# Free Random Variables II: Partitioned Permutations

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# freeness and free cumulants

- $(\mathcal{A}, \varphi)$ is a non-commutative probability space
- freeness gives us a universal rule for computing mixed moments from individual moments
- $a_1, a_2 \in \mathcal{A}$ are freely independent so by the vanishing of mixed cumulants we have for $i_1, i_2, i_3, \dots, i_n \in \{1, 2\}$ that $\kappa_n(a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n}) = 0$ unless $i_1 = i_2 = \cdots = i_n$

$$\kappa_n(a_1 + a_2, \dots, a_1 + a_n) = \sum_{i_1,\dots,i_n \in \{1,2\}} \kappa_n(a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n})$$

$$= \kappa_n(a_1, \dots, a_1) + \kappa_n(a_2, \dots, a_2)$$

- using this we can find the distribution of $a_1 + a_2$ from that of $a_1$ and $a_2$
- there is also a rule for products: suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are free, and $a_1, \dots, a_n \in \mathcal{A}_1$, $b_1, \dots, b_n \in \mathcal{A}_2$

$$\varphi(a_1 b_1 \cdots a_n b_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \dots, a_n) \kappa_{K(\pi)}(b_1, \dots, b_n)$$

- $K(\pi)$ is the *Kreweras complement* of $\pi$ (to be explained later)

# higher order non-commutative probability spaces

[Collins, Mingo, Nica, Śniady, Speicher, 2004-07]

- $\mathcal{A}$ is a unital algebra over $\mathbf{C}$, $\varphi : \mathcal{A} \to \mathbf{C}$ is a tracial linear map with $\varphi(1) = 1$, $\varphi_2 : \mathcal{A} \times \mathcal{A} \to \mathbf{C}$ is bi-linear, tracial in each variable, and $\varphi_2(1, a) = \varphi_2(a, 1) = 0$ for all $a \in \mathcal{A}$
- a random variable, $a$, is an element of $\mathcal{A}$, its *moments* are the elements of the sequence $\{\varphi(a^n)\}_{n \geqslant 1}$, its *fluctuation moments* are $m_{k,l} = \varphi_2(a^k, a^l)$
- if $X_N$ is a random matrix model with large $N$ limit we set
  - $m_l = \lim_N \mathrm{E}(\mathrm{tr}(X_N^l))$ (tr $= \dfrac{1}{N}$Tr is the normalized trace)
  - $m_{l_1,l_2} = \lim_N k_2(\mathrm{Tr}(X_N^{l_1}), \mathrm{Tr}(X_N^{l_2}))$
  - $m_{l_1,l_2,l_3} = \lim_N N\, k_3(\mathrm{Tr}(X_N^{l_1}), \mathrm{Tr}(X_N^{l_2}), \mathrm{Tr}(X_N^{l_3}))$
- $\mathcal{A} = \mathbf{C}[t]$ polynomials in $t$
  - $\varphi(p) = \lim_N \mathrm{E}(\mathrm{tr}(p(X_N)))$
  - $\varphi_2(p_1, p_2) = \lim_N k_2(\mathrm{Tr}(p_1(X_N)), \mathrm{Tr}(p_2(X_N)))$

# how to compute mixed moments

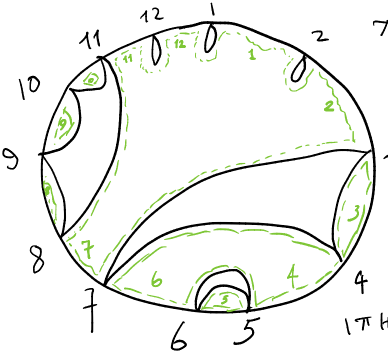
- we need a notion of freeness for higher order moments
- we assume that 'natural' random matrix models satisfy a universal rule; use this to decide what higher order freeness means
- by this we mean that independent ensembles give asymptotically free random variables
- we rewrite everything in terms of the symmetric group, $S_n$, of permutations of $[n]$

# partitions and permutations

- any permutation $\pi \in S_n$ can be written as a disjoint union of cycles; these permutations produce a partition of $[n]$ (also denoted by $\pi$)
- we let $\#(\pi)$ be the number of cycles in the cycle decomposition of $\pi$ (including cycles of length 1)
- $|\pi| = n - \#(\pi)$ is the minimal number of transpositions needed to factor $\pi$ into a product of transpositions
- $|\pi\sigma| \leqslant |\pi| + |\sigma|$ (triangle inequality)
- $\gamma_n = (1, 2, \ldots, n) \in S_n$ plays a special role, $\#(\gamma_n) = 1$, $|\gamma_n| = n - 1$

(Biane:) $|\pi| + |\pi^{-1}\gamma_n| = |\gamma_n| \Leftrightarrow$ the partition $\pi$ is non-crossing

- for $\pi$ non-crossing, $\pi^{-1}\gamma_n$ is the *Kreweras complement* of $\pi$




$\pi = (1)(2)(3,4,7)(5,6)(8,9,10,11)(12)$

$\#(\pi) = 6, \; |\pi| = 6$

$\pi^{-1}\gamma = (1,2,7,11,12)(3)(4,6)(5)(8)(9)(10)$

$\#(\pi^{-1}\gamma) = 7, \; |\pi^{-1}\gamma| = 5$

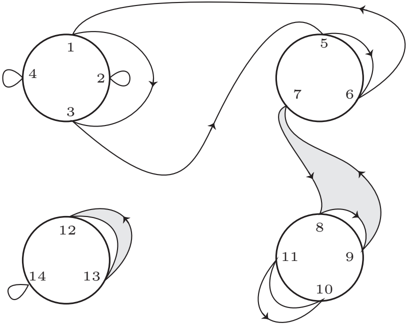
$|\pi| + |\pi^{-1}\gamma| = 11 = 12-1 = |\gamma|$

$\gamma = \gamma_{12} = (1,2,3,4,5,6,7,8,9,10,11,12)$

$\pi \vee \gamma = \gamma$

# partitioned permutations [Collins, M., Śniady, Speicher, 2007]

- if $\mathcal{U} \in \mathcal{P}(n)$ is a partition, $\pi \in S_n$ is a permutations such that every cycle of $\pi$ is contained in some block of $\mathcal{U}$, we write $\pi \leqslant \mathcal{U}$
- we let $\mathcal{P}S_n = \{(\mathcal{U}, \pi) \mid \pi \leqslant \mathcal{U}\}$
- $(\mathcal{U}, \pi)(\mathcal{V}, \sigma) = (\mathcal{U} \vee \mathcal{V}, \pi\sigma)$ defines a map $\mathcal{P}S_n \times \mathcal{P}S_n \to \mathcal{P}S_n$
- we define $|(\mathcal{U}, \pi)| = 2|\mathcal{U}| - |\pi|$, we have
  - $|(\mathcal{U} \vee \mathcal{V}, \pi\sigma)| \leqslant |(\mathcal{U}, \pi)| + |(\mathcal{V}, \sigma)|$, (triangle inequality)
- let $r \geqslant 1$, $m_1, \dots, m_r \geqslant 1$, and $m = m_1 + \cdots + m_r$
- let $\gamma_{m_1,m_2,\dots,m_r}$ be the permutation with $r$ cycles
  $$\gamma_{m_1,m_2,\dots,m_r} = (1, \dots, m_1)(m_1 + 1, \cdots, m_1 + m_2) \cdots (m_1 + \cdots + m_{r-1} + 1, \dots, m_1 + \cdots + m_r)$$
- let $\gamma = \gamma_{m_1,m_2,\dots,m_r}$, if $\pi \in S_m$ then
  $$|\pi| + |\pi^{-1}\gamma| \leqslant |(\pi \vee \gamma, \gamma)|$$
  with equality only if $\pi$ is *planar* with respect to $\gamma$
- if $\gamma$ has only 1 cycle, then $\pi \vee \gamma = \gamma$ and we get Biane's criterion




In this example

$$\gamma = (1, 2, 3, 4)\,(5, 6, 7)(8, 9, 10, 11)(12, 13, 14), \text{ and}$$

$$\pi = (1,3,5,6)\,(2)(4)(7,8,9)(10,11)(12,13)(14)$$

is shown on the left and

$$\pi^{-1}\gamma = (1,2)(3,4,6,9,11,7)(5)(8)(10)(12)(13,14).$$

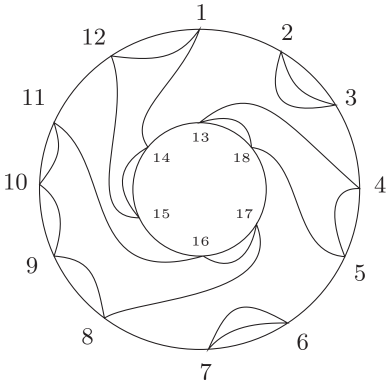
We have drawn the 4 cycles of $\gamma$ as circles and the 7 cycles of $\pi$ in a non-crossing way on these 4 circles. Note that $|\pi| = 7$, $|\pi^{-1}\gamma| = 7$ , $|\gamma| = 10$, and $|\pi \vee \gamma| = 12$. Thus $|\pi| + |\pi^{-1}\gamma| = 2|\pi \vee \gamma| - |\gamma| = |(\pi \vee \gamma, \gamma)|$.

# non-crossing annular permutations

- let $\gamma = \gamma_{m_1, m_2, \ldots, m_r}$, if $\pi \in S_m$ then

$$|\pi| + |\pi^{-1}\gamma| \leqslant |(\pi \vee \gamma, \gamma)|$$

  with equality only if $\pi$ is *planar* with respect to $\gamma$
- suppose $\gamma = \gamma_{m_1, m_2}$ and $\pi \vee \gamma = 1_{m_1 + m_2}$, then



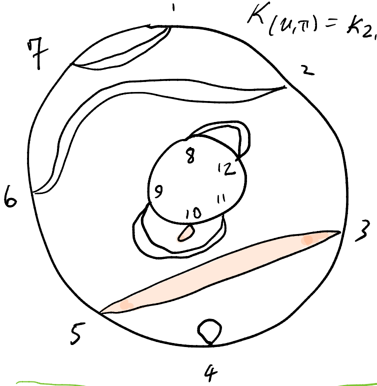

we get a non-crossing annular permutation $\pi = (1, 14, 15, 12)(2, 3)(4, 5, 18, 13)(6, 7)(8, 9, 10, 11, 16, 17)$

- we let $S_{NC}(m, n)$ be the non-crossing permutations on a $(m, n)$-annulus

# second order cumulants: $\kappa_{(\mathcal{U},\pi)}$

- let $\pi_1 \in NC(m)$ and $\pi_2 \in NC(n)$
- let $\pi = \pi_1 \times \pi_2 \in NC(m,n)$
- let $\gamma = \gamma_{m,n}$
- choose a cycle $c_1 \in \pi_1$ and a cycle $c_2 \in \pi_2$
- let $\mathcal{U}$ be the partition of $[m+n]$ where one block is $c_1 \cup c_2$ and all other blocks are either a cycle of $\pi_1$ or a cycle of $\pi_2$
- then $\#(\mathcal{U}) = \#(\pi_1) + \#(\pi_2) - 1$
- $|\mathcal{U}| = |\pi_1| + |\pi_2| + 1$, $|(\mathcal{U},\pi)| = |\pi| + 2$
- $\mathcal{U} \vee \pi^{-1}\gamma = \mathcal{U} \vee \pi \vee \pi^{-1}\gamma = \mathcal{U} \vee \gamma = 1_{m+n}$
- $(\mathcal{U},\pi)(0_{\pi^{-1}\gamma}, \pi^{-1}\gamma) = (1_{m+n}, \gamma)$
- $|(0_{\pi^{-1}\gamma}, \pi^{-1}\gamma)| = 2|\pi^{-1}\gamma| - |\pi^{-1}\gamma| = |\pi^{-1}\gamma|$
- $|(\mathcal{U},\pi)| + |(0_{\pi^{-1}\gamma}, \pi^{-1}\gamma)| = |\pi| + |\pi^{-1}\gamma| + 2 = |\gamma| + 2 = |(1_{m+n}, \gamma)|$
- for such a $(\mathcal{U},\pi)$ we set $\kappa_{(\mathcal{U},\pi)} = \kappa_{|c_1|,|c_2|} \prod_{\substack{c\in\pi \\ c\neq c_1,c_2}} \kappa_{|c|}$

$\kappa_{(\mathcal{U},\pi)} = \kappa_{2,1}\,\kappa_2^4\,\kappa_1$




$\gamma = (1,2,3,4,5,6,7)(8,9,10,11,12)$

$|\gamma| = 10$

$\pi = (1,7)(2,6)(3,5)(4)(8,12)(9,11)(10)$

does not connect the two circles

$\mathcal{U} = \{(1,7)(2,6)(3,5,10)(4)(9,11)(8,12)\}$

$|\pi| = 5, \quad |\pi^{-1}\gamma| = 5$

$|\pi| + |\pi^{-1}\gamma| = |\gamma|$

$|\mathcal{U}| = |\pi| + 1 = 6$

$|(\mathcal{U},\pi)| = 12 - 5 = 7 = |\pi| + 2$

$$\kappa_{(\mathcal{U},\pi)} = \kappa_{2,1}\kappa_2^4\kappa_1$$

# second order moment-cumulant formula

- $S_{NC}(m,n)$ are the non-crossing annular permutations that connect the two circles
- $\mathcal{P}S_{NC}(m,n)'$ are the non-crossing partitioned permutations $(\mathcal{U},\pi)$ where $\mathcal{U} \vee \gamma = 1_{m,n}$ and $\pi = \pi_1 \times \pi_2 \in NC(m) \times NC(n)$.
- $\mathcal{P}S_{NC}(m,n) = S_{NC}(m,n) \cup \mathcal{P}S_{NC}(m,n)'$
- if $(\mathcal{A}, \pi, \varphi_2)$ is a second order non-commutative probability space and $a \in \mathcal{A}$ we define the cumulants of $a$ by
    - $\varphi(a^n) = \sum_{\pi \in NC(n)} \kappa_\pi$
    - $\varphi_2(a^m, a^n) = \sum_{(\mathcal{U},\pi) \in \mathcal{P}S_{NC}(m,n)} \kappa_{(\mathcal{U},\pi)}$

# moment-cumulant examples

$\alpha_n = \varphi(a^n)$ and $\alpha_{m,n} = \varphi_2(a^m, a^n)$

$$\alpha_{1,1} = \kappa_{1,1} + \kappa_2$$

$$\alpha_{2,1} = \kappa_{1,2} + 2\kappa_1\kappa_{1,1} + 2\kappa_3 + 2\kappa_1\kappa_2$$

$$\alpha_{2,2} = \kappa_{2,2} + 4\kappa_1\kappa_{2,1} + 4\kappa_1^2\kappa_{1,1} + 4\kappa_4 + 8\kappa_1\kappa_3 + 2\kappa_2^2 + 4\kappa_1^2\kappa_2$$

$$\alpha_{1,3} = \kappa_{1,3} + 3\kappa_1\kappa_{2,1} + 3\kappa_2\kappa_{1,1} + 3\kappa_4 + 6\kappa_1\kappa_3 + 3\kappa_2^2 + 3\kappa_1^2\kappa_2$$

$$\alpha_{2,3} = \kappa_{2,3} + 2\kappa_1\kappa_{1,3} + 3\kappa_1\kappa_{2,2} + 3\kappa_2\kappa_{1,2} + 9\kappa_1^2\kappa_{1,2} + 6\kappa_1\kappa_2\kappa_{1,1} + 6\kappa_1^3\kappa_{1,1} + 6\kappa_5 + 18\kappa_1\kappa_4 + 12\kappa_2\kappa_3 + 18\kappa_1^2\kappa_3 + 12\kappa_1\kappa_2^2 + 6\kappa_1^3\kappa_2$$

$$\alpha_{3,3} = \kappa_{3,3} + 6\kappa_1\kappa_{2,3} + 6\kappa_2\kappa_{1,3} + 6\kappa_1^2\kappa_{1,3} + 9\kappa_1^2\kappa_{2,2} + 18\kappa_1\kappa_2\kappa_{1,2} + 18\kappa_1^3\kappa_{1,2} + 9\kappa_2^2\kappa_{1,1} + 18\kappa_1^2\kappa_2\kappa_{1,1} + 9\kappa_1^4\kappa_{1,1} + 9\kappa_6 + 36\kappa_1\kappa_5 + 27\kappa_2\kappa_4 + 54\kappa_1^2\kappa_4 + 9\kappa_3^2 + 72\kappa_1\kappa_2\kappa_3 + 36\kappa_1^3\kappa_3 + 12\kappa_2^3 + 36\kappa_1^2\kappa_2^2 + 9\kappa_1^4\kappa_2$$

# cumulant-moment formula

$$\kappa_{1,1} = \alpha_1^2 - \alpha_2 + \alpha_{1,1}$$

$$\kappa_{1,2} = -4\alpha_1^3 + 6\alpha_1\alpha_2 - 2\alpha_3 - 2\alpha_1\alpha_{1,1} + \alpha_{1,2}$$

$$\kappa_{2,2} = 18\alpha_1^4 - 36\alpha_1^2\alpha_2 + 6\alpha_2^2 + 16\alpha_1\alpha_3 - 4\alpha_4 + 4\alpha_1^2\alpha_{1,1} - 4\alpha_1\alpha_{1,2} + \alpha_{2,2}$$

$$\kappa_{1,3} = 15\alpha_1^4 - 30\alpha_1^2\alpha_2 + 6\alpha_2^2 + 12\alpha_1\alpha_3 - 3\alpha_4 + 6\alpha_1^2\alpha_{1,1} - 3\alpha_2\alpha_{1,1} - 3\alpha_1\alpha_{1,2} + \alpha$$

$$\kappa_{2,3} = -72\alpha_1^5 + 180\,\alpha_1^3\,\alpha_2 - 72\alpha_1\alpha_2^2 - 84\,\alpha_1^2\alpha_3 + 24\alpha_2\alpha_3 + 30\alpha_1\alpha_4 - 6\alpha_5$$

$$-12\alpha_1^3\alpha_{1,1} + 6\alpha_1\alpha_2\alpha_{1,1} + 12\alpha_1^2\alpha_{1,2} - 3\alpha_2\alpha_{1,2} - 2\alpha_1\alpha_{1,3} - 3\alpha_1\alpha_{2,2} + \alpha_{2,3}$$

$$\kappa_{3,3} = 300\alpha_1^6 - 900\alpha_1^4\alpha_2 + 576\alpha_1^2\alpha_2^2 - 48\alpha_2^3 + 432\alpha_1^3\alpha_3 - 288\alpha_1\alpha_2\alpha_3 + 18\alpha_3^2$$

$$-180\alpha_1^2\alpha_4 + 45\alpha_2\alpha_4 + 54\alpha_1\alpha_5 - 9\alpha_6 + 36\alpha_1^4\alpha_{1,1} - 36\alpha_1^2\alpha_2\alpha_{1,1} + 9\alpha_2^2\alpha_{1,1}$$

$$-36\alpha_1^3\alpha_{1,2} + 18\alpha_1\alpha_2\alpha_{1,2} + 12\alpha_1^2\alpha_{1,3} - 6\alpha_2\alpha_{1,3} + 9\alpha_1^2\alpha_{2,2} - 6\alpha_1\alpha_{2,3} + \alpha_{3,3}$$

# generating function relation

- $R(z) = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$
- $G(z) = \dfrac{1}{z} + \dfrac{\alpha_1}{z^2} + \dfrac{\alpha_2}{z^3} + \cdots$
- $z = \dfrac{1}{G(z)} + R(G(z))$
- $G(z,w) = \displaystyle\sum_{m,n \geqslant 1} \frac{\alpha_{m,n}}{z^{m+1} w^{n+1}}$
- $R(z,w) = \displaystyle\sum_{m,n \geqslant 1} \kappa_{m,n} z^{m-1} w^{n-1}$
- $G(z,w) =$
  $$G'(z)G'(w)R(G(z),G(w)) + \frac{\partial^2}{\partial z \partial w} \log\left(\frac{1/G(z) - 1/G(w)}{z-w}\right)$$
  $$\frac{\partial^2}{\partial z \partial w} \log\left(\frac{1/G(z) - 1/G(w)}{z-w}\right) = \frac{G'(z)G'(w)}{(G(z)-G(w))^2} - \frac{1}{(z-w)^2}$$