About Dulac's Problem in $\mathbb{R}^3$ for perturbations of linear non-degenerated centers

Geometry and Model Theory Seminar

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FIELDS

Contents

Fixing notation

- ξ denotes a **real analytic vector field** defined in some analytic manifold M.
- $\hat{\xi}$ will denote a formal vector field.
- In this work, we study the **periodic orbits** or **cycles** of ξ.
- A **limit cycle** is a cycle isolated from others.

Introduction to Dulac's problem

The Dulac problem is understood as a local version of the Hilbert's 16$^{\text{th}}$ Problem.

Hilbert's 16$^{\text{th}}$ Problem

Let ξ be a polynomial vector field in dimension 2.

1. ξ has a finite number of limit cycles.
2. This number is uniformly bounded by a function of the degree.

Dulac's problem

Let ξ be an analytic vector field in dimension 2 and p such that $\xi(p) = 0$. Then there is a finite number of limit cycles in a neighborhood of p.

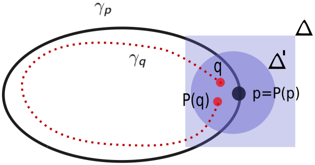
Dulac gave a positive answer for this assertion, which was proven to be inaccurate in the 1980's. Ilyashenko and Ecalle proved it independtly in the 1990's. As a consequence:

Let ξ be an analytic field in dimension 2. Then, either ξ has a finite number of cycles around a singularity p, or it is a center.

Poincaré First Return map

Poincaré First Return map

Let γ be a cycle of ξ in M. Then, for every transverse hyperplane Δ, there exists $\Delta' \subseteq \Delta$ and an analytic map $P : \Delta' \longrightarrow \Delta$, such that $P(q)$ is defined by the first intersection of the orbit starting at q with Δ.



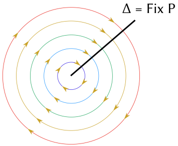
If $\dim M = 2$, we have a **bijection** between $\mathrm{Fix}(P)$ and the cycles of ξ that intersect Δ'.

In other case, we have that cycles define **periodic points**, which are fixed points of some P^n.

An example in $\mathbb{R}^2$

In dimension 2, there are only two possible behaviours of the vector field around a singularity if a Poincaré type map is defined.

- A **center** configuration is achieved when the Poincaré map is the identity.



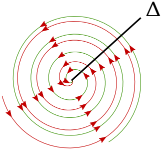
Example

One simple example in $\mathbb{R}^2$ is:

$$\xi = -by\frac{\partial}{\partial x} + bx\frac{\partial}{\partial y}, \quad b \neq 0,$$

which can be called a **linear non-degenerated center**.

- On the other hand, if in some neighborhood of 0, P is defined but $\mathrm{Fix}P = \emptyset$, we have a **focus** configuration, that is, trajectories spiral around the singularity.



To distinguish between these two possible situations is known as the **center-focus problem**, which is not solvable studying a finite $k-$jet.

An example in $\mathbb{R}^3$

Example

$$\xi = -by\frac{\partial}{\partial x} + bx\frac{\partial}{\partial y} + cz\frac{\partial}{\partial z}, \quad b \in \mathbb{R}\setminus\{0\}, c \in \mathbb{R}.$$

The distribution of cycles can be characterized studying the sign of c:

- A surface composed of cycles ($\{z = 0\}$) if $c \neq 0$.
- $\mathbb{R}^3 \setminus \{x = y = 0\}$ filled with cycles if $c = 0$.

Question

How are cycles distributed for perturbations of these vector fields?

Contents

Perturbations of linear non-degenerated centers

The study of the distribution of cycles in $\mathbb{R}^3$ is a wide problem, therefore, we will study vector fields whose linear part is fixed.

Perturbations of linear non-degenerated centers

We say that a vector field ξ is a *perturbation of a linear non-degenerated center* if

$$\xi = (-by + A_1(x,y,z))\frac{\partial}{\partial x} + (bx + A_2(x,y,z))\frac{\partial}{\partial y} + (cz + A_3(x,y,z))\frac{\partial}{\partial z},$$

where $b \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}$ and A_i is an analytic function of order greater or equal to 2 for $i = 1, 2, 3$.

Notice that these vector fields are perturbations of the two dimensional linear non-degenerated centers.
The existence and distribution of cycles is an unstable property, that is, small perturbations can completely change the structure of the cycles.
$\sim$ Center-focus problem

The local sets of cycles

As it can be the case in $\mathbb{R}^2$, the local study of cycles around a singularity depends on the set chosen. We will work with neighborhoods U of $0 \in \mathbb{R}^3$.

$$\mathcal{C}_U := \bigcup_{\substack{C \text{ is a cycle of } \xi \\ \text{contained in } U}} C$$

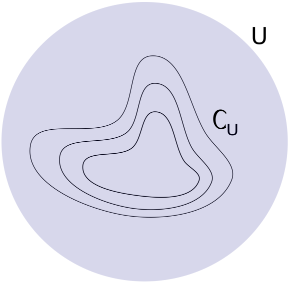
Remark

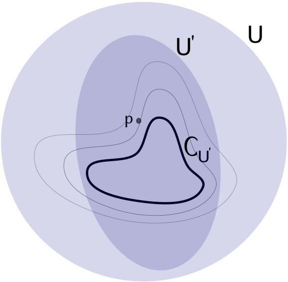
The sets $\mathcal{C}_U$ do not behave like germs because, if $U' \subset U$, we can find $\mathcal{C}_{U'} \subsetneq \mathcal{C}_U \cap U'$.

The local sets of cycles

Example

Take U and U' as in the following figures. The point p is in $U' \cap \mathcal{C}_U$. Nevertheless, $p \notin \mathcal{C}_{U'}$



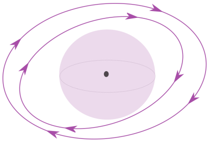


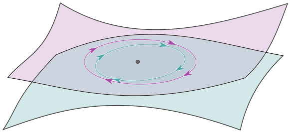
Dulac's property

Dulac's property

Let $\xi \in \mathfrak{X}^\omega(\mathbb{R}^3, 0)$, we say that ξ fulfills *Dulac's Property* if one of the following situations holds:

- There exists some open neighborhood V of 0 such that $\mathcal{C}_V = \emptyset$.
- There exists some open neighborhood U of 0 and a finite number of invariant topological surfaces S_i for $i = 1, \ldots, m$ with $S_i \cap S_j = \{0\}$ for $i \neq j$, such that $\overline{\mathcal{C}_U} = S_1 \cup \cdots \cup S_m$.





Dulac's property for perturbations of linear non-degenerated centers

The objective of the talk is to prove the following result:

Theorem [Corral, Sanz, M.]

Let ξ be a perturbation of a linear non-degenerated center with isolated singularity. Then, ξ fulfills Dulac's property.

A consequence of this theorem is that the number of limit cycles around the singularity is finite.

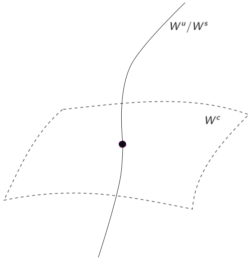
Contents

Case 1. $c \neq 0$

The singularity in 0 is **semi-hyperbolic**. By the theory of invariant manifolds:

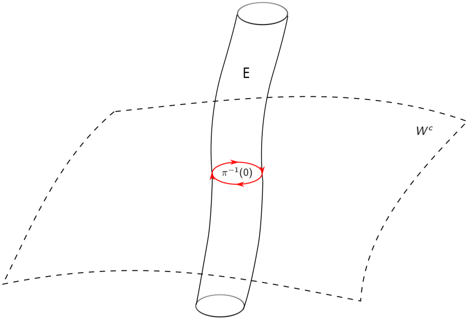
- An analytic invariant one-dimensional unstable manifold W^u ($c > 0$) or stable manifold W^s ($c < 0$).
- For every $r \in \mathbb{N}$, a $\mathcal{C}^r$ center manifold W_r^c of dimension 2 defined in some open neighborhood V_r of 0.
- Moreover, each W_r^c contains all the trajectories globally contained in V_r. As a consequence:

$$\boxed{\mathcal{C}_{V_r} \subseteq W_r^c}$$



Blowing-up of the (un)stable manifold

- We can perform a blowing-up π with center the (un)stable manifold because it is an invariant analytic submanifold.
- Inside the divisor E, $\pi^{-1}(0)$ is a periodic orbit.
- In fact, $\pi^{-1}(0) = E \cap W_r^c$ for all $r \in \mathbb{N}$.



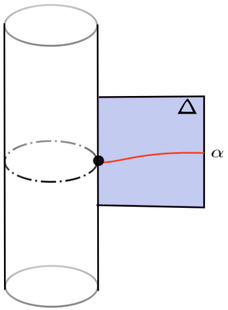
Definition and analyticity of the 1^{st} return of Poincare

Fixing $W^c = W^c_r$, $U = V_r$ and a plane Δ transverse to it.

- As we find a closed orbit in the divisor, there is $\Delta' \subseteq \Delta$ where the Poincaré map $P : \Delta' \longrightarrow \Delta$ is defined.
- P is analytic.
- Therefore, Fix$P := \{P(q) = q\}$ is by definition an analytic set.
- On the other hand, $\mathcal{C}_U \subseteq W^c$ and $\dim W^c = 2$.

$$\Rightarrow \text{Fix}P \subseteq \alpha := W^c \cap \Delta.$$

- As $\dim W^c = 2$, there is a bijection between Fix$P$ and the periodic orbits.



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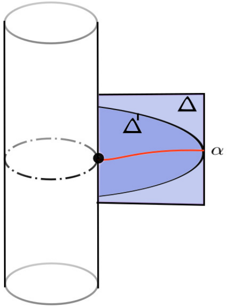
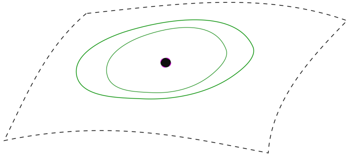
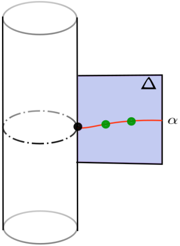


Fulfillment of Dulac's property

Now, there are two cases:

- **There is a finite number of cycles in a neighborhood of the singularity.** In this case we directly have Dulac's property, there is some U such that $\mathcal{C}_U = \emptyset$.





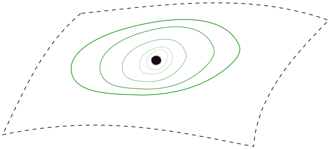
Fulfillment of Dulac's property

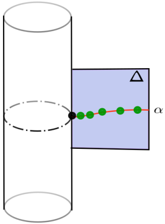
Now, there are two cases:

ii **There is a family of cycles $\{\gamma_i\}_i$ that accumulates to the singularity.**

- These cycles give a family of points $\{p_i\}_i \subset \mathrm{Fix}P$ that accumulate to the singularity.
- As $\mathrm{Fix}P \subseteq \alpha$ is analytic, we have that it must be a curve and therefore $\mathrm{Fix}P = \alpha$.
- The continuum of fixed points also gives a continuum of periodic orbits.

Then, we have the second case of Dulac's property, and $W^c = \mathcal{C}_U \cup \{0\}$.





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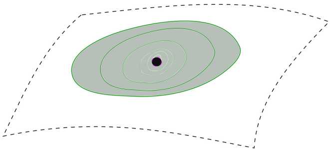
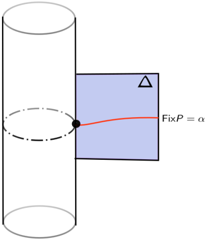




Conclusions for the semihyperbolic case

In case there is a continuum of orbits. Since the previous results are valid for any choice of W_r^c with $r \in \mathbb{N}$ and plane Δ, we have that:

- $W_{r_1}^c = W_{r_2}^c$ in $V_{r_1} \cap V_{r_2}$ for any $r_1, r_2 \in \mathbb{N}$.
- As $\pi^{-1}(W^c) = \tilde{W}^c$ is locally analytic in every point in $W^c \cap E$, we have analycity of $W^c \setminus \{0\}$.

A vector field ξ that is a perturbation of a semi-hyperbolic non-degenerated center fulfills Dulac's property.

Case 2. $c = 0$

The vector field has isolated singularity, so it can be written:

$$X = (-by + A(x,y,z))\frac{\partial}{\partial x} + (bx + B(x,y,z))\frac{\partial}{\partial y} + (c_\nu z^\nu + C(x,y,z))\frac{\partial}{\partial z},$$

where $\nu > 1$ indicates that there is minimum order such that z^μ appears in $\dot{z}$. We can also assume that $b = 1$.

Taken's normal form theorem gives formal coordinates $\hat{x}, \hat{y}, \hat{z}$ of $\mathbb{R}^3$ such that the transformed vector field is written in the following **formal normal form**.

$$\hat{\xi} = \left(\sum_{i=1}^{\infty}\sum_{i=2r+j} a_{irj}(\hat{x}^2+\hat{y}^2)^r\hat{z}^j\right)\cdot\left(\hat{x}\frac{\partial}{\partial\hat{x}} + \hat{y}\frac{\partial}{\partial\hat{y}}\right)$$
$$+\left(1+\sum_{i=1}^{\infty}\sum_{i=2r+j} b_{irj}(\hat{x}^2+\hat{y}^2)^r\hat{z}^j\right)\cdot\left(-\hat{y}\frac{\partial}{\partial\hat{x}} + \hat{x}\frac{\partial}{\partial\hat{y}}\right)$$
$$+\left(\sum_{i=2}^{\infty}\sum_{i=2r+j} c_{irj}(\hat{x}^2+\hat{y}^2)^r\hat{z}^j\right)\frac{\partial}{\partial\hat{z}}$$

This theorem also gives an **analytic normal form** ξ_ℓ for every $\ell < \infty$ in coordinates x_ℓ, y_ℓ, z_ℓ whose $\ell-$jet coincides with the formal $\ell-$jet.

Strategy

The strategy of the work will be the following:

1. We study the problem using the formal normal form as a guide.
2. We prove that the arguments are valid in the analytic case, considering a normal form with a sufficiently large $\ell-$jet.

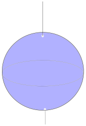
$$
\begin{tikzcd}
 & (\mathbb{R}^3_{\{\hat{x},\hat{y},\hat{z}\}},0) \\
(\mathbb{R}^3_{\{x,y,z\}},0) \arrow[ur, "\hat{\varphi}"] \arrow[dr, "\varphi_l"'] & \\
 & (\mathbb{R}^3_{\{x_l,y_l,z_l\}},0)
\end{tikzcd}
$$

$\hat{\varphi}$ defines a formal diffeomorphism and φ_ℓ an analytic one. These diffeomorphisms define the vector fields $\hat{\xi} = (\hat{\varphi})_*(\xi)$ and $\xi_\ell = (\varphi_\ell)_*(\xi)$.

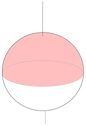
Blowing-up of the singularity

Consider the real blowing-up $\pi : (M, E) \longrightarrow (\mathbb{R}^3, 0)$ of the origin (valid for both sets of coordinates $\hat{x}, \hat{y}, \hat{z}$ and x_ℓ, y_ℓ, z_ℓ), with charts:

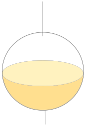
$$C_1 = \begin{cases} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= \rho\bar{z} \end{cases} \qquad \rho \geq 0, \theta \in [0, 2\pi], z' \in \mathbb{R}$$



$$C_2 = \begin{cases} x &= x'z' \\ y &= y'z' \\ z &= z' \end{cases} \qquad x', y' \in \mathbb{R}, z' \geq 0$$



$$C_3 = \begin{cases} x &= x'z' \\ y &= y'z' \\ z &= -z' \end{cases} \qquad x', y' \in \mathbb{R}, z' \geq 0.$$



The total transform of the vector field in the coordinate chart C_1

The total transform of the formal vector field $\hat{\xi}$ in the coordinate chart C_1 is the following, renaming $z = \bar{z}$:

$$\hat{\xi}_1 = \left[\sum_{i=1}^{\infty} \sum_{2r+j=i} a_{irj} \rho^{2r+j+1} z^j\right] \frac{\partial}{\partial \rho} + \left[b + \sum_{i=1}^{\infty} \sum_{2r+j=i} b_{irj} \rho^{2r+j} z^j\right] \frac{\partial}{\partial \theta}$$
$$+ \left[\sum_{i=2}^{\infty} \sum_{2r+j=i} c_{irj} \rho^{2r+j-1} z^j - \sum_{i=1}^{\infty} \sum_{2r+j=i} a_{irj} \rho^{2r+j} z^{j+1}\right] \frac{\partial}{\partial z}$$

Remarks

- $\dot{\theta}$ is a unit.
- $\rho = 0$ is the equation of the divisor and ρ divides $\dot{\rho}$ and $\dot{z}$.

Definition of a two dimensional system

From the formal vector field we can define an autonomous formal two dimensional system as:

$$\text{Formal vector field } \hat{\eta}_1 \quad \left\{ \begin{array}{rcl} \dfrac{d\rho}{d\theta} & = & \dot{\theta}^{-1}\rho^n \hat{A}_\rho(\rho, z) \\ \dfrac{dz}{d\theta} & = & \dot{\theta}^{-1}\rho^n \hat{A}_z(\rho, z) \end{array} \right. ,$$

where $\hat{A}_i \in \mathbb{R}[z][[\rho]]$. For this vector field, θ acts as the time.

For $\ell < \infty$, we are able to define a non-autonomous two dimensional system:

$$\text{Analytic vector field } \eta_{\ell,1} \left\{ \begin{array}{rcl} \dfrac{d\rho}{d\theta} & = & \dot{\theta}^{-1}\rho^n A^\ell_\rho(\rho, z, \cos\theta, \sin\theta) \\ \dfrac{dz}{d\theta} & = & \dot{\theta}^{-1}\rho^n A^\ell_z(\rho, z, \cos\theta, \sin\theta) \end{array} \right. ,$$

where $A^\ell_i \in \mathbb{R}[z, \cos\theta, \sin\theta]\{\rho\}$. In fact, $j_\ell \hat{A}_i = j_\ell A^\ell_i$ for $i = \rho, z$.

Singularities of the two dimensional system

Lemma

The sets

$$\{(\hat{\rho}, \hat{z}, \hat{\theta}) \in \hat{M} : \hat{A}_z(0, \hat{z}, \cos\hat{\theta}, \sin\hat{\theta}) = \hat{A}_\rho(0, \hat{z}, \cos\hat{\theta}, \sin\hat{\theta}) = 0\}$$

and

$$\{(\rho_\ell, z_\ell, \theta) \in M_\ell : A_z^\ell(0, z_\ell, \cos\theta, \sin\theta) = A_\rho^\ell(0, z_\ell, \cos\theta, \sin\theta) = 0\}$$

are equal when $n < \ell$. In addition, they only depend on z and have a finite number of connected components.

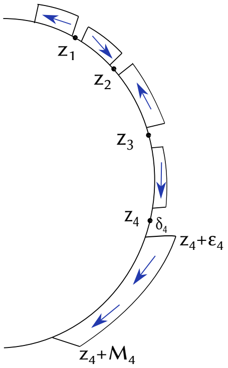
- A_i^ℓ and $\hat{A}_i$ have the same $\ell-$jet for $i = \rho, z$.
- As $n < \ell$, there is not angular dependence in this functions.
- These functions are polynomials in z.

Definition of flow-boxes

- Assume that the blowing up is **non-dicritical**, ρ divides A_ρ.
- Take $\{z_1, \ldots, z_m\}$ the coordinates of the singularities of the two dimensional system.
- Take $M_0, M_m > 0$ to be arbitrarily large and $\varepsilon_i > 0$ for $i = 1, \ldots, m$ to be arbitrarily small. Then, there exist $\delta_0, \ldots, \delta_m > 0$ small enough such that the following:

$$\begin{aligned} V_0 &= (z_1 - M_0, z_1 - \varepsilon_1) \times (0, \delta_0) \\ V_i &= (z_i + \varepsilon_i, z_{i+1} - \varepsilon_{i+1}) \times (0, \delta_i) \\ V_m &= (z_m + \varepsilon_m, z_m + M_m) \times (0, \delta_m) \end{aligned}$$

are boxes where the vector field is $z-$increasing or $z-$decreasing.

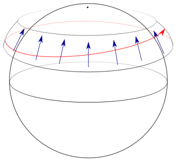


Definition of flow-boxes

Lemma

The sets $\tilde{V}_i = V_i \times \mathbb{S}^1$ define flow-boxes of the vector field $\xi_{\ell,1}$.

Considering that ρ is always greater than zero, the qualitative behavior of a trajectory in $\tilde{V}_i$ is z-**increasing** or z-**decreasing**. Therefore, we are able to define $\partial_1 \tilde{V}_i$ and $\partial_2 \tilde{V}$ as the faces of the flow-box where the vector fields points either inwards or outwards the flow-box.
An example of a trajectory:



Application of the flow-boxes to $\mathcal{C}_U$

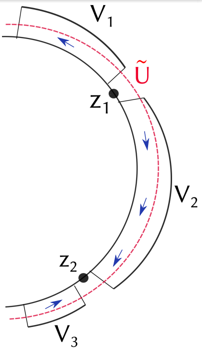
Proposition

Suppose $\tilde{U} = \mathbb{S}^2 \times [0, \delta)$ with $\delta < \min\{\delta_0, \ldots, \delta_m\}$ and $U = \pi(\tilde{U})$. Then,

$$\pi^{-1}(\mathcal{C}_U) \cap \left(\bigcup_{i=0}^{m} \tilde{V}_i \right) = \emptyset.$$

To prove this proposition:

- Suppose there is $p \in \mathcal{C}_U \cap \tilde{V}_i$ for some $0 \leq i \leq m$ and let $\gamma_p \subset U$ be the trajectory defined by ξ in p.
- As $\delta < \min\{\delta_0, \ldots, \delta_m\} \leq \delta_i$, $\tilde{U} \setminus \tilde{V}_i = U_1 \dot{\cup} U_2$ is not connected.
- The fact that $\tilde{V}_i$ is a flow-box implies that γ_p traverses some $\partial_j \tilde{V}_i$ and $\gamma_p \cap U_j \neq \emptyset$. We find a contradiction because the trajectory cannot enter that flow-box again.

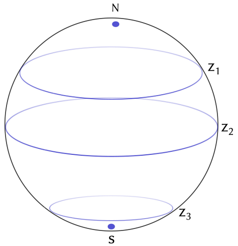


Application of the flow-boxes to $\mathcal{C}_U$

Corollary

$$\overline{\pi^{-1}(\mathcal{C}_U)} \cap E \subseteq \bigcup_{i=1}^{m} (\{0\} \times \{z_i\} \times \mathbb{S}^1) \cup \{N\} \cup \{S\}$$

where N and S denote the origins of the charts C_2 and C_3 respectively.



$\rightarrow$ We define the **characteristic orbits** $\gamma_i := \{0\} \times \{z_i\} \times \mathbb{S}^1$ as the orbits where the accumulation of cycles is possible.
Using similar arguments, the same result is achieved when the blowing up is dicritical.

Desingularization along characteristic orbits

Now, we work with the formal two dimensional vector field $\hat{\eta}_1$.

$$\left\{ \begin{array}{lcl} \dfrac{d\rho}{d\theta} & = & \dot{\theta}^{-1}\rho^n \hat{A}_\rho(\rho, z) \\ \dfrac{dz}{d\theta} & = & \dot{\theta}^{-1}\rho^n \hat{A}_z(\rho, z) \end{array} \right. .$$

This vector field can have curves of singularities but it admits an adapted **resolution of singularities**.

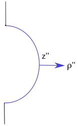
- A resolution of singularities of the curves of singularities.
- A resolution of singularities of the reduced vector field $\mathrm{Red}(\hat{\eta}_1)$.

That is, after a finite number of blowing-ups of points, the transformed reduced vector field has a finite number of simple singularities.

Blowing-ups

The blowing-ups $\pi_2, \ldots, \pi_n$ that we will use can be expressed in the following charts:

Central chart: $C_c = \begin{cases} \rho & = & \rho' \\ z & = & \rho'(z' + z_i) \end{cases} \rho'' \geq 0, z'' \in \mathbb{R},$

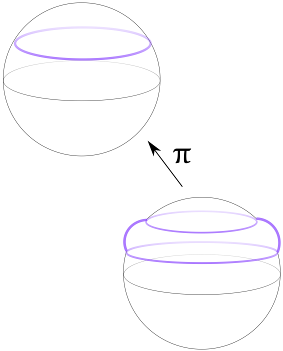


Upper chart: $C_u = \begin{cases} \rho & = & \rho'' z'' \\ z & = & z'' + z_i \end{cases} \rho'', z'' \geq 0,$

Lower chart: $C_l = \begin{cases} \rho & = & \rho'' z'' \\ z & = & -z'' + z_i \end{cases} \rho'', z'' \geq 0.$

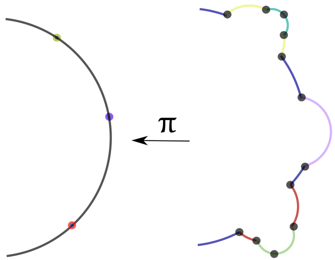
Blowing-ups

In the three dimensional system, they correspond to the blowing-up of the characteristic orbits.



Compositions of blowing-ups

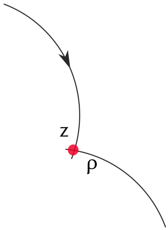
After a composition of blowing-ups: $\pi_2, \ldots, \pi_n$, the situation is the following. The different colors represent different components of the total divisor.



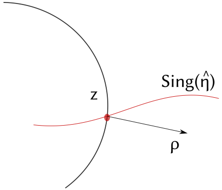
The singularities of the final situation are simple, that is, with non nilpotent linear part and whose eigenvalues λ, μ fulfill $\frac{\lambda}{\mu} \notin \mathbb{Q}_+$.

Final situations

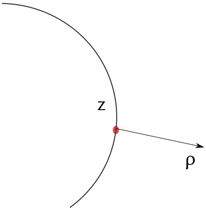
Consider $\hat{\eta}_n$ the transformed by $\pi = \pi_n \circ \ldots \circ \pi_2$ of the formal two dimensional vector field $\hat{\eta}_1$. Each singularity is in one of the following situations:



Case 1: An isolated singularity in the corner whose separatrices are the components of the divisor.



Case 2: Singularity in a curve of singular points.



Case 3: Non corner isolated singularity of $\mathrm{Red}(\hat{\eta}_n)$.

Remark

In the last case, the singularity $\hat{\eta}$ is crossed by a formal invariant separatrix.

Invariant surfaces

For $\hat{\eta}_n$, we find some invariant curves:

- The branches of $\mathrm{Sing}(\hat{\eta}_n)$ define invariant curves Y_i.
- On the other hand, the formal separatrices in singularities of the third case define invariant curves Y_j.

Therefore, we can define formal **invariant surfaces** for the three dimensional vector field $\hat{\xi}_n$ as

$$S_{Y_i} = Y_i \times \mathbb{S}^1.$$

These surfaces are invariant for this formal vector field considering that $L_{\frac{\partial}{\partial \theta}} \hat{\xi} = 0$.

The analytic case

For the resolution of singularities of the formal two dimensional system, only a **finite** $k-$**jet** of the vector field has been used.

Take $\ell > k$ and the vector field ξ_ℓ.

- Following the same sequence of blowing-ups $\pi = \pi_n \circ \cdots \circ \pi_1$ and $E = \pi^{-1}(0)$, we find that
$$\xi_\ell|_E \equiv \hat{\xi}|_E,$$
in their proper coordinates.
- Therefore, they will have the same characteristic orbits in the divisor.

Accumulation of cycles

- In the first case accumulation of cycles is not possible, because one of the components of the divisor is a (un)stable manifold and the other is the center manifold, which already contains cycles.
- In the second and third case, consider the formal diffeomorphism

$$\varphi_\ell \circ \hat{\varphi}^{-1} : (\mathbb{R}^3_{\{\hat{x},\hat{y},\hat{z}\}}, 0) \longrightarrow (\mathbb{R}^3_{\{x_\ell, y_\ell, z_\ell\}}, 0).$$

  The surfaces $S_{\ell,i} = (\varphi_\ell \circ \hat{\varphi}^{-1})(S_{Y_i})$ are invariant for the vector field $\xi_\ell = (\varphi_\ell \circ \hat{\varphi}^{-1})_*(\hat{\xi})$.
- In any other intermediate point, there is a flow-box that avoids accumulation of cycles.

Accumulation of cycles is possible only in the characteristic orbits corresponding to the cases 2 and 3.

The Poincaré map

Suppose γ_i is a characteristic orbit of the divisor and define $\Delta := \{(\rho, z, \theta) : \theta = \alpha\}$ a half-plane transverse to γ. Then, the Poincaré map is an analytic diffeomorphism tangent to the identity and defined in $P : \Delta' \subseteq \Delta \longrightarrow \Delta$.

Lemma

$\Gamma_i := S_{\ell,i} \cap \Delta$ is an invariant curve for P.

Remark

If $S_{\ell,i}$ is a convergent surface, Γ_i is a convergent curve. Otherwise, Γ_i is a formal non-convergent curve.

Cases:

- Γ_i is a formal invariant curve not of fixed points.
- Γ_i is a curve of fixed points.

Γ is not a curve of fixed points

There are two possibilites:

- P has an eigenvalue not equal to 1.
- P is tangent to the identity.

In the first case, we find that the center manifold is the divisor, and therefore, there are not other periodic points.
In the second case, we are in the conditions to apply the following result:

Theorem [López, Sanz, 2018]

Let F be a tangent to the identity diffeomorphism in $(\mathbb{C}^2, 0)$. Given a formal invariant curve Γ of F not contained in the set of fixed points, there exists at least one (attracting or repelling) parabolic curve for F asymptotic to Γ.

Using techniques found in this article, we find that there are not periodic points in a cone sufficiently small around the parabolic curve.

When P does not have periodic points, the vector field does not have cycles intersecting Δ'.

Γ is a curve of fixed points

Suppose that $\Gamma_i = \hat{S_{\ell,i}} \cap \Delta$ is a **curve of fixed points**. Then, this curve is analytic and we can consider $\Gamma := \{z - h(\rho) = 0\}$.

After a change of coordinates, $\Gamma := \{z = 0\}$ and the diffeomorphism can be written in the following way:

$$P(\rho, z) = \left(\rho + \rho^a z^b (A_1(\rho) + z^{r_1} \cdot B_1(\rho, z)), z + \rho^c z^d (A_2(\rho) + z^{r_2} \cdot B_2(\rho, z))\right),$$

where $a, b, c, d > 0$, $r_1, r_2 \geq 0$, $A_1, A_2 \in \mathbb{R}\{\rho\}$ and $B_1, B_1 \in \mathbb{R}\{\rho, z\}$.

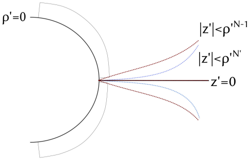
One important property is that if $z > 0$, $z \circ P(\rho, z) > 0$ and viceversa, since S_ℓ is invariant and cannot be crossed by any orbit.

Γ is a curve of fixed points

- We have that cycles and thus, periodic points of the Poincaré map, cannot intersect flow-boxes and are **confined in an open cone** $|z| < \rho^N$ for some N as big as necessary.
- Then, N for the cone can be chosen in such a way that $r_1 \cdot N >$ order$(A_1) = s_1$. Inside the cone, it can be seen that:

$$\frac{z^{r_1} B_1(\rho, z)}{A_1(\rho)} \to 0 \text{ as } \rho \to 0$$

- This implies, when $z_0 \neq 0$, that if $(\rho_n, z_n) = P(\rho_{n-1}, z_{n-1})$, then either $\rho_{n-1} < \rho_n$ or $\rho_{n-1} > \rho_n$.



The red cone intersects a flow-box.

Therefore, inside the cone the unique periodic points are those in $\{z = 0\}$, which implies that the periodic orbits are confined in the surface $S_{\ell,i}$.

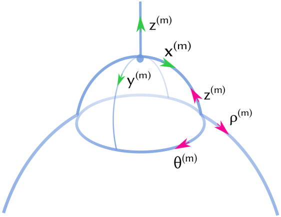
Accumulation of cycles in the z-axis direction

Considering the flow-boxes argument, accumulation of orbits is still possible in the positive (or negative) direction of the $z-$axis.

We perform an **iteration of blowing-ups** using the following charts:

$$C_{1^{(m)}} = \begin{cases} x^{(m-1)} &= \rho^{(m)} \cos\theta^{(m)} \\ y^{(m-1)} &= \rho^{(m)} \sin\theta^{(m)} \\ z^{(m-1)} &= \rho^{(m)} z^{(m)} \end{cases} \quad \rho^{(m)}, z^{(m)} \geq 0, \theta^{(m)} \in [0, 2\pi],$$

$$C_{2^{(m)}} = \begin{cases} x^{(m-1)} &= x^{(m)} z^{(m)} \\ y^{(m-1)} &= y^{(m)} z^{(m)} \\ z^{(m-1)} &= z^{(m)} \end{cases} \quad z^{(m)} \geq 0, x^{(m)}, y^{(m)} \in \mathbb{R}.$$



Accumulation of cycles in the z-axis direction

The fact that the singularity is isolated and that the formal $z-$axis is an invariant curve for the formal field $\hat{\xi}$ lead to

$$\dot{z}^{(m)} = (z^{(m)})^n(c_n + F(x^{(m)}, y^{(m)}, z^{(m)})) \tag{1}$$

after a sufficiently large number of blowing ups.

Taking ℓ to be large enough, (1) can be achieved in the analytic case.

This implies that z defines a monotonic function near the singularity, and thus, the following lemma.

Lemma

After a large enough number of $z-$direction blowing ups,there exists a small enough neighborhood V of the origin in which $\pi^{-1}(\mathcal{C}_U) \cap V = \emptyset$.

Accumulation of cycles in the z-axis direction

In the intermediate divisor components, we perform a reduction to a two dimensional system in order to obtain flow-boxes and a **finite number of characteristic orbits** in the divisor, as we did in the study of the initial C_1.

Final conclusion

In the final situation we have:

- In the chart C_1 of the first blowing-up, we have achieved a finite number of characteristic orbits. In some of them, there are surfaces composed by cycles.
- After several blowing-ups of the origins of C_2 and C_3, there can be a finite number of surfaces tangent to the $z-$axis. In addition, the contact with the z-axis is of finite order.

A perturbation of a linear non-degenerated center fulfills Dulac's property.

Thanks for your attention!