Wightman axioms

- [von Neumann 32] axioms for quantum mechanics
- [Wightman, Gårding 50s] unique extension to quantum fields
  = unbounded operator valued distributions $f \mapsto \Phi(f) : \mathcal{D} \to \mathcal{D} \subset \mathcal{H}$

Wightman axioms (sketched, see [Streater, Wightman 64])

- **Relativistic invariance.** Poincaré group represented by unitary operators on $\mathcal{H}$ under which field operators transform as $U(a,L)\Phi(f)U(a,L)^* = S(a,L)\Phi(f^{a,L})$
- **Spectrum condition.** Joint spectrum of generators P_i of translations contained in forward lightcone, i.e. $P_0^2 - \sum_{i=1}^{D-1} P_i^2 > 0$ and $P^0 > 0$
- **Vacuum.** There is a cyclic and Poincaré-invariant vector $\Omega \in \mathcal{D}$.
- **Locality.** $\Phi(f)\Phi(g) = \pm\Phi(g)\Phi(f)$ if f, g have causally disjoint support.

Reformulated by [Haag, Kastler 64] in terms of C^* and von Neumann algebras (today the preferred approach)

Wightman and Schwinger functions

Consider Wightman distributions

$$(f_1, \ldots, f_n) \mapsto W_n(f_1, \ldots, f_n) := \langle \Omega, \Phi(f_1) \cdots \Phi(f_n)\Omega \rangle$$

Axioms for $\Phi(f)$ induce axioms for W_n, and conversely. Moreover:

Theorem [Hall, Wightman 57], Bargmann, [Bros, Epstein, Glaser 67]

Wightman functions $\langle \Omega, \Phi(x_1) \cdots \Phi(x_n)\Omega \rangle$ are boundary values of holomorphic functions.

- Their restriction to real subspace of Euclidean points (minus diagonals) defines Schwinger functions S_n.
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity.

Osterwalder-Schrader axioms

Reflection positivity

Let $(x^0, \vec{x})^\theta := (-x^0, \vec{x})$ be the reflection in $\mathbb{R}^D$ at the plane $x^0 = 0$. Let f_n be a test function on $(\mathbb{R}^D)^n \ni (x_1, ..., x_n)$ whose support is contained in $0 < x_1^0 < x_2^0 < \cdots < x_n^0$. Then for any tuple $f_0, f_1, \cdots, f_N$ of such test functions with ordered time support one has

$$\sum_{m,n=0}^{N} \int dx_1 ... dx_m \, dy_1 .. dy_n S_{m+n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \overline{f_m(x_1^\theta, \ldots, x_m^\theta)} f_n(y_1, \ldots, y_n) \geq 0$$

- [Nelson 73] formulated axioms for random distributions where a Markov property was essential.
- [Osterwalder, Schrader 74] gave a first proof that the natural properties of Schwinger functions are also sufficient to imply the Wightman axioms.
- [Glaser 74] noticed a gap in the proof. [Osterwalder, Schrader 75] found a growth condition which together with their other axioms implies the Wightman axioms.
- Reflection positivity is related to the Hausdorff-Bernstein-Widder theorem.

Probability measures on distributions

Some axioms are automatic if $S_n(f_1, \dots, f_n) = \int_{\mathcal{V}'} d\mu(\Phi)\, \Phi(f_1) \cdots \Phi(f_n)$ are moments of a measure $d\mu$ on distributions, where $\mathcal{V} = C_0^\infty(\mathbb{R}^D)$ or $\mathcal{V} = \mathcal{S}(\mathbb{R}^D)$.

Consider the Fourier transform $\mathcal{Z}(f) := \int_{\mathcal{V}'} d\mu(\Phi)\, e^{\mathrm{i}\Phi(f)}$

[Glimm, Jaffe 87] axioms for a probability measure of Euclidean quantum fields

- OS0 **Analyticity.** For any $f_1, \dots, f_N \in \mathcal{V}$, $\mathbb{C}^N \ni (z_1, \dots, z_N) \mapsto \mathcal{Z}(z_1 f_1 + \cdots + z_N f_N)$ is holomorphic.
- OS1 **Regularity.** $|\mathcal{Z}(f)| \leq \exp(c_1 \|f\|_1 + c_p \|f\|_p^p)$ for some $1 < p \leq 2$.
- OS2 **Euclidean invariance.** $\mathcal{Z}(f) = \mathcal{Z}(f^{R,a})$ where $f^{R,a}(x) = f(R^{-1}(x-a))$.
- OS3 **Reflection positivity.** For any real $f_1, \dots, f_N \in \mathcal{V}$, the $N \times N$ matrix $M_{ij} = \mathcal{Z}(f_i - f_j^\theta)$ is positive semidefinite, where $f^\theta(x) := f(x^\theta)$.
- OS4 **Ergodicity.** The time translation subgroup acts ergodically on $(\mathcal{V}', d\mu)$.

Gaußian measure

GEOMETRY: DEFORMATIONS AND RIGIDITY

Theorem (Bochner, Minlos, Schur)

Any continuous inner product $\langle\ ,\ \rangle$ on a real nuclear vector space $\mathcal{V}$ defines a unique probability measure $d\mu_0$ on $\mathcal{V}'$ with $\exp(-\frac{1}{2}\langle f, f\rangle) = \int_{\mathcal{V}'} d\mu_0(\Phi)e^{i\Phi(f)}$.

A measure of this type defines a Gaußian field $\Phi \in \mathcal{V}'$.

Example

$\mathcal{V} = \mathcal{S}(\mathbb{R}^D)$ and $\langle f, g\rangle = \int_{\mathbb{R}^{2D}} dx\, dy\, f(x)(-\Delta + m^2)^{-1}(x, y)\, g(y)$

- One would like to define interacting fields by a deformation $d\mu(\Phi)$ "$:=$" $\frac{1}{\mathcal{Z}}e^{-P(\Phi)}d\mu_0(\Phi)$ of the Gaußian measure $d\mu_0$, for P a polynomial of degree > 2 bounded from below.
- This is not straightforward! At a technical level, a product of distributions in $P(\Phi)$ is not defined.
- More concretely, various types of divergences occur whose treatment (renormalisation) requires a sophisticated analysis.

Selected success stories

The construction of interacting measures succeded in only a few cases, all in dimension < 4.

- A few exactly solvable 2D models, such as the [Thirring 58] model (fermions with quartic self-interaction) and the [Schwinger 62] model (QED in 2D).
- The $\lambda\Phi^4_2$-model in 2D, $P(\Phi) = \frac{\lambda}{4!}\int_{\mathbb{R}^2} dx\, (\Phi(x))^4$, first by hard work in relativistic formulation [Glimm, Jaffe 68–72].
- The $P[\Phi]_2$ model, i.e. any polynomial interaction in 2D. First spectacular success of the Euclidean method [Simon 74].
- Many conformal field theories [Belavin, Polyakov, Zamolodchikov 84] in 2D.
- Gross-Neveu model in 2D by fermionic summation techniques [Gawędzki-Kupiainen 85, Feldman-Magnen-Rivasseau-Sénéor 86].
- The $\lambda\Phi^4_3$-model in 3D [Feldman, Osterwalder 76].
  Recently the target of spectacular developments in probability theory [Hairer 14; Gubinelli, Imkeller, Perkowski 15; Mourrat, Weber 17].

Triviality

- [Landau, Abrikosov, Khalatnikov 54] gave a heuristic argument that quantum electrodynamics cannot exist as a renormalised QFT.

  The problem is called Landau ghost, triviality, positivity of β-function.

- This was considered as death of QFT, rescued only by the discovery of asymptotic freedom in non-Abelian Yang-Mills theory by Gross, Politzer, Wilczek in 1973.
- [Aizenman 81] and [Fröhlich 82] gave a rigorous proof of triviality for the $\lambda\Phi^4$-model in $D = 4 + \epsilon$ dimensions.
- Almost 40 years later, [Aizenman, Duminil-Copin 19] proved triviality of $\lambda\Phi^4$ in $D = 4$.

The situation of 4D QFT is disappointing

The only interacting model which seems to exist is non-Abelian Yang-Mills theory, but the proof is one of the Millenium Prize problems!

This motivates us to look at QFT on noncommutative geometries.

Noncommutative geometry

[Murray, von Neumann 36] introduced operator algebras and studied their properties. They defined the types of von Neumann factors.

- [Gelfand, Naimark 43] proved that commutative C^*-algebras are isometrically isomorphic to $C(X)$ for a locally-compact Hausdorff space X.

Noncommutative geometry relaxes the commutativity assumption and studies general operator algebras.

- [Takesaki 70] worked out Tomita's theory of modular automorphisms. These allowed [Connes 73] to finish the classification of amenable von Neumann algebras.

K-theory and various (co)homology theories were extended to the noncommutative world.

- [Kasparov 80] introduced KK-theory as bivariant functor on $C*$-algebras.
- [Connes 81] developed cyclic cohomology to achieve an index pairing with K-theory.

Non-commutative differential geometry as established research area [Connes 85].

First appearance of NCG in physics

- [Bellissard 86] understood the integer quantum Hall effect in terms of K-theory
- [Connes, Rieffel 87] defined Yang-Mills for non-commutative two-tori
- [Dubois-Violette, Kerner, Madore 90] worked out noncommutative differential geometry of matrix algebras
- [Connes 90] showed that noncommutative geometry on two points gives the Higgs potential; details worked out by [Connes, Lott 91]
- [Madore 91] proposed the fuzzy sphere, [Grosse, Madore 92] formulated a noncommutative version of the Schwinger model
- [Doplicher, Fredenhagen, Roberts 95] discussion (later more)
- [Filk 96] introduced Feynman rules for noncommutative quantum fields and showed that planar diagrams have the same divergences

First appearance of NCG in physics

- [Connes 96] introduced spectral triples and showed later that commutative spectral triples are manifolds [Connes 08]. Contributions by [Figueroa, Gracia-Bondía, Varilly 00] and [Rennie, Varilly 06].
- [Connes, Chamseddine 96] proposed the spectral action principle as unification of gravity and standard model of particle physics. Improvements by [Connes, Chamseddine, Marcolli 07] and [Connes, Chamseddine, van Suijlekom 13]
- [Kreimer 97] understood that the combinatorics of renormalisation in quantum field theory is encoded in the antipode of a Hopf algebra. It is closely related to a Hopf algebra which governs the local index formula [Connes, Moscovici 98] for foliations.