Regularisation

The Dyson-Schwinger equation needs a reinterpretation in view of quantum field theory.

- Recall $e_k = \frac{k}{\mathcal{N}^{2/D}} + \frac{\tilde{M}^2}{2}$ and $r_k = 1$ for $D = 2$ and $r_k = k + 1$ for $D = 4$.
- We restrict the sums to $\sum_{k=0}^{\Lambda^2 \mathcal{N}^{2/D}}$. The limit $\Lambda^2 \to \infty$ can only exist if $\tilde{M}(\Lambda)$ is a carefully adapted function of Λ and if $G \mapsto Z(\Lambda)G$ is carefully rescaled.
- We better write $e_k = \tilde{e}_k + \frac{\tilde{M}^2}{2}$, $\zeta = \tilde{\zeta} + \frac{\tilde{M}^2}{2}$, $\eta = \tilde{\eta} + \frac{\tilde{M}^2}{2}$ and $G(\zeta, \eta) = \tilde{G}(\tilde{\zeta}, \tilde{\eta})$

The result is for $D = 4$ (omitting the tilde)

$$\Big(\zeta + \eta + M^2(\Lambda) + \frac{\lambda}{\sqrt{\mathcal{N}}} \sum_{k=0}^{\Lambda^2\sqrt{\mathcal{N}}} \frac{k+1}{\sqrt{\mathcal{N}}} Z(\Lambda) G^{(0)}(\zeta, \tfrac{k}{\sqrt{\mathcal{N}}})\Big) Z(\Lambda) G^{(0)}(\zeta, \eta)$$

$$= 1 + \frac{\lambda}{\sqrt{\mathcal{N}}} \sum_{k=0}^{\Lambda^2\sqrt{\mathcal{N}}} \frac{k+1}{\sqrt{\mathcal{N}}} \frac{Z(\Lambda)\big(G^{(0)}(\frac{k}{\sqrt{\mathcal{N}}}, \eta) - G^{(0)}(\zeta, \eta)\big)}{\frac{k}{\sqrt{\mathcal{N}}} - \zeta}$$

For $\sqrt{\mathcal{N}} = \frac{\theta}{4}$ large enough, this is arbitrarily close to integral equation

$$\Big(\zeta + \eta + M^2 + \lambda \int_0^{\Lambda^2} dt\, t Z G^{(0)}(\zeta, t)\Big) Z G^{(0)}(\zeta, \eta) = 1 + \lambda \int_0^{\Lambda^2} dt\, t \frac{Z\big(G^{(0)}(t, \eta) - G^{(0)}(\zeta, \eta)\big)}{t - \zeta}$$

Integral equation

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We can arrange the two-point function of a large family of matrix and QFT models with quartic interaction into the integral equation

$$\Big(\zeta+\eta+M^2+\lambda\int_0^\infty dt\,\varrho_0(t)ZG^{(0)}(\zeta,t)\Big)ZG^{(0)}(\zeta,\eta)=1+\lambda\int_0^\infty dt\,\varrho_0(t)\frac{Z(G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta))}{t-\zeta}$$

- large-θ 4D Moyal: $\varrho_0(t)=t\chi_{[0,\Lambda^2]}$ and $M=M(\Lambda)$, $Z=Z(\Lambda)$
- large-θ 2D Moyal: $\varrho_0(t)=\chi_{[0,\Lambda^2]}$ and $M=M(\Lambda)$, $Z=1$
- $N\times N$ matrix model $\varrho(t)=\frac{1}{N}\sum_{k=1}^d r_k\delta(t-e_k)$, $r_1+...+r_d=N$, $M=0$, $Z=1$

The spectral measure encodes a spectral dimension $\delta:=\inf(p\ :\ \int_0^\infty\frac{\varrho_0(t)}{(1+t)^{p/2}}<\infty)$

- In [Panzer, W 18] we solved the case $\varrho_0(t)=1$. Key step was to extrapolate a computer algebra evaluation of iterated integrals.
- In [Grosse, Hock, W 19a] we succeeded in solving the integral equation for any Hölder-continuous measure ϱ_0 of spectral dimension $\delta<6$.

Solution

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19a]

1. Ansatz $G^{(0)}(\zeta,\eta) = \dfrac{e^{\mathcal{H}_\zeta[\tau_\eta(\bullet)]}\sin\tau_\eta(\zeta)}{Z\lambda\pi\varrho_0(\zeta)}$, $\quad\mathcal{H}_\zeta[f] := \frac{1}{\pi}\fint_0^{\Lambda^2}\frac{dp\, f(p)}{p-\zeta}$ finite Hilbert transf.
2. $\tau_\eta(\zeta) = \operatorname{Im}\log\big(\eta + I(\zeta+\mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R_D(-m^2 - R_D^{-1}(\zeta))$
3. $\boxed{R_D(z) = z - \lambda(-z)^{D/2}\displaystyle\int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(m^2+t)^{D/2}(t+m^2+z)}}$ $\qquad D = 2[\frac{\delta}{2}]$
4. ϱ_λ is implicit solution of $\varrho_0(R_D(\zeta)) = \varrho_\lambda(\zeta)$.

- Proof: [Cauchy 1831] residue theorem,
  [Lagrange 1770] inversion theorem, [Bürmann 1799] formula
- $\varrho_0(t) \equiv 1$ (2D Moyal, $m = 1$) in terms of Lambert-W satisfying $W(z)e^{W(z)} = z$:
  $I(\zeta) := \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - \lambda\log\big(1 - \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big)\big)$

$D=4$ Moyal space: $\varrho_0(t)=t$ [Grosse-Hock-W 19b]

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- $\varrho_\lambda(x) \equiv \varrho_0(R_4(x)) = R_4(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(m^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R_4(x)$ bounded above. Consequently, R_4^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$R_4(x) \equiv \varrho_\lambda(x) = x \cdot {}_2F_1\Big(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\Big| -\frac{x}{m^2}\Big) \qquad \alpha_\lambda = \begin{cases} \dfrac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \dfrac{1}{2} + \mathrm{i}\dfrac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem (in the planar sector).

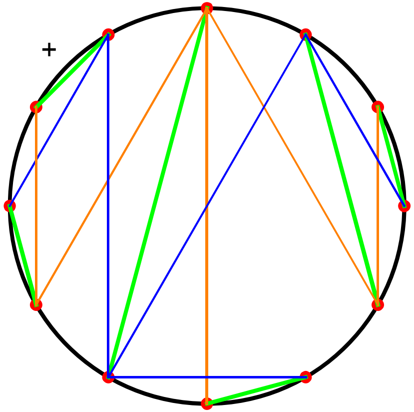
Gives non-perturbative integral representation for $G^{(0)}(\xi,\eta)$.

All planar correlation functions

- A Catalan tuple is a tuple $\tilde{p} = (p_0, p_1, \dots, p_k)$ with $p_i \geq 0$, $\sum_{j=0}^{l} p_j > l$ if $l < k$ and $\sum_{j=0}^{k} p_j = k$. We let $|\tilde{p}| = k$.
- We call a collection $\mathcal{T} = \langle \tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_{n+1} \rangle$ of Catalan tuples a nested Catalan table (of length n) if its length tuple $(|\tilde{p}_0| + 1, |\tilde{p}_1|, \dots, |\tilde{p}_{n+1}|)$ is itself a Catalan tuple.
- There are $\frac{1}{n+1}\binom{3n+1}{n}$ nested Catalan tables of length n.

Theorem [de Jong, Hock W 19]

The planar n-point function is a sum of terms of the form $\frac{\pm G^{(0)}_{b_p b_q} \cdots G^{(0)}_{b_r b_s}}{(E_{b_t} - E_{b_u}) \cdots (E_{b_v} - E_{b_w})}$ which are in bijection with nested Catalan tables of length $\frac{n-2}{2}$.



- A green chord between a, b encodes a factor $G_{|ab|}$.
- An edge of blue or orange rooted plane tree from k to l encodes a factor $\frac{1}{E_k - E_l}$.
- There is a natural bijection between blue and orange rooted plane trees.

The Kontsevich model

Consider the Gaußian measure $d\mu_0$ on $\mathcal{V}'$, for $\mathcal{V} = H_N$ Hermitean $N \times N$-matrices, induced by $\langle f, g\rangle = \frac{1}{N}\sum_{k,l=1}^{N} \frac{f_{kl} g_{lk}}{E_k+E_l}$.

- Deforming it to $d\mu_K(\Phi) := \frac{1}{\mathcal{Z}} e^{\frac{\mathrm{i}}{6} N\,\mathrm{Tr}(\Phi^3)} d\mu_0(\Phi)$ gives the [Kontsevich 92] model which generates intersection numbers of ψ- and κ-classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves. Proves a conjecture by [Witten 90].
- This is a key example for topological recursion [Eynard, Orantin 07]. Its spectral curve is $x(z) = z^2$, $y(z) = -z + \frac{1}{N}\sum_{k=1}^{N} \frac{1}{\hat{E}_k(\hat{E}_k - z)}$ and $\omega_{0,2}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$ where $\hat{E}_k = \sqrt{E_k^2 + c}$ and $c = \frac{2}{N}\sum_{k=1}^{N} \frac{1}{\sqrt{E_k^2 + c}}$.
- One can turn the Kontsevich model into a QFT on Moyal space (or other NCG) [Grosse, Steinacker 05/06], [Grosse, Sato, W 16], [Grosse, Hock, W 19c]. But $d\mu_K$ is not a valid measure.

The quartic Kontsevich model

We will now deform the same $d\mu_0$ of the Kontsevich model by a quartic potential $d\mu_\lambda(\Phi) := \frac{1}{\mathcal{Z}} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_0(\Phi)$.

- We called this the quartic Kontsevich model. There are another variations of the Kontsevich model, e.g. the family of generalised Kontsevich models which relate to r-spin intersection numbers [Belliard, Charbonnier, Eynard, Garcia-Failde 21].
- One can also take for $\mathcal{V}$ all complex $N \times N$-matrices [Langmann, Szabo, Zarembo 03]. Very recently, [Hock, Branahl 22] proved that the complex quartic model obeys topological recursion, whereas the real model (discussed below) needs blobbed TR. They clearly located the differences responsible for the blobs.

Recall that the planar 2-point function of the real quartically deformed model satisfies

$$\Big(\eta+\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-\zeta}\Big)G^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{G^{(0)}(e_k,\eta)}{e_k-\zeta}$$

'Elementary' solution

Suppose there is a rational function R of degree $d+1$, with simple pole at ∞ of residue -1 and

$$R(z)+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(R(z),e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-R(z)}=-R(-z)$$

Setting $\zeta=R(z)$, $\eta=R(w)$ and $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}(z,w)$ and chosing $\varepsilon_k\in R^{-1}(e_k)$, the non-linear equation becomes

$$(R(w)-R(-z))\mathcal{G}^{(0)}(z,w)=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}$$

Theorem [Schürmann, W 19]

1. $R(z)=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$ where $R(\varepsilon_k)=E_k$ and $R'(\varepsilon_k)\varrho_k=r_k$.
2. $\mathcal{G}^{(0)}(z,w)=\dfrac{P(R(z),R(w))}{(R(z)-R(-w))(R(w)-R(-z))}$ where

$$P(R(z),R(w))=\frac{\prod_{u\in R^{-1}(\{w\})}(R(z)-R(-u))}{\prod_{k=1}^{d}(R(z)-R(\varepsilon_k))}\equiv P(R(w),R(z))$$

Auxiliary functions [Branahl, Hock, W 20]

Recall the complexified Dyson-Schwinger equation

$$(\zeta+\eta)G(\zeta,\eta) = 1 + \frac{\lambda}{N}\sum_{k=1}^{N}\frac{G(E_k,\eta)-G(\zeta,\eta)}{E_k-\zeta} + \frac{\lambda}{\mathcal{N}^2}\frac{G(\eta|\eta)-G(\zeta|\eta)}{\eta-\zeta}$$
$$- G(\zeta,\eta)\Big(\frac{\lambda}{N}\sum_{k=1}^{N}G(\zeta,E_k) + \frac{\lambda}{N^2}G(\zeta|\zeta)\Big) - \frac{\lambda}{N^2}T(\zeta\|\zeta,\eta|)$$

where $T(E_a\|\zeta,\eta|) := -N\frac{\partial}{\partial E_a}G(\zeta,\eta)$. To get an equation for $T(\xi\|\zeta,\eta|)$ we differentiate again, and so on. Let $I=(\zeta_1,..,\zeta_n)$ and $T(\emptyset\|\xi,\eta|)=G(\xi,\eta)$, $T(\emptyset\|\xi|\eta|)=G(\xi|\eta)$.

Definition

- $T(\zeta,I\|\xi,\eta|)$ is the complexification of $T(E_a,I\|\xi,\eta):=-N\frac{\partial}{\partial E_a}T(I\|\xi,\eta)$
- $T(\zeta,I\|\xi|\eta|)$ is the complexification of $T(E_a,I\|\xi|\eta|):=-N\frac{\partial}{\partial E_a}T(I\|\xi|\eta)$
- $\tilde{\Omega}_1(\zeta) := \frac{\lambda}{N}\sum_{k=1}^{N}G(\zeta,E_k)+\frac{\lambda}{N^2}G(\zeta|\zeta)$
- $\tilde{\Omega}_{n+1}(\zeta,I)$ is the complexification of $\tilde{\Omega}_{n+1}(E_a,I):=-N\frac{\partial}{\partial E_a}\tilde{\Omega}_n(I)+\frac{\delta_{n,1}}{(E_a-\zeta_1)^2}$

System of Dyson-Schwinger equations I

Apply the change of variables via R encoded in the 2-point function:

$$\tilde{\Omega}_n(R(z_1),..,R(z_n)) =: \Omega_n(z_1,...,z_n),$$
$$T(R(z_1),..,R(z_n)\|R(w_1),R(w_2)|) =: \mathcal{T}(z_1,...,z_n\|w_1,w_2|),$$
$$T(R(z_1),..,R(z_n)\|R(w_1)|R(w_2)|) =: \mathcal{T}(z_1,...,z_n\|w_1|w_2|)$$

and the formal genus expansion $\Omega_n(I) = \sum_{g=0}^{\infty} N^{-2g}\Omega_n^{(g)}(I)$ etc.

Equation (I)

$$(R(w)-R(-z))\mathcal{T}^{(g)}(I\|z,w|) - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k\mathcal{T}^{(g)}(I\|\varepsilon_k,w|)}{R(\varepsilon_k)-R(z)}$$
$$= \delta_{0,m}\delta_{g,0} - \lambda\Big\{\sum_{\substack{I_1\uplus I_2=I,\ g_1+g_2=g\\(g_1,I_1)\neq(0,\emptyset)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,z)\mathcal{T}^{(g_2)}(I_2\|z,w|) + \mathcal{T}^{(g-1)}(I,z\|z,w|)$$
$$+\sum_{i=1}^{m}\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(I\setminus u_i\|u_i,w|)}{R(u_i)-R(z)} + \frac{\mathcal{T}^{(g-1)}(I\|z|w|)-\mathcal{T}^{(g-1)}(I\|w|w|)}{R(w)-R(z)}\Big\}$$

System of Dyson-Schwinger equations II

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Equation (II)

$$
\begin{aligned}
&(R(z)-R(-z))\mathcal{T}^{(g)}(I\|z|w|) - \frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{\mathcal{T}^{(g)}(I\|\varepsilon_k|w|)}{R(\varepsilon_k)-R(z)} \\
&= -\lambda\Big\{ \sum_{\substack{I_1\uplus I_2=I,\ g_1+g_2=g \\ (I_1,g_1)\neq(\emptyset,0)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,z)\mathcal{T}^{(g_2)}(I_2\|z|w|) + \mathcal{T}^{(g-1)}(I,z\|z|w|) \\
&+ \sum_{i=1}^{m} \frac{\partial}{\partial R(u_i)} \frac{\mathcal{T}^{(g)}(I\setminus u_i\|u_i|w|)}{R(u_i)-R(z)} + \frac{\mathcal{T}^{(g)}(I\|z,w|)-\mathcal{T}^{(g)}(I\|w,w|)}{R(w)-R(z)} \Big\}
\end{aligned}
$$

System of Dyson-Schwinger equations III

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Equation (III)

$$R'(z)\mathfrak{G}_0(z)\Omega^{(g)}_{|I|+1}(I,z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d} r_n r_k \frac{\mathcal{T}^{(g)}(I\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$

$$= \frac{\delta_{g,0}\delta_{|I|,1}}{(R(z)-R(u_1))^2} - \sum_{\substack{I_1\uplus I_2=I,\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)\neq(I_2,g_2)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,z)\frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g_2)}(I_2\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}$$

$$-\sum_{j=1}^{m}\frac{\partial}{\partial R(u_j)}\frac{\frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g)}(I\backslash u_j\|u_j,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}}{R(u_j)-R(z)} - \frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g-1)}(I,z\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)} + \mathcal{T}^{(g-1)}(I\|z|z|)$$

$$-\frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g-1)}(I\|z|\varepsilon_n|)-\mathcal{T}^{(g-1)}(I\|\varepsilon_n|\varepsilon_n|)}{(R(\varepsilon_n)-R(z))(R(\varepsilon_n)-R(-z))} - \sum_{j=1}^{m}\frac{\partial}{\partial R(u_j)}\mathcal{T}^{(g)}(I\setminus u_j\|u_j,z|)\ ,$$

where $\mathfrak{G}_0(z) := \mathrm{Res}_{v\to -z}\,\mathcal{G}^{(0)}(z,v)dv$.

Dyson-Schwinger equation for $\Omega_2^{(0)}(u,z)$

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$$\Omega_2^{(0)}(u,z)R'(z)\mathfrak{G}_0(z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d} \frac{r_k r_n \mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$
$$= -\frac{\partial}{\partial R(u)}\big(\mathcal{G}^{(0)}(u,z)+\mathcal{G}^{(0)}(u,-z)\big)$$

- Seems to need $\mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)$ which itself needs $\Omega_2^{(0)}$.
- But poles separate by partial fraction decomposition

$$\mathcal{G}^{(0)}(z,u) = \frac{\mathfrak{G}_0(z)}{u+z} + \frac{\lambda^2}{N^2}\sum_{k,l,m,n=1}^{d} \frac{C_{k,l}^{m,n}}{(z+\widehat{\varepsilon}_l{}^n)(z-\widehat{\varepsilon}_k{}^m)(u-\widehat{\varepsilon}_l{}^n)}$$

Proposition

$$\Omega_2^{(0)}(u,z) = \frac{1}{R'(u)R'(z)}\Big(\frac{1}{(u-z)^2}+\frac{1}{(u+z)^2}\Big)$$

One recognises the Bergman kernel of topological recursion!

Contact with topological recursion

Set $\omega_{g,m}(z_1,\dots,z_m)=\lambda^{2-2g-m}\Omega^{(g)}_m(z_1,\dots,z_m)\prod_{k=1}^m dR(z_i)$. A lengthy calculation gives:

$$\omega_{0,3}(u_1,u_2,z)=-\sum_{i=1}^{2d}\frac{\left(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\right)\left(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\right)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$
$$+\left[d_{u_1}\left(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\right)+u_1\leftrightarrow u_2\right]$$

$$\omega_{1,1}(z)=\sum_{i=1}^{2d}\frac{dz}{R'(-\beta_i)R''(\beta_i)}\left\{-\frac{1}{8(z-\beta_i)^4}+\frac{\frac{1}{24}x_{1,i}}{(z-\beta_i)^3}+\frac{(x_{2,i}+y_{2,i}-x_{1,i}y_{1,i}-x_{1,i}^2-\frac{6}{\beta_i^2})}{48(z-\beta_i)^2}\right.$$
$$-\frac{dz}{8(R'(0))^2z^3}+\frac{R''(0)dz}{16(R'(0))^3z^2}$$

where $\beta_{1,\dots,2d}$ solve $dR(\beta_i)=0$ (ramification pnts), $x_{n,i}:=\frac{R^{(n+2)}(\beta_i)}{R''(\beta_i)}$, $y_{n,i}:=\frac{(-1)^nR^{(n+1)}(-\beta_i)}{R'(-\beta_i)}$

Observation

The blue terms correspond to topological recursion for $x(z)=R(z)$ & $y(z)=-R(-z)$, the magenta terms signal an extension to blobbed topological recursion [Borot-Shadrin 15].

Conjecture: NC $\lambda\Phi^4$-model obeys BTR!

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When trying to prove the conjecture for $g = 0$ we noticed surprising identities between $\omega_{0,m+1}(u_1, \ldots, u_m, -z)$ and $\omega_{0,k+1}(u_1, \ldots, u_k, z)$. They were generalised in [Hock 22] to a general approach to the x-y symmetry in TR.

Definition

Let $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ramified covering with ramification points $\beta_1, \ldots, \beta_r$. For a global involution $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which neither fixes nor permutes the β_i, let $y(z) := -x(\iota z)$. Then a family $\{\omega_{0,n}\}_{n\geq 2}$ of meromorphic differentials is introduced by

$$\omega_{0,2}(w,z) = \frac{1}{2}\frac{dw\,dz}{(w-z)^2} + \frac{1}{2}\frac{d(\iota w)\,d(\iota z)}{(\iota w - \iota z)^2} - \frac{1}{2}\frac{dw\,d(\iota z)}{(w-\iota z)^2} - \frac{1}{2}\frac{d(\iota w)\,dz}{(\iota w - z)^2}$$

and for $m \geq 2$ by the involution identity (here $I := \{u_1, \ldots, u_m\}$)

$$\omega_{0,m+1}(I,z) + \omega_{0,m+1}(I,\iota z) = \sum_{s=2}^{m} \sum_{I_1 \uplus \ldots \uplus I_s = I} \frac{1}{s} \operatorname*{Res}_{w\to z}\left(\frac{dy(z)dx(w)}{(y(z)-y(w))^s} \prod_{i=1}^{s} \frac{\omega_{0,|I_i|+1}(I_i,w)}{dx(w)}\right).$$

Recursion of the genus-0 sector

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Theorem [Hock-W 21]

The involution identity has the unique solution

$$\omega_{0,m+1}(I,z) = \sum_{i=1}^{r} \operatorname*{Res}_{q\to\beta_i} K_i(z,q) \sum_{I_1\uplus I_2=I} \omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))$$

$$-\sum_{k=1}^{m} d_{u_k}\Big[\operatorname*{Res}_{q\to\iota u_k} \sum_{I_1\uplus I_2=I} \tilde{K}(z,q,u_k) d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big].$$

(for $m\geq 2$). Here σ_i is the local Galois involution near β_i, i.e. $x(z)=x(\sigma_i(z))$, $\sigma_i(\beta_i)=\beta_i$, $\sigma_i\neq \mathrm{id}$. The recursion kernels are given by

$$K_i(z,q) := \frac{\frac{1}{2}\big(\frac{dz}{z-q} - \frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))}, \qquad \tilde{K}(z,q,u) := \frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q} - \frac{d(\iota z)}{\iota z-u}\big)}{dx(q)(y(q)-y(\iota u))}.$$

For the choice $\iota z=-z$ and $x(z)=R(z):=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$, the solution of the involution identity coincides with the solution of the system for $(\Omega_n^{(0)},\mathcal{T}^{(0)})$ found in [BHW 20].

Linear and quadratic loop equations

The best strategy seems to prove that the Dyson-Schwinger equations imply linear and quadratic loop equations [Borot, Eynard, Orantin 13]

$$\sum_{j=0}^{d} \omega_{g,|I|+1}(I, \hat{z}^j) = f_1(R(z); I)$$

$$\sum_{\substack{j,k=0 \\ j\neq k}}^{d} \Big(\omega_{g-1,|I|+2}(I, \hat{z}^j, \hat{z}^k) + \sum_{\substack{g_1+g_2=g \\ I_1 \uplus I_2 = I}} \omega_{g_1,|I_1|+1}(I_1, \hat{z}^j) \omega_{g_2,|I_2|+1}(I_2, \hat{z}^k) \Big) = f_2(R(z); I)$$

where f_1, f_2 are *holomorphic* at ramification points of R and $R^{-1}(\{R(z)\}) = \{\hat{z}^0 \equiv z, \hat{z}^1, ..., \hat{z}^d\}$.

- Blobbed TR is the general solution of such loop equations [Borot, Shadrin 15]. To reduce to TR one needs more.
- In our case f_1, f_2 are of particular structure which completely fixes the poles at $z = -u_k$ and $z = 0$.
- Work in progress: extension to higher g (kernel K_0 for pole at $z = 0$ already found).

Outlook I: Intersection numbers

Eynard proved in 2011 a general formula which expresses $\omega_{g,n}$ for *any* spectral curve $(x : \Sigma \to \Sigma_0, \omega_{0,1}, \omega_{0,2})$ with simple ramification points in terms of intersection numbers of ψ- and κ-classes on several copies of $\overline{\mathcal{M}}_{g,n}$.

- Extensions to higher-order ramifications are known.
- According to [Borot, Shadrin 15], Eynard's formula survives with some modifications to blobbed TR.

Thus, expressing our $\omega_{g,n}$ in terms of intersection numbers is an achievable goal in very near future.

- This expression can be interesting (like the ELSV formula) or not.
- We hope it captures aspects of the involution $z \mapsto -z$ which plays a decisive rôle in the residue formula.

Outlook II: Integrability

Consider $\tau(\{t_i\}) = N^2\mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \sum_{g=2}^{\infty} N^{2-2g}\omega_{g,0}$

- In the [Kontsevich 92] model, for $t_i = -\frac{(2i-1)!!}{N}\mathrm{Tr}(E^{-2i-1})$, τ satisfies the Hirota bilinear PDE of the KdV-hierarchy.
- [Eynard, Orantin 07] describe how to obtain from any spectral curve of TR a formal Hirota equation (order by order in $1/N^2$).
- Whether or not integrability extends to blobbed TR is not known.
- We remain optimistic for our case because the recursion formula with residue kernel is very close to TR.
- $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ [Branahl, Hock 21] have been found.