FINITENESS THEOREMS FOR LIMIT CYCLES

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PLAN

1. Definition of STAR-1
2. Definition of index
3. Lower Estimate Theorem (LET)

Theorem

Cochains of class $\mathcal{FC}^1$ admit the following lower estimate:

$$|F| \succ \exp(-C\xi) \text{ on } (\mathbb{R}^+, \infty)$$

for some $C > 0$.

Example

How the multiple exponentials occur? Consider a composition

$$\ln \circ (id + \varphi) \circ \exp, \; \varphi = \exp(-\xi).$$

We have:

$$\ln(\exp \zeta + \exp(-\exp \zeta)) = \zeta + \ln(1 + \exp(-exp\zeta - \zeta))$$

$$= \zeta + \Sigma \frac{(-1)^{k-1}}{k} exp(-k \exp \zeta - k\zeta)$$

$$= \zeta + \Sigma(-1)^{k-1} \frac{\exp(-k\zeta)}{k} \exp(-k \exp \zeta).$$

Where is the exponent and where is the coefficient?

STAR-1: first steps of the definition

STAR-1 has the form

$$\Sigma = \Sigma_1^\infty a_j \exp \mathbf{e}_j, \tag{1}$$

where $\mathbf{e}_j$ are exponents, and a_j are coefficients, all to be defined. A STAR-1 has an index: a finite subset of $(0, 1)$. The index is not uniquely defined: the same series may have different representations, and the index depends on the representation.

Exponents and coefficients

The set of exponents is denoted by E^1, the set of coefficients is $\mathcal{K}^1$.
The coefficients have non-negative rank. The set of all the coefficients of rank r is denoted by $\mathcal{K}^{1,r}$.

Exponents of class 1

Definition (exponents and principal exponents)

The set E^1 of exponents of the series of class 1 does not depend on the rank. It is a special set of all partial sums of the generalized exponential Dulac series with non-negative exponents no greater than 1. Namely,

$$E^1 = \{\mathbf{e} \mid \mathbf{e} = \sum P_j(\zeta) \exp \mu_j \zeta\},$$

where the sum is finite, P_j are real polynomials, and $\mu_j \in [0, 1]$. Moreover, if $\mu_j = 1$, then $P_j = const$. The principal exponent of the term $\exp \mathbf{e}$ is the limit

$$\nu(\mathbf{e}) = \lim_{(\mathbb{R}^+, \infty)} \frac{\mathbf{e}(\xi)}{\exp \xi}.$$

This limit exists by definition of E^1.

Coefficients and series of class 1,0

Definition

$$\mathcal{K}^{1,0} = \mathcal{FC}^0, \;\; \mathit{ind}\,(a \in \mathcal{K}^{1,0}) = \emptyset.$$

Definition

$\mathcal{E}^{1,0}$ is a set of all STAR-(1,0):

$$(\Sigma \in \mathcal{E}^{1,0}) = \Sigma a_j \exp \mathbf{e}_j, \;\; \mathbf{e}_j \in E^1, \;\; a_j \in \mathcal{K}^{1,0}$$

Standard domains of class 1 and decomposability

Definition

Standard domain of class 1 is a domain of the form

$$\Omega_C = \Phi_C(\mathbb{C}^+_C),$$

where

$$\Phi_C : \zeta \to \zeta(1 + \frac{C}{\ln \zeta}), \; = \mathbb{C}^+ \setminus K_C, \; K_c = \{\zeta || \zeta| \leq C\}$$

The set of all such domains is denoted by Ω_1.

LET-1,0 for $\mathcal{FC}^{1,0}$

Definition (plus-decomposability)

A sectorial cochain F is plus-decomposable in a STAR-(1,0) Σ provided that there exist a standard domain Ω of class 1 and $\varepsilon > 0$ such that for any $\nu > 0$ there exists a partial sum Σ_N of Σ that approximates $F \circ \exp$ with an accuracy $\exp(-\nu \exp \xi)$:

$$|F \circ \exp - \Sigma_N| \prec \exp(-\nu \exp \xi) \qquad (2)$$

in the intersection $\ln \Omega \cap \mathbb{H}^\varepsilon$. Here $\mathbb{H}^\varepsilon$ is the ε-neighborhood of the upper half-plane.

Theorem

Cochains of class $\mathcal{FC}^{1,0}$ admit the following lower estimate:

$$|F| \succ \exp(-C\xi) \text{ on } (\mathbb{R}^+, \infty)$$

for some $C > 0$.

Proof.

Take $F \in \mathcal{FC}^{1,0}$. Let Σ be an asymptotic series for F. We may assume that the exponents $\mathbf{e}_j$ have no terms $\alpha\zeta + \beta$ or else they will be absorbed by the coefficients:

$$b_j = a_j \exp(\alpha\zeta + \beta), a_j \in \mathcal{FC}^0 \Longrightarrow b_j \in \mathcal{FC}^0.$$

□

Proof.

Take a partial sum Σ_N of Σ that approximates F with the accuracy R rapidly decreasing. Let $\mathbf{h}$ be the maximal exponent of Σ_N. For any other exponent $\mathbf{e}$,

$$\frac{\mathbf{e}-\mathbf{h}}{\xi} \to -\infty \text{ on } (\mathbb{R}^+, \infty) \tag{3}$$

Then

$$\Sigma_N = \exp \mathbf{h}(a + R_1),$$

$$R_1 = \Sigma a_j \exp(\mathbf{e}_j - \mathbf{h})$$

□

Proof.

By (3), R decreases on $(\mathbb{R}^+, \infty)$ faster than any exponent, because a_j grows no more than exponentially. The same holds for $\exp(\mathbf{e}_j - \mathbf{h})$. By LET for $\mathcal{FC}^0$,
$|a_1| \succ \exp(-C\xi)$ on $(\mathbb{R}^+, \infty)$ for some $C > 0$. Hence,

$$\Sigma_N \succ \exp(\nu - \varepsilon) \exp \xi,$$

where ν is the principle exponent of $\mathbf{h}, \varepsilon > 0$ is arbitrary. □

Plan of the definition of $\mathcal{FC}^{1,r}$

This definition is given by induction in r. Base of induction and the step from 0 to 1 are already done.

Given the set $\mathcal{K}^{1,r}$ (coefficients of class 1 and rank r) we define the set $\mathcal{E}^{1,r}$ of STAR-(1,r) as the set of series of the form

$$\Sigma = a_j \exp \mathbf{e}_j, \mathbf{e}_j \in E^1, \; a_j \in \mathcal{K}^{1,r}.$$

Given the set $\mathcal{E}^{1,r}$ we define the set $\mathcal{F}^{1,r}$.

Given the set $\mathcal{F}^{1,r}$ we define the set $\mathcal{K}^{1.r+1}$, and thus complete the induction step.

Definition (plus-decomposability)

A sectorial cochain F is plus-decomposable in a STAR-(1,0) Σ provided that there exist a standard domain Ω of class 1 and $\varepsilon > 0$ such that for any $\nu > 0$ there exists a partial sum Σ_N of Σ that approximates $F \circ \exp$ with an accuracy $\exp(-\nu \exp \xi)$:

$$|F \circ \exp - \Sigma_N| \prec \exp(-\nu \exp \xi) \quad (4)$$

in the intersection $\ln \Omega \cap \mathbb{H}^{\varepsilon}$. Here $\mathbb{H}^{\varepsilon}$ is the ε-neighborhood of the upper half-plane.

The sets $\mathcal{K}^{1,r}, \mathcal{E}^1, \mathcal{FC}^1$

Definition

Set

$$\mathcal{K}^{1,r+1} = \mathcal{K}^{1,r} \cup_{\mu\in(0,1)} \mathcal{FC}^{1,r} \circ \exp \circ \mu. \quad (5)$$

Thus, by induction in r we defined the series and cochains of classes $\mathcal{E}^{1,r}$, $\mathcal{FC}^{1,r}$ respectively for all $r \in \mathbb{Z}^+$. Equalities

$$\mathcal{E}^1 = \cup_0^\infty \mathcal{E}^{1,r}, \ \mathcal{FC}^1 = \cup_0^\infty \mathcal{FC}^{1,r}$$

complete the definitions of series of class STAR-1 and cochains of class $\mathcal{FC}^1$.
This completes the definition of the class $\mathcal{FC}^1$, modulo the definition of index.

Index

A typical coefficient a of a STAR-1 is

$$a = F \circ \exp \circ \mu, \;\; \mu \in (0, 1), \;\; F \in FC^1.$$

The index of a partial sum of a STAR-1 takes care of all the μ's that occur in the coefficients of this sum. But some μ's are hidden in F.

Definition of the index

It is given by induction in r.

Definition

(base of induction) Let $F \in \mathcal{FC}^{1,0}$*. Then* $\mathit{ind}\ F = \emptyset$*.*

Definition

(induction step) Let ind be defined on $\mathcal{K}^{1,r}$*. Let us define it on* $\mathcal{E}^{1,r}$*. Let* $\Sigma = \Sigma a_j \exp \mathbf{e}_j,\ a_j \in \mathcal{K}^{1,r}$*. Then*

$$\mathit{ind}\ \Sigma = \cup \mathit{ind}\ a_j.$$

Definition of the index of a cochain

Definition

Let $F \in \mathcal{FC}^{1,r}$, *and* $\Sigma \in \mathcal{E}^{1,r}$ *be an asymptotic series for* $F \circ \exp$. *Then*

$$\text{ind } F = \text{ind } \Sigma.$$

Remark

An asymptotic series for $F \circ \exp$ *is not uniquely defined. So in fact we define an index of a pair* F, Σ. *Yet we write* ind F *for simplicity.*

Definition of the index for $\mathcal{K}^{1,r+1}$

Definition

Let $a \in \mathcal{K}^{1,r+1}$. Then either $a \in \mathcal{K}^{1,r}$ (and then ind a is defined by the induction assumption) or

$$a = F \circ \exp \circ \mu, \;\; F \in \mathcal{FC}^{1,r}, \;\; \mu \in (0,1).$$

Then

$$\textit{ind } a = (\textit{ind } F) \cdot \mu \cup \{\mu\},$$

This completes the induction step in the definition of the index.

Principle representation theorem

Theorem

Let Σ_N be a non-zero non-contructable partial sum of a STAR-1 with an index

$$\text{ind}\,\Sigma_N = (\mu_1, \dots, \mu_i),\ \ \mu_k \in (0,1),\ \ \mu_k \searrow,$$

Then Σ_N admits the following representation:

$$\Sigma_N = \exp \mathbf{h}_1(\exp \mathbf{h}_2(\dots \exp \mathbf{h}_{i+1}(a + R_{i+1}) \dots) + R_1),$$

where $a \in \mathcal{FC}_0$,

Theorem

$$\mathbf{h}_1 \in E^1,\ \mathbf{h}_k \in E^1 \circ \mu_{k-1}, \tag{6}$$

and either $\mathbf{h}_k \equiv 0$, *or*

$$\frac{\mathbf{h}_k}{\exp \circ \mu_k} \to \infty \text{ on } (\mathbb{R}^+, \infty),\ k = 1, \dots, i,$$

$$\frac{\mathbf{h}_{i+1}}{\xi} \to \infty \text{ on } (\mathbb{R}^+, \infty).$$

The reminder terms are partial sums of STAR-1 and satisfy the relations

$$|R_k| \preceq \exp(-\nu \exp \circ \mu_k) \forall\ \nu > 0 \text{ on } (\mathbb{R}^+, \infty),\ k = 1, \dots, i,$$

$$|R_{i+1}| \preceq \exp(-\nu\xi)\ \forall \nu > 0 \text{ on } (\mathbb{R}^+, \infty).$$

Lower estimate theorem LET

Theorem

Cochains of class $\mathcal{FC}^{1,0}$ admit the following lower estimate:

$$|F| \succ \exp(-C\xi) \text{ on } (\mathbb{R}^+, \infty)$$

for some $C > 0$.

Proof.

Let us take a partial sum of a STAR-1 Σ_N that approximates $F \circ \exp$. Suppose that

$$\text{ind } \Sigma_N = (\mu_1, ..., \mu_i).$$

By the previous theorem

$$\Sigma_N = \exp \mathbf{h}_1(\exp \mathbf{h}_2(\ldots \exp \mathbf{h}_{i+1}(a + R_{i+1}) \ldots) + R_1),$$

Base of induction

Proof.

Let

$$S_{k-1} = \exp \mathbf{h}_k(\exp \mathbf{h}_{k+1}(\dots \exp \mathbf{h}_{i+1}(a + R_{i+1}) \dots) + R_k),$$

Then $\Sigma_N = S_0$, $S_i = \exp \mathbf{h}_{i+1}(a + R_{i+1})$.
We will prove the theorem by the inverse induction in k:

$$|S_{k-1}| \succ \exp(-C \exp \circ \mu_{k-1}) \text{ on } (\mathbb{R}^+, \infty), \;\; \mu_0 = 1.$$

Base of induction: $|S_i| \succ \exp(-C \exp \circ \mu_i)$.

$$\mathbf{h}_{i+1} \in E^1 \circ \mu_i \Longrightarrow \mathbf{h}_{i+1} \succ -C \exp(\mu_i \xi) \Longrightarrow$$
$$|\exp \mathbf{h}_{i+1}(a + R_{i+1})| \succ \exp(-C \exp \circ \mu_i). \quad (7)$$

□

Induction step

Proof.

Induction assumption:

$$|S_k| \succ \exp(-C \exp \circ \mu_k) \text{ on } (\mathbb{R}^+, \infty). \tag{8}$$

We want to prove:

$$|S_{k-1}| \succ \exp(-C \exp \circ \mu_{k-1}) \text{ on } (\mathbb{R}^+, \infty). \tag{9}$$

We have:

$$S_{k-1} = \exp \mathbf{h}_k (S_k + R_k).$$

By (8) $R_k = o(S_k)$.
Together with relation $\mathbf{h}_k \in E^1 \circ \mu_{k-1}$ this implies (9). □

General outlook

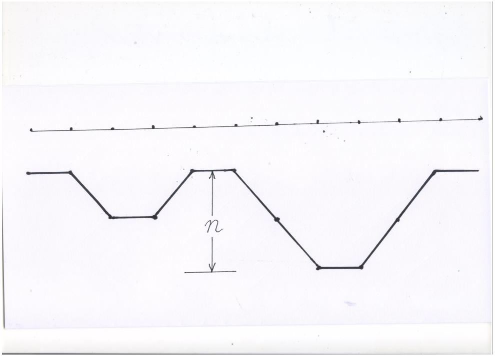

Figure 1: Graph of the characteristic of a composition