Free Random Variables III:
Moment-Cumulant Generating Functions

Jamie Mingo (Queen's)

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first order moment-cumulant relation

- let us recall that the moment-cumulant relation
- $m_n = \sum_{\pi \in NC(n)} \kappa_\pi$ yields the relation
- $m_n = \sum_{s=1}^{n} \kappa_s \sum_{\substack{i_1,\dots,i_s \geqslant 0 \\ i_1+\cdots+i_s+s=n}} m_{i_1} \cdots m_{i_s} \qquad (m_0 = 1)$, which in turn yielded
- $M(z) = C(zM(z))$ where
- $M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n$ and $C(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n$

second order generating function relation

- $R(z) = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$
- $G(z) = \frac{1}{z} + \frac{\alpha_1}{z^2} + \frac{\alpha_2}{z^3} + \cdots$
- $z = \frac{1}{G(z)} + R(G(z))$
- $G(z, w) = \sum_{m,n \geqslant 1} \frac{m_{m,n}}{z^{m+1} w^{n+1}}$
- $R(z, w) = \sum_{m,n \geqslant 1} \kappa_{m,n} z^{m-1} w^{n-1}$
- $G(z, w) =$
  $G'(z)G'(w)R(G(z), G(w)) + \frac{\partial^2}{\partial z \partial w} \log\left(\frac{1/G(z) - 1/G(w)}{z - w}\right)$

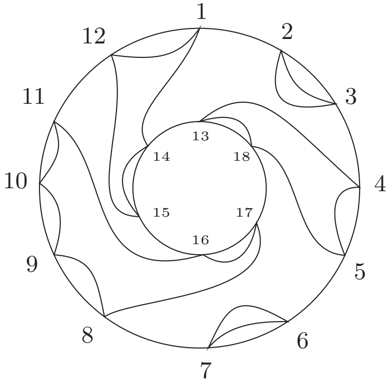
$$\frac{\partial^2}{\partial z \partial w} \log\left(\frac{1/G(z) - 1/G(w)}{z - w}\right) = \frac{G'(z)G'(w)}{(G(z) - G(w))^2} - \frac{1}{(z - w)^2}$$

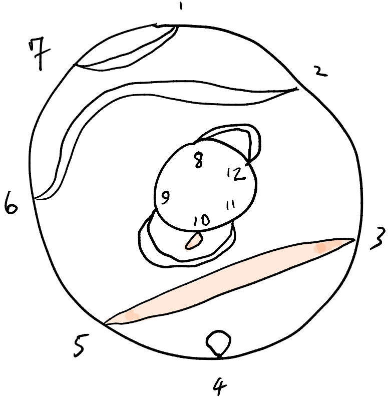
let $\zeta = G(z)$, $\omega = G(w)$ then we rewrite:

$$[(z - w)^{-2} + G(z, w)]\, dz\, dw = [(\zeta - \omega)^{-2} + R(\zeta, \omega)]\, d\zeta\, d\omega$$

second order free cumulants

- $S_{NC}(m,n) = \{\pi \in S_{m+n} \mid |\pi| + |\pi^{-1}\gamma_{m,n}| = |\gamma_{m,n}| + 2\}$ is the set of non-crossing annular permutations



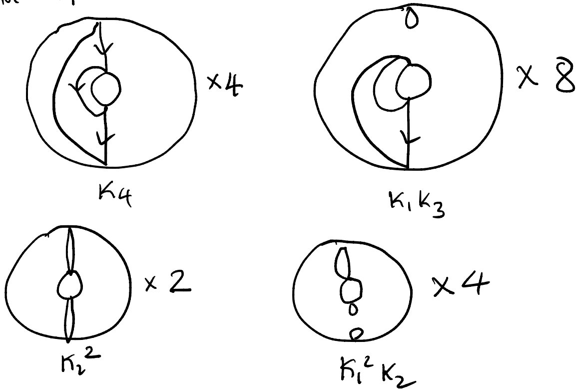


- $\mathcal{PS}_{NC}(m,n)'$ are the non-crossing partitioned permutations $(\mathcal{U},\pi)$ where $\mathcal{U} \vee \gamma = 1_{m,n}$ and $\pi = \pi_1 \times \pi_2 \in NC(m) \times NC(n)$.
- $\mathcal{PS}_{NC}(m,n) = S_{NC}(m,n) \cup \mathcal{PS}_{NC}(m,n)'$

step I-1

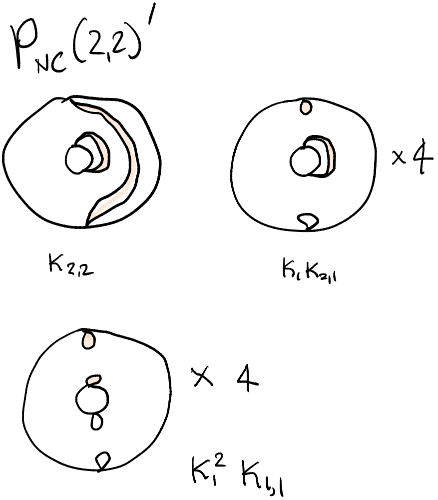
$$m_{2,2} = \kappa_{2,2} + 4\kappa_1\kappa_{2,1} + 4\kappa_1^2\kappa_{1,1} + 4\kappa_4 + 8\kappa_1\kappa_3 + 2\kappa_2^2 + 4\kappa_1^2\kappa_2$$

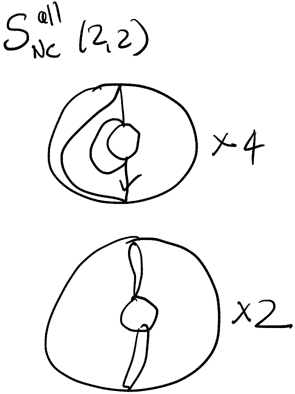
$|S_{NC}(2,2)| = 18$



$$\sum_{\pi \in S_{NC}(2,2)} \kappa_\pi = 4\kappa_4 + 8\kappa_1\kappa_3 + 2\kappa_2^2 + 4\kappa_2\kappa_1^2$$

step I-2





step I-3

Let $\tilde{\kappa}_n = \kappa_n$, $\tilde{\kappa}_{m,n} = \sum_{\pi \in S_{NC}^{all}(m,n)} \kappa_\pi$

$$\tilde{\kappa}_{2,2} = 4\kappa_4 + 2\kappa_2^2$$

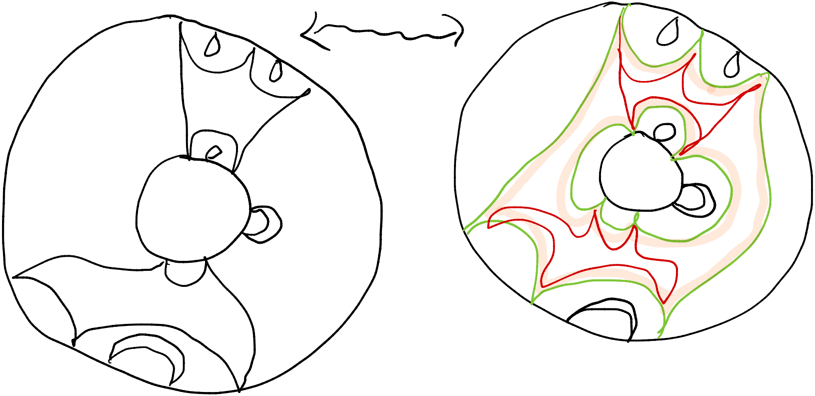
$$\tilde{\kappa}_{2,1} = 2\kappa_3$$

$$\tilde{\kappa}_{1,1} = \kappa_2$$

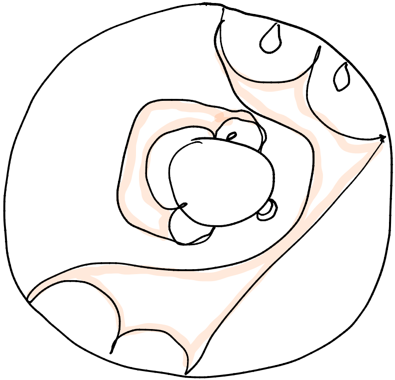
Lemma $\displaystyle\sum_{\pi \in S_{NC}(m,n)} \kappa_\pi = \sum_{(\mathcal{U},\pi) \in \mathcal{P}S_{NC}(m,n)'} \tilde{\kappa}_{(\mathcal{U},\pi)}$

step I-4

Proof



step I-5

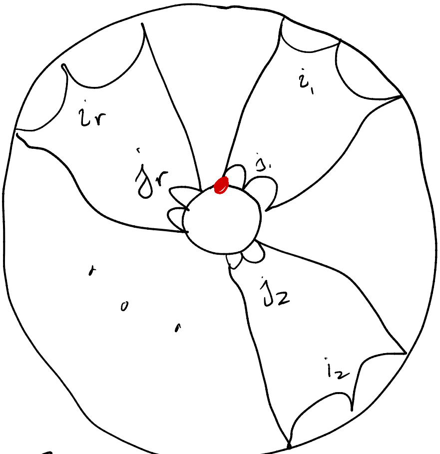


step I-6, for a general power series F

- let $F(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n$ be a formal power series
- let $\widehat{F}(z,w) = \sum_{i,j \geqslant 1} \alpha_{i+j} z^i w^j$
- then $\widehat{F}(z,w) = 1 - \frac{zF(w) - wF(z)}{z - w}$
- let $S_{NC}^{\text{all}}(m,n) = \{\pi \in S_{NC}(m,n) \mid$ all cycles of π are through cycles$\}$
- let $\tilde{\alpha}_{m,n} = \sum_{\pi \in S_{NC}^{\text{all}}(m,n)} \alpha_\pi$
- let $\widetilde{F}(z,w) = \sum_{m,n \geqslant 1} \tilde{\alpha}_{m,n} z^m w^n$
- then $\widetilde{F}(z,w) = -zw \frac{\partial^2}{\partial z \partial w} \log\left(1 - \widehat{F}(z,w)\right)$

proof!: $\tilde{\alpha}_{m,n} = \sum_{r \geqslant 1} \sum_{\substack{i_1,\ldots,i_r \geqslant 1 \\ i_1+\cdots+i_r = m}} \sum_{\substack{j_1,\ldots,j_r \geqslant 1 \\ j_1+\cdots+j_r = n}} i_1 n \alpha_{i_1+j_1} \cdots \alpha_{i_r+j_r}$

step I-7



$\#(\pi) = r$

i_1 = n° ways to place 1 in block 1

n = n° ways to position ●

contribution of k^{th} cycle is

$\kappa_{i_k + j_k}$

$$\tilde{\alpha}_{m,n} = \sum_{r \geq 1} \sum_{\substack{i_1, \dots, i_r \geq 1 \\ i_1 + i_2 + \dots + i_r = m}} \sum_{\substack{j_1, \dots, j_r \geq 1 \\ j_1 + j_2 + \dots + j_r = n}} i_1 n \, \alpha_{i_1 + j_1} \cdots \alpha_{i_r + j_r}$$

step II

- suppose we have cumulant sequences $\{\kappa_n\}_{n\geqslant 1}$ and $\{\kappa_{m,n}\}_{m,n\geqslant 1}$
- let $M(z) = 1 + \sum_{n\geqslant 1} m_n z^n$ where $m_n = \sum_{\pi\in NC(n)} \kappa_\pi$.
- let $H(z,w) = \sum_{m,n\geqslant 1} \overline{\kappa}_{m,n} z^m w^n$, where $\overline{\kappa}_{m,n} = \kappa_{m,n} + \tilde{\kappa}_{m,n}$
- then $m_{m,n} = \sum_{(\mathcal{U},\pi)\in\mathcal{PS}_{NC}(m,n)'} \overline{\kappa}_{(\mathcal{U},\pi)}$
- then $M(z,w) = H(zM(z), wM(w))\left(1 + \frac{zM'(z)}{M(z)}\right)\left(1 + \frac{wM'(w)}{M(w)}\right)$

proof!: $\overline{\kappa}_{m,n} =$

$$\sum_{k,l\geqslant 1} \sum_{\substack{i_1,\dots i_k\geqslant 0 \\ i_1+\cdots+i_k+k=m}} \sum_{\substack{j_1,\dots j_l\geqslant 0 \\ j_1+\cdots+j_l+l=n}} \overline{\kappa}_{k,l}\kappa_{i_1}\cdots\kappa_{i_k}\kappa_{j_1}\cdots\kappa_{j_l}(1+i_1+j_1+i_1j_1)$$

then use that $H(z,w) = C(z,w) + \widetilde{C}(z,w)$