FINITENESS THEOREMS FOR LIMIT CYCLES

Yu. Ilyashenko

National Research University Higher School of Economy
Independent University of Moscow

PLAN

1. Recalling the Structural theorem
2. Classes $\mathcal{FC}^0$, $\mathcal{FC}^1$ and the Additive Decomposition theorem (ADT)
3. From ADT to the Finiteness theorem
4. Constructive definition of $\mathcal{FC}^0$
5. Phragmen-Lindelof theorem for $\mathcal{FC}^0$

Maps TO and FROM

The Dulac map for the real saddle-node TO the central manifold extended to the complex domain has the form:

$$\Delta = g \circ \Delta_{st} \circ F_{norm},$$

where F_{norm} is a normalizing cochain for the monodromy map of the saddle-node, Δ_{st} is the same as above, g is holomorphic germ at a fixed point 0, and $g'(0) > 0$.

$$FROM = TO^{-1}.$$

Structural theorem

Theorem

The monodromy map of a polycycle of an analytic vector field is a composition of the maps TO, FROM for real saddle-nodes, and of almost regular germs that are real on the real axis.

Characteristic of a composition

Definition

Characteristic of a composition Δ *above is a continuous function* χ *on a segment* $[-N, 0]$ *which is linear between two subsequent integers,* $\chi(0) = 0$*, and*

$$\chi(-j) = \chi(-j+1) \text{ for } \Delta_j \in R,$$

$$\chi(-j) = \chi(-j+1) - 1 \text{ for } \Delta_j \in TO,$$

$$\chi(-j) = \chi(-j+1) + 1 \text{ for } \Delta_j \in FROM.$$

The composition Δ is balanced iff $\chi(-N) = \chi(0) = 0$.

Graph of the characteristic of a composition

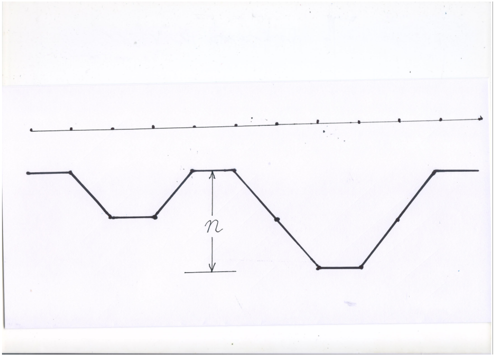


Figure 1: Graph of the characteristic of a composition

Transition to the logarithmic chart $\zeta = -\ln z$

	Mapping in a natural chart	The same mapping in the logarithmic chart
1	Power: $z \mapsto Cz^{\nu}$	Affine: $\zeta \mapsto \nu\zeta - \ln C$
2	Standard flat: $z \mapsto \exp(-1/z)$	Exponential: $\zeta \mapsto \exp \zeta$
3	A mapping defined in a sector with vertex 0 and expandable in a convergent or asymptotic Taylor series $\hat{f} = z(1 + \sum_1^{\infty} a_j z^j)$	A mapping defined in a horizontal half-strip and expandable in a convergent or asymptotic Dulac (exponential) series $\tilde{f} = \zeta + \sum_1^{\infty} b_j \exp(-j\zeta)$

Transition to the logarithmic chart (continued)

4	$h_{k,a}\colon z \mapsto kz^k(1 - az^{-k}\ln z)^{-1}$	$\tilde h_{k,a}\colon \zeta \mapsto k\zeta - \ln k - \ln(1 - a\zeta\exp(-k\zeta))$
5	An almost regular mapping with asymptotic Dulac series at zero $z \mapsto Cz^\nu + \sum P_j(z)z^{-\nu_j}$, where $C > 0$, $\nu > 0$, $0 < \nu_j \nearrow \infty$, and and the P_j are real polynomials	An almost regular mapping with asymptotic Dulac exponential series at infinity $\zeta \mapsto \nu\zeta - \ln C + \sum Q_j(\zeta)\cdot\exp(-\mu_j\zeta)$ where $C > 0$, $\nu > 0$, $0 < \mu_j \nearrow \infty$, and the Q_j are real polynomials

Normalizing cochains in the logarithmic chart

Upon transition to the logarithmic chart the normalizing cochain F_{norm} becomes a map-cochain defined in a half-plane $\mathbb{C}_a^+ : \xi \geq a$; a depends on the cochain.

1. *Partition* The corresponding partition is a partition of $\mathbb{C}_a^+$ into half-strips by the rays $\eta = \pi m/k$, $m \in \mathbb{Z}$, $\xi > a$. This partition for $k = 1$ is called *standard* and denoted by Ξ_{st}.

2. *Extension* The components of the map-cochain extend analytically to the ε-neighborhoods of the corresponding half-strips in the partition for arbitrary $\varepsilon \in (0, \pi/2k)$ (a depends also on ε).

Normalizing cochains in the logarithmic chart

3. *Growth* These components have an exponentially decreasing correction (difference with the identity).

4. *Coboundary* The modulus of the coboundary has the upper estimate

$$C \exp(-C' \exp k\xi)$$

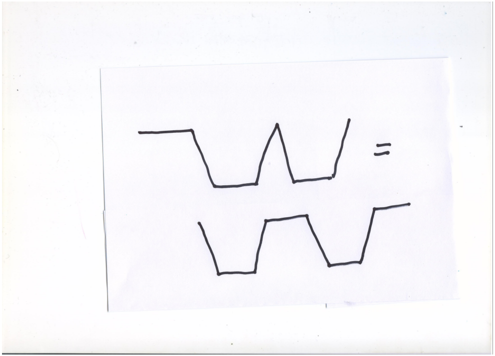
for some $C, C' > 0$ depending on the cochain.

5. *Decomposition* The mappings making up F_{norm} (components of F) can be expanded in a common asymptotic Dulac exponential series; see row 5 of the table.

Classes $\mathcal{FC}^0$ and $\mathcal{FC}^1$

We extend class of normalizing cochains to a wider class of simple cochains denoted by $\mathcal{FC}^0$.

We also extend class of almost regular germs to a much wider class of sectorial cochains denoted by $\mathcal{FC}^1$ so that the following transposition becomes possible:



Lower Estimate theorem (LET)

Theorem

Cochains from $\mathcal{FC}^0$ *and* $\mathcal{FC}^1$ *admit the following lower estimate:*

$$|ReF| \succ \exp(-\varkappa\xi)$$

in a standard domain for some $\varkappa > 0$*, provided that* $F^u \not\equiv 0$*.*

Additive Decomposition Theorem (ADT)

These extensions will allow us to decompose the monodromy map Δ of an alternant polycycle as the following theorem claims.

Theorem

(ADT)

$$\Delta = a + F_0 + \Sigma_1^N F_j \circ \exp \circ \mu_j \qquad (1)$$

where a is real affine, all F's are cochains exponentially decreasing at infinity,

$$0 < \mu_j \nearrow, \ F_0 \in \mathcal{FC}^0, \ F_j \in \mathcal{FC}^1.$$

ADT implies Non-accumulation Theorem

Theorem

Either $\Delta = id$, *or*

$$\Delta \notin Fix_\infty \tag{2}$$

Proof.

Case 1: $a \neq \mathrm{id}$ in (1)
Then

$$|a - \mathrm{id}| > C \neq 0, \;\; \Delta - a \to 0.$$

Hence, (2) holds.
Case 2: $a = \mathrm{id}, F_0 \not\equiv 0$ in (1).
Then $|F_0| \succ \exp(-\varkappa\xi)$ on $(\mathbb{R}^+, \infty)$, and

$$|\Delta - a - F_0| \prec \exp(\nu\xi) \forall \nu > 0 \text{ on } (\mathbb{R}^+, \infty).$$

Again, (2) holds. □

ADT implies Non-accumulation Theorem

Proof.

Case 3: $a = \mathrm{id}, F_0 \equiv 0$ in (1), but $\Delta \not\equiv \mathrm{id}$.
Then
$$|F_1 \circ \exp \circ \mu_1| \succ \exp(-\varkappa \exp \mu_1 \xi).$$
Then, for some $\lambda > 0$,
$$|\Delta - \zeta - F_1 \circ \exp \mu_1 \xi)| \prec \exp(-\lambda \exp \mu_2 \xi) = o(F_1 \circ \exp \mu_1 \xi).$$
□

Definition of $\mathcal{FC}^0$

These cochais are similar to the normalizing ones.

1. *Partition*

Definition

A simple partition of the type

$$\sigma = (\mu_1, \dots, \mu_N), \ \mu_1 > \mu_2 \dots > \mu_N > 0 \tag{3}$$

is a product of images of the standard partition:

$$\Xi_\sigma = \prod_1^N \mu_{j*} \Xi_{st}.$$

2. *Extension* The components of the cochain extend analytically to the ε-neighborhoods of the corresponding half-strips of the partition for some $\varepsilon >$.

Definition of $\mathcal{FC}^0$

3. *Growth* All the components of F grow no faster than some exponent

4. *Coboundary* Recall that the boundary of the partition Ξ is the union of the boundaries of all its domains; notation: $\partial\Xi$.

Definition

Consider a quadratic standard domain and its simple partition of type (3). *A rigging cochain* $m_{\varepsilon,C,C'}$ *is defined in the generalized* ε*-neighborhood of the boundary* $\partial\Sigma_\sigma$*, and in a generalized* ε*-neighborhood of a boundary curve* $\mathcal{L}$*, it takes the form:*

$$m_{\varepsilon,C,C'} = \sum C \exp(-C' \exp \mu_j^{-1}\xi), \tag{4}$$

the summation is over those j *for which* $\mathcal{L} \subset \mu_j \partial\Xi_{st}$.

Definition of $\mathcal{FC}^0$

The coboundary of F admits an upper estimate by some rigging cochain (4): $\exists\ \varepsilon, C, C'$ such that $|\delta F| < m_{\varepsilon,C,C'}$ in the generalized ε-neighborhoods of $\partial \Xi_\sigma$.

5. *Decomposition* The mappings making up F_{norm} can be expanded in a common asymptotic Dulac exponential series; see row 5 of the table.

This is a sketch; an actual definition contains more details.

Genesis

We consider $F \in \mathcal{NC}$, $a = \alpha\zeta$ affine, and a composition $F \circ a$. This composition corresponds to a partition $\alpha^{-1}\Xi_{st}$. We want to add such cochains for different $\alpha' s$, and thus get products of partitions. The rigging cochain for the partition $\alpha^{-1}\Xi_{st}$ is the pullback of the rigging cochain for the standard partition by the map a^{-1}.

Sectorial cochains

These cochains are defined in the similar way, but the partition has the form

$$\Xi = \prod_1^N \mathsf{exp} \circ \mu_{j*} \Xi_{\mathsf{st}}.$$

The rigging cochains are the sums of the pullbacks of the rigging cochain for a standard partition. The decomposition part is much more complicated, and will be delivered in the next lecture. Here the Super Exact Asymptotic Series occur.

Phragmen-Lindelof theorem for $\mathcal{FC}^0$

Theorem

If a cochain $F \in \mathcal{FC}^0$ decreases on $(\mathbb{R}^+, \infty)$ faster than any exponent, then $F^u \equiv 0$ (F^u is the component of F that is defined in the strip of the corresponding partition that is adjacent to $(\mathbb{R}^+, \infty)$ from above).

Lower Estimate theorem for $\mathcal{F}\mathcal{C}^0$

Proof.

The proof is similar to that for the almost regular germs. The cochain F corresponds to an exponential Dulac series Σ in some quadratic standard domain Ω:

$$\Sigma = \Sigma_1^\infty P_j(\zeta) \exp \nu_j \zeta, \ \ \nu_j \searrow -\infty.$$

For any $\nu > 0$ there exists N such that

$$\Sigma_N = \Sigma_1^N P_j(\zeta) \exp \nu_j \zeta,$$

and

$$|F - \Sigma_N| \prec \exp(-\nu \xi)$$

in Ω.

If $\Sigma \equiv 0$ then by Phragmen-Lindelof theorem $F^u \equiv 0$, a contradiction. □

Phragmen-Lindelof theorem for two quadrants

Proof.

If $\Sigma \not\equiv 0$ then

$$F = P_1(\zeta)(\exp \nu_1 \zeta)(1 + o(1).$$

This cochain does not oscillate. □

Theorem

If a holomorphic function f defined in $\mathbb{C}^+$ increases no faster than an exponential $\exp \nu\xi, \nu > 0$, and is bounded on the positive semiaxis of the real axis, and on the imaginary axis, then it is bounded in $\mathbb{C}^+$, and

$$\sup_{\mathbb{C}^+} = \sup_{\partial\mathbb{C}^+}.$$

Phragmen-Lindelof theorem for decreasing holomorphis functions

Theorem

If a holomorphic function f defined in $\mathbb{C}^+$ increases no faster than an exponential $\exp \nu\xi, \nu > 0$, is bounded on the imaginary axis and tends to zero faster than any expronent on the positive semiaxis of the real axis, then $f \equiv 0$.

Proof.

Take $\zeta_0 \in \mathbb{C}^+, \xi_0 = Re\zeta_0 \neq 0, f(\zeta_0) \neq 0$. Consider

$$f_\lambda = f \exp \lambda\zeta, \lambda > 0$$

This function is bounded on $i\mathbb{R} \cup \mathbb{R}^+$. Hence,

$$|f(\zeta) \exp \lambda\zeta| \leq \sup_{i\mathbb{R}} |f| = M.$$

Proof.

Hence,

$$|f(\zeta_0)| < M \exp(-\lambda\xi_0) \forall\lambda > 0,$$

a contradiction. □

$\mathcal{NC}$ is a class of simple cochains corresponding to the standard partition.

Theorem

If a cochain F of class $\mathcal{NC}$ decreases on $\mathbb{R}^+$ faster than any exponential $\exp(-\nu\xi)$, $\nu > 0$, then $F^u \equiv 0$.

Trivialization of a cocycle

Lemma

Let F be an ε -extendable cochain of class $\mathcal{NC}$ defined in an ε-neighborhood Ω^ε of a standard domain Ω. Let $\Xi^{F,\varepsilon}$ (Ξ^F) be the partition of Ω^ε (respectively, Ω) that corresponds to this cochain . Let the coboundary of F be estimated from above by $m = \exp(-C \exp \xi)$, and

$$m_0 = \sup_{\Omega^\varepsilon} m, \qquad \int_{\partial \Xi^{F,\varepsilon}} m\, ds = I < \infty.$$

Then there exists an ε–extendable functional cochain Φ defined in Ω, such that

$$\delta F = \delta \Phi \qquad \max_{\Omega} |\Phi| \leq C\varepsilon^{-1}(m_0 + I).$$

Maximum modulo principle for cochains

Lemma

In assumptions of the previous lemma, let Ω be a standard domain of class 1, F be a simple or sectorial or rotated sectorial cochain that grows no faster than the exponent $\exp \nu\xi$ in Ω and is bounded on $\partial\Omega$ and on a positive real axis. Then F is bounded in Ω, and

$$\sup_{\Omega} |F| \leq \sup_{\partial\Omega} |F| + 2C\varepsilon^{-1}(m_0 + I).$$

where C, ε, m_0, I are the same as in the previous lemma.

Preliminary estimate

Lemma

Let F be a cochain of the class $\mathcal{NC}$ defined in a standard domain Ω , that decreases on $\mathbb{R}^+$ faster than any exponential. Then for any sector S_α: $|\arg\zeta| < \alpha < \pi/2$ and any $\delta > 0$ there exists $C > 0$ such that for any $\zeta \in S_\alpha$ the following estimate holds:

$$|F(\zeta)| < \exp(-C\exp(1-\delta)\xi).$$

The classical Phragmen-Lindelof theorem for a halfstripe now implies the Phragmen-Lindelof theorem for the cochains of the class $\mathcal{NC}$.