# FINITENESS THEOREMS FOR LIMIT CYCLES

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# PLAN

1. Hilbert 16th problem
2. Non-accumulation theorem for hyperbolic polycycles. Almost regular maps
3. Singular points of real planar vector fields
4. Complex saddle-nodes and parabolic germs
5. Structural theorem

# Limit cycles

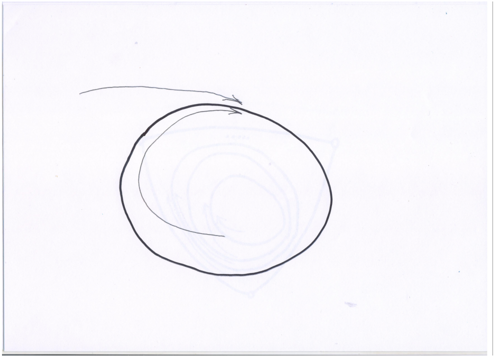

Figure 1: Limit cycle

**Definition**

*Limit cycle is an isolated closed orbit*

# Hilbert 16th problem

## Explicit estimate

What may be said about the number and location of limit cycles of a polynomial vector field of degree $n$ in the plane?

There are two simplifications of this problem

## Uniform estimate

Does there exist an upper bound of the number $H(n)$ of limit cycles of a polynomial vector field of degree $n$?

Even the existence of $H(2)$ is not yet proved. A vast program was initiated by Dumortier, Roussarie and Rousseau; it is largely pushed forward, but not completed

## Finiteness problem

Is it correct that the number of limit cycles of a polynomial vector field of degree $n$ is finite?

# Non-accumulation theorem

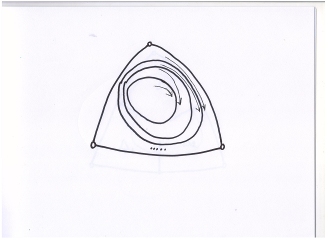

Figure 2: This is possible for the $C^\infty$ case and impossible for $C^\omega$

**Theorem**

*Limit cycles of an analytic vector field cannot accumulate to a polycycle of this field.*

It is easy to prove that this theorem is equivalent to the positive answer to the last question.

# Monodromy and Dulac maps

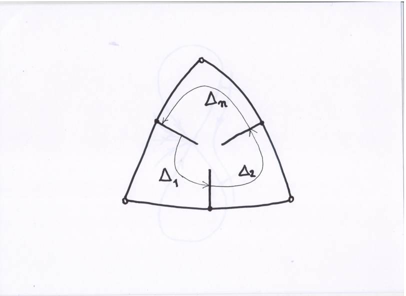


Figure 3: The monodromy map is a product of the Dulac maps

**Definition**

*The monodromy map of a polycycle is the first return map (analog of the Poincare map). The Dulac map is a map of a cross-section to one separatrix of a hyperbolic sector of a singular point to a cross-section to another one.*

# Semi-regular germs and Dulac's theorem

## Definition

*A Dulac series is a formal series of the form*

$$\sigma = cx^{\nu_0} + \sum_{1}^{\infty} P_j(\ln x)x^{\nu_j},$$

*where* $c > 0$, $0 < \nu_0 < \cdots < \nu_j < \cdots$, $\nu_j \to \infty$, *and the* $P_j$ *are polynomials.*

## Definition

*The germ of a mapping* $f : (\mathcal{R}^+, 0) \to (\mathcal{R}^+, 0)$ *is said to be semiregular if it can be expanded in an asymptotic Dulac series. In other words, for any* $N$ *there exists a partial sum* $S$ *of the above series such that* $f(x) - S(x) = o(x^N)$.

Examples.

1. $\Delta$ is a $C^\infty$ diffeo.
2. $\Delta(x) = x^\lambda$.
3. $\Delta = h_{1,1} = \frac{x}{1 - x \ln x}$.
4. Dulac maps of the hyperbolic saddles.

A composition of the semiregular maps is semi-regular again. Hence, the monodromy map of a hyperbolic polycycle is semi-regular.

**Theorem**

*(Dulac) A monodromy map of a polycycle of an analytic vector field is either flat, or inverse to flat, or semi-regular*

# Dulac's lemma and mistake

**Lemma**

*(Dulac) A non-identical germ of a semi-regular map cannot have fixed points near zero.*

**Proof**

$$\Delta(x) - x = P(\ln x)x^{\nu}(1 + o(1))$$

The r.h.s. does not oscillate.

Error: the previous formula is wrong if $\Delta(x) - x$ decreases exponentially. Yet $\Delta$ is semiregular, and its Dulac's series is simply $x$.

# Small discrepancy theorem

**Theorem**

*For any n there exists an analytic vector field with a polycycle whose monodromy map has the form*

$$\Delta = id + R, \ \ R = o(exp(-exp^{[n]}), R \not\equiv 0$$

*in the log coordinate* $\xi = -\ln x$.

A counterexample like this is impossible in the hyperbolic case.

# Going into the complex domain: almost regular germs in the logarithmic chart $\zeta = -\ln z$.

**Theorem**

*Limit cycles of an analytic vector field cannot accumulate to a hyperbolic polycycle of this field.*

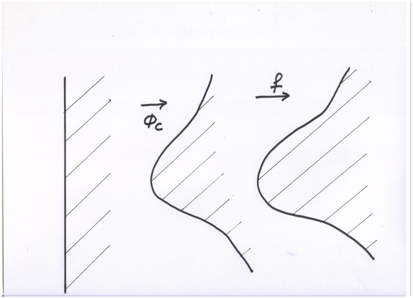


Figure 4: Quadratic standard domain and almost regular map

# Almost regular mappings

**Definition**

*A Dulac exponential series is a formal series of the form*

$$\Sigma = \nu_0\zeta + c + \sum P_j(\zeta) \exp \nu_j \zeta,$$

*where $\nu_0 > 0$, $c \in \mathbb{R}$, $0 > \nu_j \searrow -\infty$, and the $P_j$ are real polynomials; the arrow $\searrow$ means monotonically decreasing convergence.*

A quadratic standard domain is an image of the right half-plane with a disc $|z| < C$ deleted under a map $\Phi_C = \zeta + \frac{C}{\sqrt{\zeta}}$

## Definition

*An almost regular mapping is a holomorphic mapping of some quadratic standard domain $\Omega$ in $\mathbb{C}$ that is real on $\mathbb{R}^+$ and can be expanded in this domain as an asymptotic real Dulac exponential series. Expandability means that for any $\nu > 0$ there exists a partial sum approximating the mapping to within $o(\exp(-\nu\xi))$ in $\Omega$.*

## Theorem

*The Dulac mapping of a hyperbolic saddle, written in a logarithmic chart, extends to an almost regular mapping in some quadratic domain.*

Almost regular germs at infinity form a group with the operation a composition.

# Phragmen-Lindelöf theorem

**Theorem**

*If a function $g$ is holomorphic and bounded in the right half-plane and decreases on $(\mathbb{R}^+, \infty)$ more rapidly than any exponential* $\exp(-\nu\xi)$, $\nu > 0$, *then* $g \equiv 0$. *The same holds for almost regular mappings.*

**Theorem**

*An almost regular mapping is uniquely determined by its asymptotic series. In particular, an almost regular mapping with asymptotic series $\zeta$ is the identity.*

# Non-accumulation theorem for hypebolic polycycles

This implies the finiteness theorem for hyperbolic polycycles.

**Proof**

The monodromy map of a hyperbolic polycycle of an analytic vector field is almost regular. By the Phragmen-Lindelof theorem , if its Dulac series is $x$, then the map is identity. But if the Dulac series is not $x$, then the map has no fixed points near zero by the Dulac lemma.

# Conclusion

In the general case the monodromy map of a polycycle should be extended to the complex domain, and sort of Phragmen-Lindelof theorem should be proved for it. On the other hand, this map should be decomposed into an asymptotic series whose terms do not oscillate. This is a very robust scheme of the general investigation.

# Dulac's idea that looked crazy

Dulac idea: study the correspondence maps and their compositions.

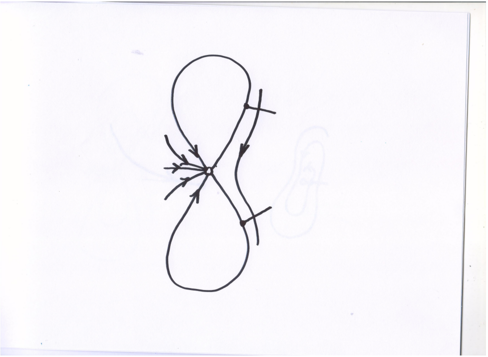

Figure 5: Dulac map of a hyperbolic sector of a complex singular point

# Monodromic and characteristic singular points

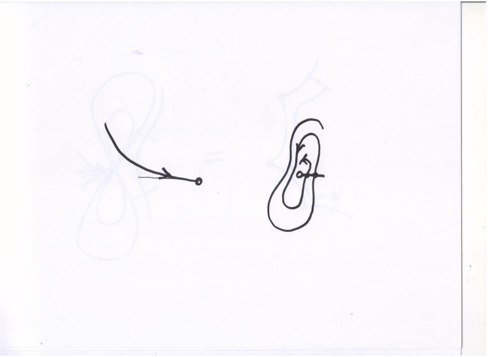

Figure 6: Characteristic and monodromic singular points

Characteristic singular points are studied quite well. Study of monodromic singular points is usually called "center – focus problem". A better name is 'distinguishing stable and unstable foci''. This problem was solved recently by late N.Medvedeva.

# Desingularization theorem

## Definition

*An elementary singular point of a planar vector field is a point with at least one non-zero eigenvalue.*

Ckassification for smooth vector fields: saddles, foci, centers by linear terms, nodes, saddlenodes

## Theorem

*(Bendixson - Lefshetz - Seidenberg - Dumortier...) An isolated singular point of an analytic vector field may be split to a finite number of elementary ones by a finite number of blow ups. Same for $C^\infty$ vector fields with the Loyasievic condition:* $|v(x)| > c|x^\lambda|$ *for some* $C, \lambda > 0$.

# Desingularization: corollary

**Corrolary**

*The Dulac map for a hyperbolic sector of a complex singular point equals a composition of the Dulac maps for saddle-nodes and hyperbolic saddles.*

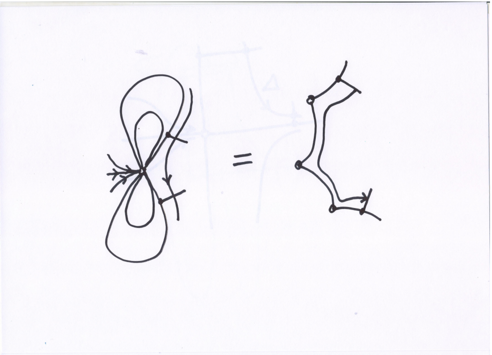

Figure 7: Desingularization of a Dulac map

# Real saddle-nodes: $\lambda_1 = 0, \lambda_2 \neq 0$

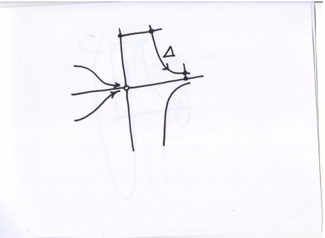


Figure 8: Phase portrait of a real saddle-node

Example: $\dot{x} = x^2, \dot{y} = -y$.
The Dulac map TO the center manifold is $C^\infty$ equivalent to

$$\Delta_{\text{st}} = \exp(-1/h_{k,a}(z)), \text{ where } h_{k,a}(z) = kz^k/(1 - akz^k \ln z).$$

The factor $\exp(-\frac{1}{x})$ gives rise to exponentially decreasing discrepancies of the monodromy maps of non-hyperbolic polycycles.

# Complex saddle-nodes

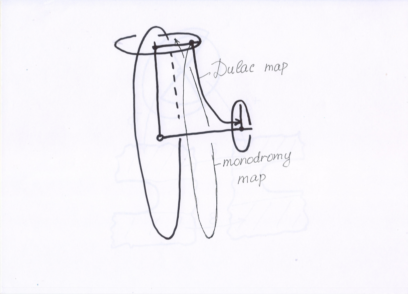


Figure 9: Complex saddle-node, its Dulac and monodromy maps

Complex saddle-node has a holomorphic invariant stable manifold, and in general, no holomorphic invariant center manifold. Circuite of zero on the stable manifold generates a monodromy map which is a parabolic germ $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0),\ f'(0) = 1.$

# Normalizing cochains for parabolic germs

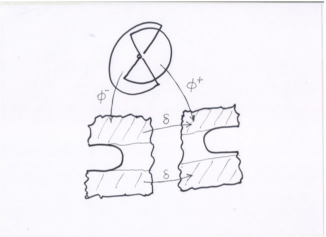


Figure 10: A normalizing cochain for a parabolic germ

The maps $\Phi^{\pm}$ are so called Fatou coordinates. They conjugate $f$ with the shift by 1: $\Phi^{\pm} \circ f = \Phi^{\pm} + 1$. The transition map $\Xi = \Phi^{+} \circ \Psi^{-}$ commutes with the shift by 1. Its difference with identity is one-periodic, and thus decreases exponentially at infinity. The map $\Xi$ is *Ecalle-Voronin modulus.* The tuple $\Phi^{\pm}$ forms a normalizing cochain $F_{norm}$ for $f$, and $\Xi$ is its *coboundary.*

# Maps TO and FROM

The Dulac map for the real saddle-node TO the central manifold extended to the complex domain has the form:

$$\Delta = g \circ \Delta_{st} \circ F_{norm},$$

where $F_{norm}$ is a normalizing cochain for the monodromy map of the saddle-node, $\Delta_{st}$ is the same as above, $g$ is holomorphic germ at a fixed point 0, and $g'(0) > 0$.

$$FROM = TO^{-1}.$$

# Structural theorem

**Theorem**

*The monodromy map of a polycycle of an analytic vector field is a composition of the maps TO, FROM for real saddle-nodes, and of almost regular germs that are real on the real axis.*