Quantum Gravity from Integration over Dirac ensembles

by John W. Barrett

Outline

I Spectral triples

- Commutative
- Non-commutative

II Quantum models

III Random Dirac ensembles

- Finite spectral triples
- dim 3 results

I: Commutative spectral triples

$(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A}$: commutative $*$-algebra
- $\mathcal{H}$: Hilbert space with action of $\mathcal{A}$ and commuting action of $\gamma, \gamma^2 = 1$
- D: $\mathcal{H} \to \mathcal{H}$
  $D\gamma = -\gamma D$ (s even)
  $+\gamma D$ (s odd)

First-order $[[D, a], b] = 0$ for all $a, b \in \mathcal{A}$

Manifolds

$$(M, g_{\mu\nu}) \leftrightarrow (\mathcal{A}, \mathcal{H}, D)$$

M oriented
spin-$\mathbb{C}$
compact

- $\mathcal{A} = C^\infty(M, \mathbb{C})$
- $(* f)(x) = \overline{f}(x)$
- $\mathcal{H} = L^2(S, dV)$, S: bundle of spinors on M
- $\gamma =$ chirality of S
- $D = e_a^\mu(x)\gamma^a \nabla_\mu$, $e^2 = g$

Regularity
$\lambda_n \sim n^{1/d}$

Connes reconstruction: given dimension *d*, conditions on $(\mathcal{A}, \mathcal{H}, D)$ such that it is a *d*-manifold.

Hochschild cycle $\sum a_0 [D, a_1][D, a_2] \cdots [D, a_d] = \gamma$

NC spectral triple

$$(\mathcal{A}, \mathcal{H}, D)$$

- $\mathcal{A}$: $*$-algebra
- $\mathcal{H}$: Hilbert space, bimodule over $\mathcal{A}$
  and commuting action of $\gamma, \gamma^2 = 1$
- D: $\mathcal{H} \to \mathcal{H}$
  $D\gamma = -\gamma D$ (s even)
  $+\gamma D$ (s odd)
  $\big[[D, a \triangleright], \triangleleft\, b\big] = 0$ for all $a, b \in \mathcal{A}$

First order
- generalised

Real structure

$$J : \mathcal{H} \to \mathcal{H}, \text{antilinear}$$

- $J^2 = \pm 1$
- $JD = \pm' DJ$
- $J\gamma = \pm'' \gamma J$

$$\text{Signs} \leftrightarrow s \in \mathbb{Z}/8$$

Commutative case
$Ja^* J^{-1} = a$
M: spin manifold

Non-commutative case
$\triangleleft\, a = J(a^* \triangleright) J^{-1}$

Non-trivial generalization

SM internal space

$(\mathcal{A}_F, \mathcal{H}_F, D_F)$ finite real spectral triple, $s = 6$

- $\mathcal{A} = M_3(\mathbb{C}_\mathbb{R}) \oplus \mathbb{H} \oplus \mathbb{C}_\mathbb{R} = \{(m, q, \lambda)\}$

2 1 1 6 3 3 = 16 x 3 = 48

- $\mathcal{H} = \mathbb{C}^{96} = \langle l_L, e_R, \nu_R, q_L, d_R, u_R,$
  $\bar{l}_L, \bar{e}_R, \bar{\nu}_R, \bar{q}_L, \bar{d}_R, \bar{u}_R \rangle$

$\frac{48}{96}$

- $Jf = \overline{f},\ J\overline{f}=f$

- $$D_F = \begin{pmatrix} 0 & M & \overbrace{G}^{=0} & 0 \\ M^* & 0 & 0 & H \\ G^* & 0 & 0 & \overline{M} \\ 0 & H^* & M^T & 0 \end{pmatrix}$$

Majorana masses

Higgs VEV

Dirac Mass matrices

basis $(f_L, f_R, \overline{f}_L, \overline{f}_R)$

particle	left action	right action	γ_F
l_L	q	λ	1
e_R	$\overline{\lambda}$	λ	-1
q_l	q	m^T	1
d_R	$\overline{\lambda}$	m^T	-1
u_R	λ	m^T	-1
$\bar{l}_L$	λ	q^T	-1
$\bar{e}_R$	λ	$\overline{\lambda}$	1
$\bar{q}_l$	m	q^T	-1
$\bar{d}_R$	m	$\overline{\lambda}$	1
$\bar{u}_R$	m	λ	1
ν_R	λ	λ	-1
$\bar{\nu}_R$	λ	λ	1

Vacuum of SM

$(\mathcal{A}_M, \mathcal{H}_M, D_M) = \text{spacetime}$

$(\mathcal{A}, \mathcal{H}, D_0) = (\mathcal{A}_M \otimes \mathcal{A}_F, \mathcal{H}_M \otimes \mathcal{H}_F, D_M \otimes 1 + \gamma_M \otimes D_F)$

D_0 is the vacuum of SM for the spacetime.
Physical fermion fields are in $\mathcal{H}_+$: $\gamma_M \otimes \gamma_F = 1$

All bosonic fields: $D = D_0 + \underbrace{\sum_i a_i [D_0, b_i]}_{\text{internal fluctuation}}$, $\quad a_i, b_i \in \mathcal{A}$

II: Quantum models

~ Euclidean

Partition function for QG+SM:

$$Z(f) = \int e^{-S(D) + i\langle J\psi, D\psi\rangle} f(D,\psi)\, dD\, d\psi$$

$$D \in \mathcal{G}, \quad \psi \in \mathcal{H}_+$$

Grassman integration

Issues:

- What is $\mathcal{G}$?
- Is D_F fixed?
- What is S?
- Are any axioms just e.o.m.?
- Functional integration?

Quantum models simplified:

- Assume fermions integrated already
- Fix $\mathcal{H}$, $\mathcal{A}$ finite dimensional and NC
- $\mathcal{G} = \text{all } D$ satisfying real spectral triple axioms
- $S(D) = \operatorname{tr} V(D)$, bounded below
- $\int$ is ordinary integration on vector space $\mathcal{G}$

$$Z(f) = \int_{\mathcal{G}} e^{-S(D)}\, f(D)\, dD$$

Fuzzy spaces

$(\gamma^i)^2 = \pm 1$

$\# +1 = p$

$\# -1 = q$

$M(n) = n \times n$ matrices

V= module for Cliff(p,q)

s=q-p (mod 8)

- $\mathcal{A} = M(n,\mathbb{C}), M(n,\mathbb{R})$ or $M(n/2,\mathbb{H})$
- $\mathcal{H} = V \otimes M(n,\mathbb{C})$
- $\langle v \otimes m, v' \otimes m' \rangle = (v,v') \operatorname{Tr} m^* m'$
- $a \triangleright$ $\rho(a)(v \otimes m) = v \otimes (am)$
- $\Gamma(v \otimes m) = \gamma v \otimes m$
- $J(v \otimes m) = Cv \otimes m^*$

Random : H, L free data

- Type (0,0)

$$D = 0$$

- Type (1,0)

$$D = \{H,\cdot\} + \gamma^1 \otimes \{H_1,\cdot\}$$

- Type (0,1)

$$D = [H,\cdot] + \gamma^1 \otimes [L_1,\cdot]$$

- Type (2,0)

$$D = \gamma^1 \otimes \{H_1,\cdot\} + \gamma^2 \otimes \{H_2,\cdot\}$$

- Type (1,1)

$$D = \gamma^1 \otimes \{H,\cdot\} + \gamma^2 \otimes [L,\cdot]$$

- Type (0,2)

$$D = \gamma^1 \otimes [L_1,\cdot] + \gamma^2 \otimes [L_2,\cdot]$$

H, L nxn matrices

Phase transition

$$S(D) = \mathrm{tr}\, V(D)$$

$$V(D) = D^4 + g_2 D^2$$

$$S = \sum_{\lambda} \lambda^4 + g_2 \lambda^2$$

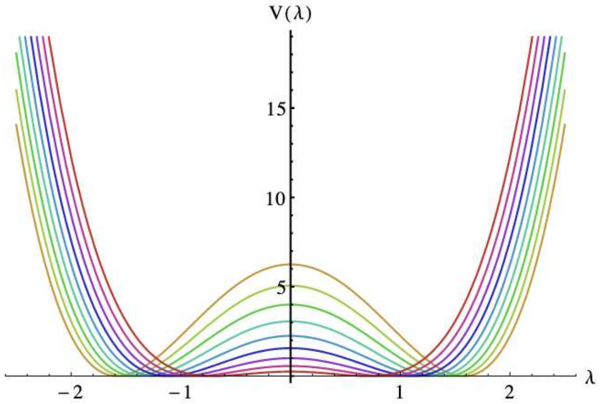
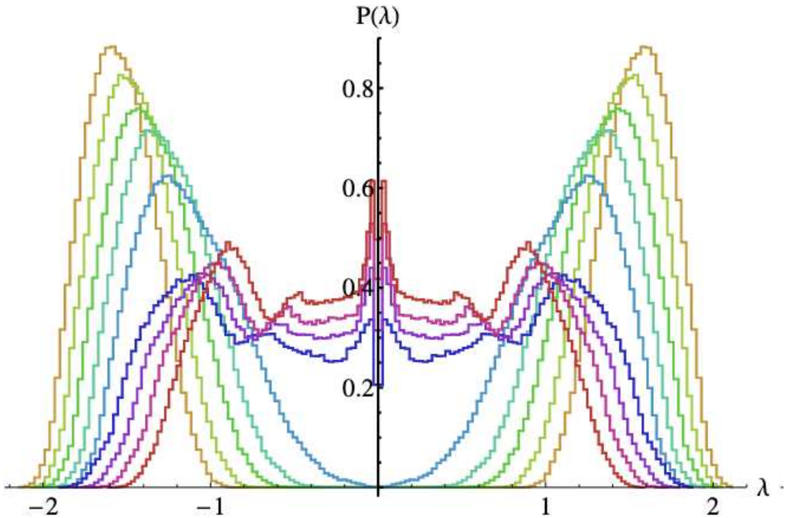


Figure 11: The potential $V = \lambda^4 + g_2\lambda^2$ for $g_2 = -1, -1.5, -2, -2.5, -3, -3.5, -4, -4.5, -5$. The lines are coloured from red ($g_2 = -1$) through to yellow ($g_2 = -5$).



(c) Type (2, 0)

Monte Carlo
Eigenvalue distribution

JWB + L. Glaser 2016

3d models

σ_i Pauli

Numerical simulation of random

Dirac operators

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Type $(p,q) = (3,0)$:

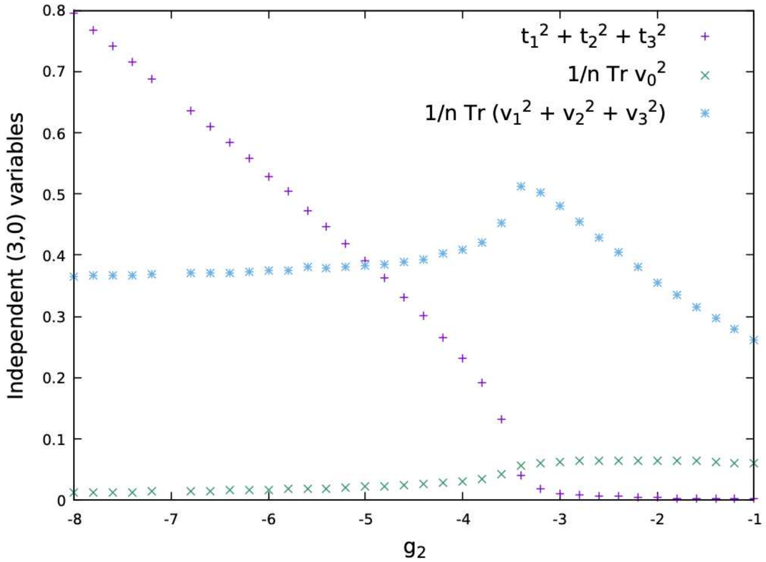
$$D = 1 \otimes [m_0,\cdot] + \sum_{1}^{3} \sigma_i \otimes \{m_i,\cdot\}$$

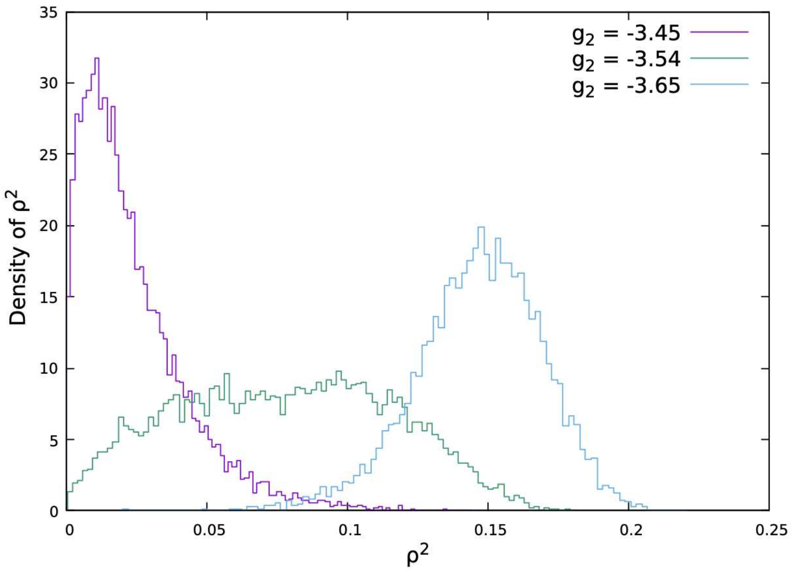
Type $(p,q) = (0,3)$:

$$D = 1 \otimes \{m_0,\cdot\} + \sum_{1}^{3} \sigma_i \otimes [m_i,\cdot]$$

Decompose $m_\mu = t_\mu 1 + v_\mu$
with $\operatorname{tr} v_\mu = 0$

Type (3,0)





$$D = 1 \otimes [m_0,\cdot] + \sum_1^3 \sigma_i \otimes \{m_i,\cdot\}$$

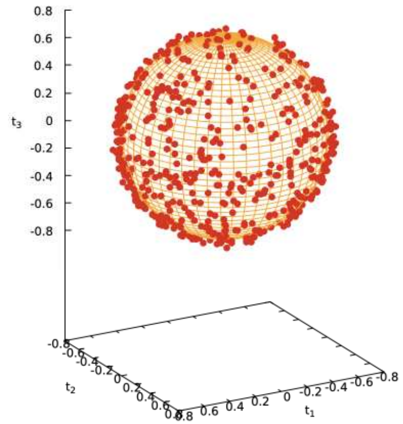
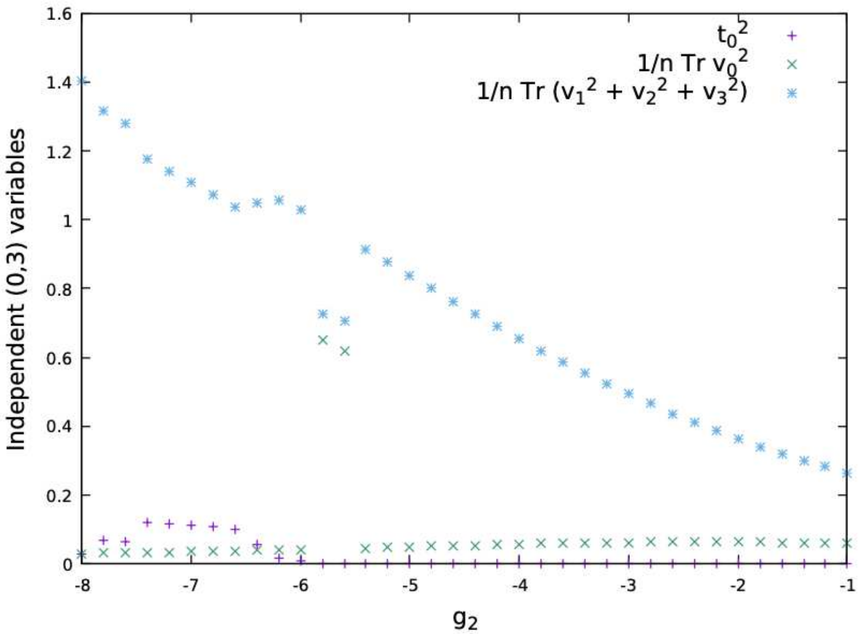


Figure 6.8: Monte Carlo history of t_1, t_2 and t_3 in region II of the (3,0) model at $g_2 = -6$, $n = 8$. The solid orange sphere is a guide for the eyes.

2nd order transition to commutative phase

Type (0,3)

$$D = 1 \otimes \{m_0,\cdot\} + \sum_1^3 \sigma_a \otimes [m_a,\cdot]$$



g_2	Chain 1	Chain 2	Chain 3	Chain 4	Fuzzy sphere	$-g_2/16$
-300	**18.6946(3)**	**18.6946(2)**	**18.6945(2)**	**18.6946(2)**	18.6951	18.75
-150	**9.3465(3)**	9.3740(3)	**9.3465(2)**	9.3739(2)	9.3476	9.375
-100	**6.2301(2)**	**6.2301(3)**	**6.2301(2)**	**6.2301(3)**	6.2317	6.25

Fuzzy sphere $v_a = Rl_a,\ a = 1{,}2{,}3$

$[l_a, l_b] = \sum_c\ i\epsilon_{abc} l_c$, irreducible

$$\frac{1}{n}\operatorname{Tr} v_c^2 = -\frac{g_2}{8}\frac{n^2-1}{2n^2-1} \approx -\frac{g_2}{16}, \quad c = 1, 2, 3$$

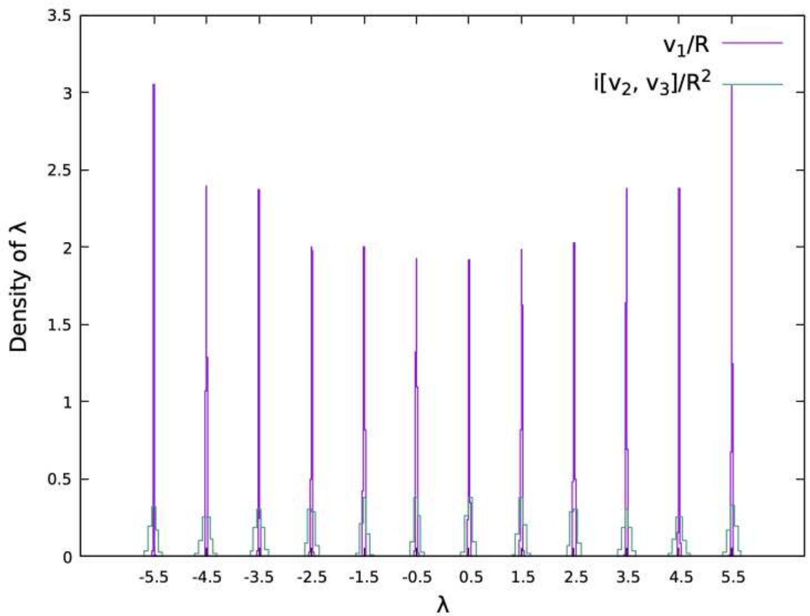


Figure 6.6: Model (0,3), eigenvalue density of v_1/R (purple) and $i[v_2, v_3]/R^2$ (green) for $n = 12$, $g_2 = -300$. The spectrum is compatible with an $su(2)$ solution.

Conclusion

- Would like to model (Euclidean) quantum spacetime with a random Dirac model.
- This supposes spacetime has some NC structure. If it does, there is a good explanation of the Planck scale.
- Understanding the vacuum in such models is crucial to explaining the physical picture.