=0}^{2\ell-2} m_k m_{2\ell-k-2} - \frac{1}{2} g m_{2\ell} - 3 m_2 m_{2\ell}.$$

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- **Step 2**: The dimension of the search space is 1.

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- Take a real polynomial $f(x) = \sum c_j x^j$. Then the positivity of the integral $\int_{\mathbb{R}} f(x)^2 \rho(x) dx$ implies the positive semi-definiteness of the Hankel matrix of moments

$$\mathcal{M} = \begin{bmatrix} 1 & m_1 & m_2 & m_3 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \\ m_2 & m_3 & m_4 & m_5 & \cdots \\ m_3 & m_4 & m_5 & m_6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

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- These positivity constraints can be applied to both the moments of the matrix ensemble and the Dirac ensemble.

The Solution of Quartic Dirac Ensemble

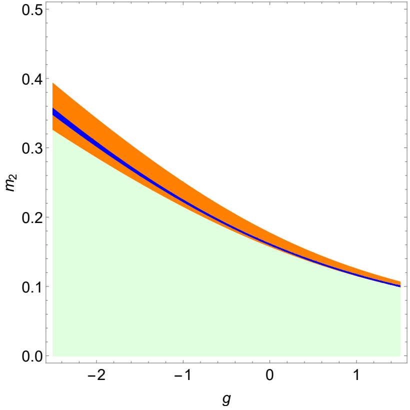

Figure: The approximate relation between m_2 and g, with g varying from -2.5 to 1.5. The different coloured regions denote different constraints applied. The more constraints the smaller the region. The relationship found is very close to the analytic relationship.

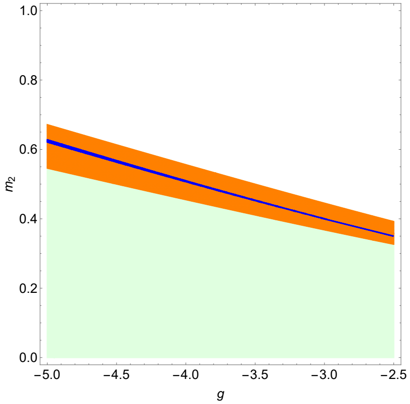

Figure: The approximate relation between m_2 and g, with g varying from -5 to -2.5.

Type (1,0) Cubic Dirac Ensemble

- Now consider the partition function of the cubic Dirac ensemble

$$Z = \int_{\mathcal{G}} e^{-S(D)} dD$$

with

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The loop equations of the model are the following equations:

$$\sum_{k=0}^{\ell-1} m_k m_{\ell-k-1} = m_{\ell+1} + m_1 m_\ell + g\left(m_{\ell+2} + 2 m_1 m_{\ell+1} + m_2 m_\ell\right).$$

The Solution of Cubic Dirac Ensemble

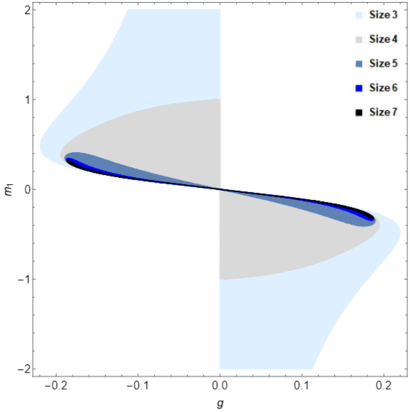

Figure: The search space region for the (1,0) cubic model. Each colour corresponds to the positivity of different number of constraints derived from principal minors. The solution space narrows as the number of constraints increases. Notice that in this example increasing the number of constraints seems to show that there exists a nonlinear relationship between g and m_1.

Type (2,0) Dirac Ensemble

- Recall type (2,0) random geometries

$$D = \gamma^1 \otimes \{A, \cdot\} + \gamma^2 \otimes \{B, \cdot\},$$

with

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where A and B are Hermitian matrices.

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$$S(D) = g \operatorname{Tr} D^2 + \operatorname{Tr} D^4.$$

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$$\operatorname{Tr} D^2 = 4N \operatorname{Tr} A^2 + 4N \operatorname{Tr} B^2 + 4(\operatorname{Tr} A)^2 + 4(\operatorname{Tr} B)^2$$

$$\begin{aligned}\operatorname{Tr} D^4 = {} & 4N \operatorname{Tr} A^4 + 4N \operatorname{Tr} B^4 + 16N \operatorname{Tr} A^2B^2 - 8N \operatorname{Tr} ABAB \\ & + 16 \operatorname{Tr} A \operatorname{Tr} A^3 + 16 \operatorname{Tr} A \operatorname{Tr} B^2A + 16 \operatorname{Tr} B \operatorname{Tr} B^3 \\ & + 16 \operatorname{Tr} B \operatorname{Tr} A^2B + 16(\operatorname{Tr} AB)^2 + 12(\operatorname{Tr} A^2)^2 \\ & + 12(\operatorname{Tr} B^2)^2 + 8 \operatorname{Tr} A^2 \operatorname{Tr} B^2.\end{aligned}$$

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- The partition function of a Dirac ensemble can be written in terms of random Hermitian $N \times N$ matrices variable A and B:

$$Z = \int_{\mathcal{H}_N \times \mathcal{H}_N} e^{-\tilde{S}(A,B)} dAdB.$$

SDEs of Type (2,0) Dirac Ensemble

- The SDE of this model with respect to word A^ℓ comes from

$$\sum_{i,j=1}^{N} \int_{\mathcal{H}_N^2} \frac{\partial}{\partial (A)_{ij}} \left((A^\ell)_{ij}\, e^{-\left(g \operatorname{Tr} D^2 + \operatorname{Tr} D^4\right)} \right) dAdB = 0,$$

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- Notation

$$m_{a,b,c,d} = \lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} A^a B^b A^c B^d \rangle.$$

-

$$\sum_{k=0}^{\ell-1} m_k m_{\ell-k-1} = (8g + 64 m_2) m_{\ell+1} + 16 m_{\ell+3} - 16 m_{\ell,1,1,1} + 32 m_{\ell+1,2},$$

- When $\ell < 9$: In the left hand side, there is no term that is product of moments that come from degree four words or higher.
  For instance, $m_{2,2}m_4$ cannot be found.

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- SDE's of words with length less than 9 can be seen as the system of linear equations and that can be solved in terms of g and m_2.

Lemma

The number of non-trivial moments (up to cyclic permutation and symmetry) that appear in the loop equations of words with length $\ell \geq 9$ is less than the number of non-zero loop equations.

The Solution for the Type(2,0) Model

Corollary

The dimension of the search space of the model is 1.

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- By generating all the loop equations for words up to order ten, we found remarkable formulas for moments up to order eight in terms of g and the second moment m_2.

$$m_4 = -\frac{1}{8}gm_2 + \frac{1}{64},$$

$$m_{2,2} = -\frac{1}{8}gm_2 - m_2^2 + \frac{1}{64},$$

$$m_{1,1,1,1} = \frac{gm_2}{8} + 2m_2^2 - \frac{1}{64},$$

$$m_6 = \frac{g^2 m_2}{64} - \frac{g}{512} - \frac{g\, m_2{}^2}{8} + \frac{3\, m_2}{64} - \frac{5\, m_2{}^3}{4}$$

The Solution for the Type(2,0) Model

$$m_{4,2} = \frac{g^2 m_2}{64} + \frac{g\, {m_2}^2}{8} - \frac{g}{512} - \frac{{m_2}^3}{4} + \frac{m_2}{64}$$

$$m_{3,1,1,1} = -\frac{g^2 m_2}{64} - \frac{3\, g\, {m_2}^2}{8} - \frac{7\, {m_2}^3}{4} + \frac{g}{512} + \frac{m_2}{64}$$

$$m_{2,1,2,1} = \frac{g^2 m_2}{64} + \frac{3\, g\, {m_2}^2}{8} - \frac{g}{512} + \frac{11\, {m_2}^3}{4} - \frac{m_2}{64}$$

$$m_8 = -\frac{g m_2}{64} + \frac{{m_2}^4}{4} + \frac{g^2}{4096} + \frac{{m_2}^2}{256} + \frac{3}{4096} - \frac{g^3 m_2}{512} + \frac{3\, g^2 {m_2}^2}{64} + \frac{g\, {m_2}^3}{2}$$

- First three nonzero moments of the signature two Dirac ensemble

$$d_2 = 8\, m_2,$$

$$d_4 = -4\, g m_2 + \frac{1}{2},$$

$$d_6 = -160\, {m_2}^3 - 16\, g\, {m_2}^2 + 6\, m_2 + 2\, g^2 m_2 - \frac{1}{4} g.$$

Consider the space of words with letters A and B, and index them in the following manner.

$$\mathcal{W} = \{\mathcal{W}_0 = \emptyset, \mathcal{W}_1 = A, \mathcal{W}_2 = B, AA, AB, BA, BB, \cdots\}$$

Positivity Constraints of Multimatrix Model

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Let

$$\mathcal{O} = \sum_{p=1}^{k} a_p \mathcal{W}_p.$$

Then

$$\frac{1}{N}\langle \mathrm{Tr}\, \mathcal{O}^t \mathcal{O} \rangle = \sum_{p,q=1}^{k} a_p a_q \mathcal{M}_{p,q} \geq 0,$$

where

$$\mathcal{M}_{p,q} = \frac{1}{N}\langle \mathrm{Tr}\, \mathcal{W}_p^* \mathcal{W}_q \rangle.$$

i.e.

$$\mathcal{M} = (\mathcal{M}_{p,q}) \geq 0.$$

Positivity Constraints of Multimatrix Model

- For instance, with the following sequence of words $\emptyset, A, B, AA, AB, BB, \cdots$ we can enforce positivity of the matrix

$$\begin{bmatrix} 1 & m_A & m_B & m_{AA} & m_{AB} & m_{BB} \\ m_A & m_{AA} & m_{AB} & m_{AAA} & m_{AAB} & m_{ABB} \\ m_B & m_{BA} & m_{BB} & m_{BAA} & m_{BAB} & m_{BBB} \\ m_{AA} & m_{AAA} & m_{AAB} & m_{AAAA} & m_{AAAB} & m_{AABB} \\ m_{BA} & m_{BAA} & m_{BAB} & m_{BAAA} & m_{BAAB} & m_{BABB} \\ m_{BB} & m_{BBA} & m_{BBB} & m_{BBAA} & m_{BBAB} & m_{BBBB} \end{bmatrix}.$$

The Solution for the Type (2,0) Model

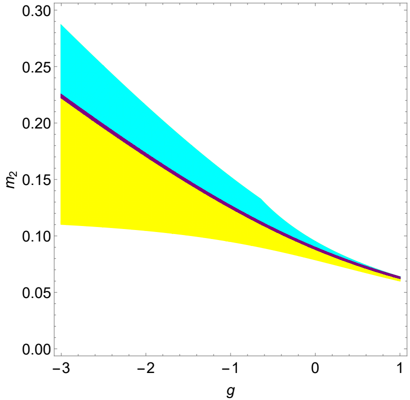

Figure: The search space region for the (2,0) quartic model where the relationship between g and m_2 is nonlinear. The different coloured regions denote different constraints applied. The more constraints the smaller the region.

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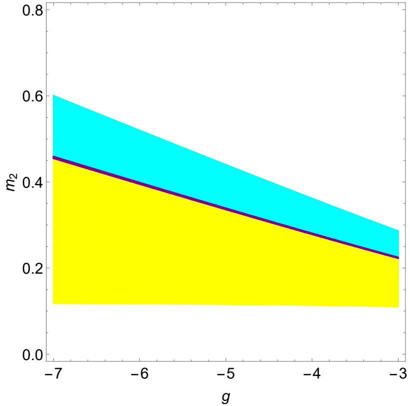

Figure: The search space region for the (2,0) quartic model where the relationship between g and m_2 becomes linear.

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subject to	$\sum_{i,j} \mathcal{W}_i A_{ij}^{(k)} \mathcal{W}_j = \sum_i B_i^{(k)} \mathcal{W}_i$	k'th loop equation,
	$\mathcal{M} \geq 0,$	positivity constraint.

Relaxing Bootstrap

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They treated $X_{ij} = \mathcal{W}_i \mathcal{W}_j$ appears in the left hand side of the loop equations as independent variables, and relaxed it to the positivity of the matrix

$$\mathcal{R} = \begin{bmatrix} 1 & \mathcal{W}_1 & \mathcal{W}_2 & \mathcal{W}_3 & \cdots \\ \mathcal{W}_1 & X_{11} & X_{12} & X_{13} & \cdots \\ \mathcal{W}_2 & X_{21} & X_{22} & X_{23} & \cdots \\ \mathcal{W}_3 & X_{31} & X_{32} & X_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \geq 0.$$

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- Find SDEs strictly in terms of Dirac moments for any random geometry.

References

1. J. W. Barrett and L. Glaser. Monte Carlo Simulations of Random Non-commutative Geometries. *J. Phys. A*, 49(24): 245001, 2016.

2. H. W. Lin. Bootstraps to strings: solving random matrix models with positivity. *J. JHEP*, 06: 090, 2020.

3. M. Khalkhali and N. Pagliaroli. Phase transition in random noncommutative geometries *J. Math. Phys.*, 54(3): 035202, 2020.

4. V. Kazakov and Z. Zheng. Analytic and Numerical Bootstrap for One-Matrix Model and "Unsolvable" Two-Matrix Model. arXiv:2108.04830, 2021.

5. C. I. Perez-Sanchez. Computing the spectral action for fuzzy geometries: from random noncommutatative geometry to bi-tracial multimatrix models. arXiv:1912.13288, 2019.

6. H. Hessam, M. Khalkhali and N. Pagliaroli. Bootstrapping Dirac Ensembles. Journal of Physics A: Mathematical and Theoretical J. Phys. A: Math. Theor. 55 015203 2022.

Questions

Thanks for your attention.