Summary so far (and outlook)

In the first lecture we sketched the path to the Euclidean formulation of quantum field theory, described by a measure on a space of distributions. We also gave a few historical notes about noncommutative geometry.

We will now bring both fields together. It is relatively straightforward to write down formal measures for Euclidean quantum fields on noncommutative spaces.

- A free field measure only requires an inner product on test functions; no product.
- Interaction is encoded in a product in a noncommutative algebra, possibly with differential forms. This gives up locality, which we argue with Doplicher, Fredenhagen & Roberts as desired.
- In contrast to first expectation [Grosse, Klimčík, Prešnajder 95], the non-locality does not rule out divergences in Feynman graps [Filk 96]. These can be treated by usual QFT methods, but another problem was discovered from the string theory side.

With Harald Grosse we understood that this problem just signals an incomplete model, which can be consistently completed. Surprisingly, this completion improves the behaviour concerning triviality.

Limits to localisability

[Doplicher, Fredenhagen, Roberts 95] combined an argument of [Wheeler 55] with noncommutative geometry (in particular the fuzzy sphere [Madore 91]).

- To resolve a structure of size ℓ, a scattering experiment with wavelength $\lambda < \ell$ is necessary. These waves carry a quantum-mechanical energy $E = \frac{\hbar c}{\lambda} > \frac{\hbar c}{\ell}$.
- Energy is a source of gravitational fields; the simplest type is the Schwarzschild solution with characteristic radius $r_S = \frac{2GE}{c^4} > \frac{2G\hbar}{c^3 \ell}$.
- To avoid trapped surfaces (in the sense of Penrose), the structure to resolve must stay outside the Schwarzschild horizon, $\ell > 2r_s > \frac{4G\hbar}{c^3 \ell}$.

Hence, only structures larger than the Planck length $\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 10^{-33}$ cm can be meaningful if quantum physics and general relativity are combined.

The detailed anlysis of Doplicher, Fredenhagen, Roberts established unvertainty relations

$$\Delta x^0(\Delta x^1 + \Delta x^2 + \Delta x^3) \geq \ell_P^2 \text{ and } \Delta x^1 \Delta x^2 + \Delta x^2 \Delta x^3 + \Delta x^3 \Delta x^1 \geq \ell_P^2 .$$

They are induced by noncommutative coordinate operators $\hat{x}^\mu = (\hat{x}^\mu)^*$ with

$$[[\hat{x}^\mu, \hat{x}^\nu], \hat{x}^\rho] = 0 , \qquad [\hat{x}_\mu, \hat{x}_\nu][\hat{x}^\mu, \hat{x}^\nu] = 0 , \qquad \big(\tfrac{1}{8}[\hat{x}^\mu, \hat{x}^\nu][\hat{x}^\rho, \hat{x}^\sigma]\epsilon_{\mu\nu\rho\sigma}\big)^2 = \ell_P^8 .$$

Input from String theory

- [Witten 86] discussed Non-commutative geometry and string field theory.
- [Banks, Fischler, Shenker, Susskind 97] and [Ishibashi, Kawai, Kitazawa, Tsuchiya 97] introduced two formulations of M-theory as matrix models.
- [Connes, Douglas, Schwarz 97] studied compactifications of M-theory on noncommutative tori.
- Yang-Mills on noncommutative 4-torus is one-loop renormalisable [Krajewski-W 99]. Similar results from Filk's rules [Martín, Sánchez-Ruiz 99].
- [Schomerus 99] discovered that *D*-branes in flat background with constant magnetic field have operator product expansion given by noncommutative Moyal product.
- [Seiberg, Witten 99] vastly extended Schomerus' ideas and found a transformation between objects on noncommutative and commutative space.
- [Minwalla, van Raamsdonk, Seiberg 99] discovered UV/IR-mixing. Thoroughly investigated by [Chepelev, Roiban 99].

Remarks on relativistic formulation

A direct relativistic (with time) construction of true quantum fields on noncommutative spaces is much harder.

- [Doplicher, Fredenhagen, Roberts 95] outline a definition of quantum fields directly on non-local space-time.
- Further developed in [Bahns, Doplicher, Fredenhagen, Piacitelli 02], with emphasis on unitarity.
- *Free* Wightman functions on quantum space remain boundary values of holomorphic functions [Bahns 09], but these differ from the usual Euclidean approach (except for commutative time [Grosse, Lechner, Ludwig, Verch 11]).
- New achievements in perturbative algebraic quantum field theory [Doplicher, Morsella, Pinamonti 20].

There are thoughts that time should be introduced differently: Noncommutative algebras come with a canonical time evolution (Tomita-Takesaki theory). See [Connes, Rovelli 94]

Approximation by matrix algebras

To define interacting quantum fields one needs to make sense of a product of distributions.

If correctly implemented, the various divergences of the naïve product (of individual Feynman graphs [Filk], UV/IR-mixing [Minwalla, van Raamsdonk, Seiberg] and triviality [Landau; Aizenman]) should all disappear.

The only known strategy consists in finite-dimensional approximations of the problem, together with a careful limiting procedure.

Finite-dimensional approximations of noncommutative algebras are matrix algebras. E.g.

- nuclear C^*-algebras in which the identity map, as a completely positive map, approximately factors through matrix algebras;
- AF-algebras, which are inductive limit of sequences of matrix algebras.

These two classes of C^*-algebras are not appropriate for us; we need (smooth) test functions, equipped which a Fréchet topology. There are enough examples where Fréchet algebras have a matrix approximation.

The Moyal algebra

Let $\Theta = -\Theta^t$ be a skew-symmetric real $D \times D$-matrix. Then by

$$(f \star g)(x) := \int_{\mathbb{R}^D \times \mathbb{R}^D} \frac{dydk}{(2\pi)^D} f(x + \tfrac{1}{2}\Theta k) g(x+y)\, e^{\mathrm{i}\langle k,y\rangle}$$

an associative non-commutative product on Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^D)$ is defined.

- Prototype example for strict deformation quantisation by $\mathbb{R}^D$-actions [Rieffel 93].
- Defining $(\hat{x}^\mu f)(x) = (x^\mu \star f)(x)$ by obvious extension, then $[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\Theta^{\mu\nu}$ is central and a simple-minded implementation of [Doplicher, Fredenhagen, Roberts 95].
- $\overline{f \star g} = \overline{g} \star \overline{f}$, $\int_{\mathbb{R}^D} dx\, (f \star g)(x) = \int_{\mathbb{R}^D} dx\, f(x) g(x)$.

The inner product $\langle f, g \rangle = \int_{\mathbb{R}^{2D}} dx\, dy\; f(x)(-\Delta + M^2)^{-1}(x,y)\, g(y)$ gives, as usual, rise to a Gauß measure $d\mu_0$ on $(\mathcal{S}(\mathbb{R}^D))'$ which now can be noncommutatively deformed to $d\mu_\lambda(\Phi) := \frac{1}{\mathcal{Z}} e^{-\frac{\lambda}{4}\int_{\mathbb{R}^D} dx\, (\Phi\star\Phi\star\Phi\star\Phi)(x)} d\mu_0(\Phi)$. The resulting QFT suffers from UV/IR-mixing.

The matrix basis

Let $D = 2$ and $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$. We recommend to check:

Exercise

1. The Gaußian $b_{00}(x) := 2e^{-\frac{\|x\|^2}{\theta}}$ is a projector, $b_{00} \star b_{00} = b_{00}$.
2. Let $a := \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2)$, $\bar{a} := \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2)$. Then $a \star b_{00} = 0 = b_{00} \star \bar{a}$ and $[\bar{a}, a]_* = \theta$.
3. Consequently, $b_{mn} := \frac{1}{\sqrt{m!n!\theta^{m+n}}}\bar{a}^{\star m} \star b_{00} \star a^{\star n}$ satisfies $b_{kl} \star b_{mn} = \delta_{lm} b_{kn}$. Moreover, $\int_{\mathbb{R}^2} dx\; b_{kl}(x) = 2\pi\theta\delta_{kl}$.
4. $b_{mn}(x) = 2(-1)^m \sqrt{\frac{m!}{n!}}\left(\sqrt{\frac{2}{\theta}}(x_1 + \mathrm{i}x_2)\right)^{n-m} L_m^{n-m}\left(\frac{2}{\theta}\|x\|^2\right)e^{-\frac{\|x\|^2}{\theta}}$ where L_m^α are associate Laguerre polynomials.

Let $\mathcal{A}_\theta = \{f = (f_{kl})_{k,l \in \mathbb{N}}\}$ be the vector space of infinite matrices, completed in the Fréchet topology induced by the family $\langle f, g\rangle_m := \sum_{k,l\in\mathbb{N}} \theta^{2m}(k + \frac{1}{2})^m(l + \frac{1}{2})^m \overline{f_{kl}} g_{kl}$.

Then $\mathcal{A}_\theta \ni f \mapsto \sum_{k,l\in\mathbb{N}} f_{kl} b_{kl}$ is an isomorphism of Fréchet algebras between $(\mathcal{A}_\theta, \cdot)$ and $(\mathcal{S}(\mathbb{R}^2), \star)$ [Gracia-Bondía, Varilly 88].

The $\lambda\Phi^{\star 4}$-model in the matrix basis

- The isomorphism extends to $\mathcal{S}(\mathbb{R}^D)$, for D even, via $b_{k_1 l_1}(x_1, x_2) \cdots b_{k_{D/2} l_{D/2}}(x_{D-1}, x_D)$. Use bijection $\pi : \mathbb{N}^{D/2} \to \mathbb{N}$, e.g. $\pi(k_2, k_2) = \frac{1}{2}(k_1{+}k_1)(k_1{+}k_1{+}1){+}k_2$ [Cantor].
- Setting $\Phi_{kl} := \Phi(b_{k_1 l_1} \otimes .. \otimes b_{k_{D/2} l_{D/2}})\big|_{k=\pi(k_1,...k_{D/2}), l=\pi(l_1,..,l_{D/2})}$, we turn the measure deformation into

$$d\mu_\lambda(\Phi) \mapsto \frac{1}{\mathcal{Z}} e^{-\frac{\lambda}{4}\sqrt{\det(2\pi\Theta)}\sum_{k,l,m,n\in\mathbb{N}} \Phi_{kl}\Phi_{lm}\Phi_{mn}\Phi_{nk}} d\mu_0(\Phi) \; .$$

We should also express the Gauß measure $d\mu_0(\Phi)$ in the matrix basis. One first notices

$$\int_{\mathbb{R}^2} dx \, f(x)((-\Delta + M^2)g)(x) = \sum_{k,l,m,n\in\mathbb{N}} \Delta_{kl;mn} f_{kl} g_{mn} \qquad \text{where}$$

$$\Delta_{kl;mn} = (M^2 + \frac{2}{\theta}(m+n+2))\delta_{nk}\delta_{ml} - \frac{2}{\theta}\sqrt{kl}\,\delta_{n+1,k}\delta_{m+1,l} - \frac{2}{\theta}\sqrt{mn}\,\delta_{n-1,k}\delta_{m-1,l}$$

- The inner product to define $d\mu_0$ needs the kernel of the inverse $(-\Delta + M^2)^{-1}$. Can be found by diagonalisation via Meixner polynomials [Grosse, W 04].
- Then UV/IR-mixing traced back to off-diagonal terms too large.

The harmonic oscillator potential [Grosse, W 04]

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Scaling down the off-diagonal term to cure the UV/IR-mixing problem amounts to introduce a harmonic oscillator potential. We thus arrive at the inner product (in $D = 4$)

$$\langle f, g\rangle := \int_{\mathbb{R}^4\times\mathbb{R}^4} dxdy\; f(y)\Big(-\Delta + \frac{4\Omega^2}{\theta^2}|x|^2 + M^2\Big)^{-1}(y,x)g(x) = \sum_{\substack{k_1,k_2,\dots\\ n_1,n_2=0}}^{\infty} f_{\substack{k_1 l_1\\ k_2 l_2}}\, g_{\substack{m_1 n_1\\ m_2 n_2}}\, C_{\substack{k_1 l_1\\ k_2 l_2};\substack{m_1 n_1\\ m_2 n_2}}$$

$$C_{\substack{k_1 l_1\\ k_2 l_2};\substack{m_1 n_1\\ m_2 n_2}} = \frac{\theta}{2(1+\Omega)^2}\delta_{m_1+k_1,n_1+l_1}\delta_{m_2+k_2,n_2+l_2}$$

$$\times \sum_{v_1=\frac{|m_1-l_1|}{2}}^{\frac{m_1+l_1}{2}} \sum_{v_2=\frac{|m_2-l_2|}{2}}^{\frac{m_2+l_2}{2}} B\Big(1+\frac{M^2\theta}{8\Omega}+\frac{1}{2}(m_1+k_1+m_2+k_2)-v_1-v_2,\, 1+2v_1+2v_2\Big)$$

$$\times\, {}_2F_1\left(\begin{matrix}1+2v_1+2v_2\,,\; \frac{M^2\theta}{8\Omega}-\frac{1}{2}(m_1+k_1+m_2+k_2)+v_1+v_2\\ 2+\frac{M^2\theta}{8\Omega}+\frac{1}{2}(m_1+k_1+m_2+k_2)+v_1+v_2\end{matrix}\,\middle|\, \frac{(1-\Omega)^2}{(1+\Omega)^2}\right)$$

$$\times \prod_{i=1}^{2}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v_i}\sqrt{\binom{n_i}{v_i+\frac{n_i-k_i}{2}}\binom{k_i}{v_i+\frac{k_i-n_i}{2}}\binom{m_i}{v_i+\frac{m_i-l_i}{2}}\binom{l_i}{v_i+\frac{l_i-m_i}{2}}}\,.$$

Perturbative renormalisation and β-function

Aim is to make sense of $\frac{1}{\mathcal{Z}}\sum_{p=0}^{\infty}\frac{1}{p!}\Big(-\frac{\lambda}{4}\sqrt{\det(2\pi\Theta)}\sum_{k,l,m,n=0}^{\infty}\Phi_{kl}\Phi_{lm}\Phi_{mn}\Phi_{nk}\Big)^p d\mu_0(\Phi)$ for $d\mu_0$ defined with inner product with kernel C.

- Restrict everything to finite $N\times N$-matrices and set $\Phi_{mn}\mapsto\sqrt{Z}(N)\Phi_{mn}$, $\lambda\mapsto\lambda(N)$ $M\mapsto M(N)$, $\Omega\mapsto\Omega(N)$ (regularisation).
- One proves [Grosse, W 05] that $\sqrt{Z}(N),\mu(N),\Omega(N),\lambda(N)$ can be found as formal power series in a (new) parameter λ such that the measure exists as formal power series in λ.
- Moreover, at lowest order in λ one finds $\frac{\lambda(N)}{\Omega^2(N)}=const$ and $\lim_{N\to\infty}\Omega(N)=1$ [Grosse W 04]. This implies existence of $\lim_{N\to\infty}\lambda(N)$, i.e. absense of the Landau ghost.
- [Disertori, Rivasseau 06] proposed to look at $\Omega\equiv 1$ independent of N. The inner product simplifies enormously, and they could check existence of $\lim_{N\to\infty}\lambda(N)$ up to third order.
- [Disertori, Gurau, Magnen, Rivasseau 06] extended this result to any order. Their method is (together with topological recursion) the key to a complete solution.

Simplifications for $\Omega = 1$

Set $\mathcal{N} := (\frac{\theta}{4})^{D/2}$. We will soon arrive at an $1/\mathcal{N}$-expansion. Note that $\mathcal{N}$ is not the size of matrices, it is the scale of noncommutativity!

The inner product on $\mathcal{V} = \mathcal{A}_\theta$ specifies for $\Omega = 1$ to

$$\langle f, g\rangle = \frac{1}{\mathcal{N}} \sum_{k,l=0}^{\infty} \frac{f_{kl} g_{lk}}{E_k + E_l}, \qquad E_k = \frac{(k_1 + \ldots + k_{D/2} + D/4)}{\mathcal{N}^{2/D}} + \frac{M^2}{2}\Big|_{(k_1,\ldots,k_{D/2}) = \pi^{-1}(k)}$$

It gives rise to a unique measure $d\tilde{\mu}_0$ on $\mathcal{V}'$ with $\int_{\mathcal{V}'} d\tilde{\mu}_0(\Phi) e^{\mathrm{i}\Phi(f)} = \exp(-\frac{1}{2}\langle f, f\rangle)$ which we (formally) deform to

$$d\tilde{\mu}_\lambda(\Phi) = \frac{1}{\mathcal{Z}} e^{-\frac{\lambda\mathcal{N}}{4}\mathrm{Tr}(\Phi^4)} d\tilde{\mu}_0(\Phi)$$

where $\mathrm{Tr}(\Phi^4) := \sum_{k,l,m,n=0}^{\infty} \Phi(e_{kl})\Phi(e_{lm})\Phi(e_{mn})\Phi(e_{nk})$.

For technical reasons we temporarily assume that all E_k are pairwise different.

Moments and cumulants

$$\langle e_{k_1 l_1} \dots e_{k_n l_n} \rangle := \int_{\mathcal{V}'} d\mu_\lambda(\Phi) \prod_{i=1}^{n} \Phi(e_{k_i l_i}) = \frac{1}{\mathrm{i}^n} \frac{\partial^n \mathcal{Z}(M)}{\partial f_{k_1 l_1} \cdots \partial f_{k_n l_n}}\Big|_{f=0}\,, \qquad \mathcal{Z}(f) := \int_{\mathcal{V}'} d\mu_\lambda(\Phi)\, e^{\mathrm{i}\Phi(f)}$$

if $f = \sum_{k,l=0}^{\infty} f_{kl} e_{kl}$ with respect to standard matrix basis (e_{kl}). These moments decompose into cumulants

$$\Big\langle \prod_{i=1}^{n} e_{k_i l_i} \Big\rangle = \sum_{\substack{\text{partitions} \\ \pi \text{ of } \{1,\dots,n\}}} \prod_{\text{blocks } \beta \in \pi} \Big\langle \prod_{i \in \beta} e_{k_i l_i} \Big\rangle_c \,.$$

They are only non-zero if n is even and every block β is of even length.

Take all k_i pairwise different. Then $\langle e_{k_1 l_1} \dots e_{k_n l_n} \rangle_c$ is only non-zero if $(l_1, \dots, l_n)$ is a permutation of $(k_1, \dots, k_n)$, and in this case the cumulant only depends on the *cycle type*:

$$N^{n_1+\dots+n_b} \Big\langle (e_{k_1^1 k_2^1} e_{k_2^1 k_3^1} \cdots e_{k_{n_1}^1 k_1^1}) \cdots (e_{k_1^b k_2^b} e_{k_2^b k_3^b} \cdots e_{k_{n_b}^b k_1^b}) \Big\rangle_c =: N^{2-b} G_{|k_1^1 \dots k_{n_1}^1| \dots |k_1^b \dots k_{n_b}^b|} \,.$$

Equations of motion

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Lemma [Schürmann, W 19]

The Fourier transform $\mathcal{Z}(f) := \int_{\mathcal{V}'} d\mu_\lambda(\Phi)\, e^{\mathrm{i}\Phi(f)}$ of the measure satisfies

1. $$\frac{1}{\mathrm{i}}\frac{\partial \mathcal{Z}(f)}{\partial f_{ab}} = \frac{\mathrm{i} f_{ba}\mathcal{Z}(f)}{\mathcal{N}(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^3(E_a+E_b)}\sum_{k,l=0}^{\infty}\frac{\partial^3\mathcal{Z}(f)}{\partial f_{ak}\partial f_{kl}\partial f_{lb}}$$

2. $$\frac{1}{\mathcal{N}}\frac{\partial \mathcal{Z}(f)}{\partial E_a} = \sum_{k=0}^{\infty}\frac{\partial^2\mathcal{Z}(f)}{\partial f_{ak}\partial f_{ka}} + \Big(\frac{1}{\mathcal{N}}\sum_{k=0}^{\infty} G_{|ak|} + \frac{1}{\mathcal{N}^2}G_{|a|a|}\Big)\mathcal{Z}(f)$$

Corollary (Ward-Takahashi identity)

3. $$-\mathcal{N}\sum_{k=0}^{\infty}(E_a-E_b)\frac{\partial^2\mathcal{Z}(f)}{\partial f_{ak}\partial f_{kb}} = \sum_{k=0}^{\infty}\Big(f_{ka}\frac{\partial\mathcal{Z}(f)}{\partial f_{kb}} - f_{bk}\frac{\partial\mathcal{Z}(f)}{\partial f_{ak}}\Big)$$

3 was discovered in [Disertori, Gurau, Magnen, Rivasseau 06]. Alex Hock will prove it (for finite matrices) in his tutorial.

Dyson-Schwinger equations

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The equation of motion ❶ induces Dyson-Schwinger equations between moments. For $n=2$:

$$\frac{1}{\mathcal{N}}G_{|ab|} := \langle e_{ab}e_{ba}\rangle \equiv -\frac{\partial^2\mathcal{Z}(f)}{\partial f_{ba}\partial f_{ab}}\Big|_{f=0} = \frac{1}{\mathcal{N}(E_a+E_b)} - \frac{\lambda}{(E_a+E_b)}\sum_{k,l=0}^{\infty}\frac{\partial^4\mathcal{Z}(f)}{\partial f_{lb}\partial f_{ba}\partial f_{ak}\partial f_{kl}}\Big|_{f=0}$$

It seems that the rhs will produce 4-point functions (in general, n-point function expressed in terms of $n+2$-point functions). Using ❸ and ❷ one can avoid this:

$$\begin{aligned}(E_a+E_b)G_{|ab|} &= 1+\lambda\sum_{\substack{l=0\\ l\neq a}}^{\infty}\frac{\partial^2}{\partial f_{lb}\partial f_{ba}}\Big[\sum_{k=0}^{\infty}\frac{1}{E_a-E_l}\Big(f_{ka}\frac{\partial\mathcal{Z}(f)}{\partial f_{kl}} - f_{lk}\frac{\partial\mathcal{Z}(f)}{\partial f_{ak}}\Big)\Big]_{f=0}\\ &\quad -\lambda\mathcal{N}\frac{\partial^2}{\partial f_{ab}\partial f_{ba}}\Big[\frac{1}{\mathcal{N}}\frac{\partial\mathcal{Z}(f)}{\partial E_a} - \Big(\frac{1}{\mathcal{N}}\sum_{k=0}^{\infty}G_{|ak|}+\frac{1}{\mathcal{N}^2}G_{|a|a|}\Big)\mathcal{Z}(f)\Big]_{f=0}\\ &= 1+\frac{\lambda}{\mathcal{N}}\sum_{\substack{k=0\\ k\neq a}}^{\infty}\frac{G_{|kb|}-G_{|ab|}}{E_k-E_a} + \frac{\lambda}{\mathcal{N}^2}\frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a} - G_{|ab|}\Big(\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\infty}G_{|ak|}+\frac{\lambda}{\mathcal{N}^2}G_{|a|a|}\Big) + \frac{\lambda}{\mathcal{N}}\frac{\partial G_{|ab|}}{\partial E_a}\end{aligned}$$

Complexification

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- The dependence of $G_{|ab|}$ on the matrix indices a, b is of the form of an evaluation $G_{|ab|} = G(E_a, E_b)$ of a function G of two complex variables at E_a, E_b.
- $G(\zeta, \eta)$ still depends on summation variables E_k. Differentiating wrt some E_c is also an evaluation at E_c of another function $-\mathcal{N}\frac{\partial}{\partial E_c} G(\zeta, \eta) = T(E_c \| \zeta, \eta|)$.
- Similarly for $G_{|a|b|} = G(E_a|E_b)$.

Complexified Dyson-Schwinger equation

$$(\zeta+\eta)G(\zeta,\eta) = 1 + \frac{\lambda}{\mathcal{N}} \sum_{k\in\mathbb{N}} \frac{G(E_k,\eta) - G(\zeta,\eta)}{E_k - \zeta} + \frac{\lambda}{\mathcal{N}^2} \frac{G(\eta|\eta) - G(\zeta|\eta)}{\eta - \zeta}$$
$$- G(\zeta,\eta)\Big(\frac{\lambda}{\mathcal{N}} \sum_{k\in\mathbb{N}} G(\zeta, E_k) + \frac{\lambda}{\mathcal{N}^2} G(\zeta|\zeta)\Big) - \frac{\lambda}{\mathcal{N}^2} T(\zeta\|\zeta,\eta|)$$

Alternatively, sum only over pairwise different e_k and include their multiplicities r_k:

$$\sum_{k\in\mathbb{N}} \frac{G(E_k,\eta) - G(\zeta,\eta)}{E_k - \zeta} = \sum_{k=0}^{\infty} r_k \frac{G(e_k,\eta) - G(\zeta,\eta)}{e_k - \zeta}, \qquad \sum_{k\in\mathbb{N}} G(\zeta, E_k) = \sum_{k=0}^{\infty} r_r G(\zeta, e_k)$$

The genus expansion

We approach the solution of such equations in a formal genus expansion (where $\mathcal{N} := (\frac{\theta}{4})^{D/2}$)

$$G(\zeta,\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta,\eta)\,, \qquad G(\zeta|\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta|\eta)\,,$$

$$T(\xi\|\zeta,\eta|) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} T^{(g)}(\xi\|\zeta,\eta|)$$

together with the convention that $\frac{1}{\mathcal{N}}$ in front of a summation is neutral.

Note that these series have zero radius of convergence! Making sense of them via Borel resummation is a main challenge for the future. Connects to resurgence.

Theorem [Grosse, W 09]

The planar two-point function satisfies the closed non-linear equation

$$\Big(\zeta + \eta + \frac{\lambda}{\mathcal{N}} \sum_{k\in\mathbb{N}} r_k G^{(0)}(\zeta, e_k)\Big) G^{(0)}(\zeta,\eta) = 1 + \frac{\lambda}{\mathcal{N}} \sum_{k\in\mathbb{N}} r_k \frac{G^{(0)}(e_k,\eta) - G^{(0)}(\zeta,\eta)}{e_k - \zeta}$$