Stratified Resolution of Singularities of Generalized Analytic Functions

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Comment

Our introduction (the basics of generalized power series and the category of generalized analytic manifolds) is largely based on the paper [van den Dries and Speissegger, 1998] due to L. van den Dries and P. Speissegger, as well as the paper [Martín-Villaverde et al., 2013] by R. Martín Villaverde, J.-P. Rolin and F. Sanz Sánchez.

Generalized power series

A *formal generalized power series* with real coefficients is a function, $s\colon \mathbb{R}^m_{\geq 0} \to \mathbb{R}$, written as

$$s := s(X) = \sum_{\alpha} s_\alpha X^\alpha\,, \quad s_\alpha := s(\alpha)\,,$$

$$X = (x_1, x_2, \ldots, x_m)\,, \quad X^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}\,,$$

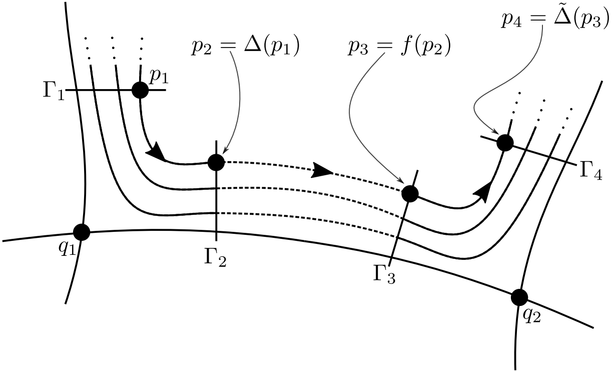
such that its support,

$$\mathrm{Supp}(s) := \{\alpha \in \mathbb{R}^m_{\geq 0} \,:\, s_\alpha \neq 0\}\,,$$

is the Cartesian product of well-ordered subsets of the non-negative real numbers.
We denote by $\mathbb{R}[[X^*]]$ the local ring of all formal generalized power series in the variables X.

Examples

1. Solutions to the Euler system $x' = t^{-1}\lambda x$, $\lambda > 0$, are given by the simplest generalized power series $x = t^{\lambda} = \exp(\lambda \ln t)$.
2. Puiseux series: These are power series with fractional exponents that appear naturally as parametrizations of irreducible local plane analytic curves.
3. Asymptotic developments of certain Dulac's maps appearing as first return maps at polycycles:



Some special subrings of $\mathbb{R}[[X^*]]$:

1. The ring of *usual* power series $\mathbb{R}[[X]]$: a power series s belongs to $\mathbb{R}[[X]]$ if and only if $\mathrm{Supp}(s) \subset \mathbb{Z}^m_{\geq 0}$.
2. The ring of *mixed* power series $\mathbb{R}[[X^*, Y]]$: its elements are all power series $s \in \mathbb{R}[[(X, Y)^*]]$ such that $\mathrm{Supp}(s) \subset \mathbb{R}^m_{\geq 0} \times \mathbb{Z}^n_{\geq 0}$; the X variables are called *generalized variables*, while the Y variables are called *analytic*.
3. Of course, the ring of *convergent* generalized power series, denoted by $\mathbb{R}\{X^*\}$.

Definition

We say that a formal generalized power series $s \in \mathbb{R}[[X^*]]$ (or a mixed one, $s \in \mathbb{R}[[Y]][[X^*]]$) is of *monomial-type* (with respect to the generalized variables X) if

$$s(X) = U(X) \cdot X^\alpha \,, \quad U(0) \neq 0 \,, \quad \alpha \in \mathbb{R}^m_{\geq 0} \,.$$

$$\left(s(X,Y) = U(X,Y) \cdot X^\alpha \,, \quad U(0,Y) \not\equiv 0 \in \mathbb{R}[[Y]] \,, \quad \alpha \in \mathbb{R}^m_{\geq 0}\right) .$$

How can we know if a given power series is of monomial-type?

(a) Minimal support

The *division order* on $\mathbb{R}^m_{\geq 0}$ is the partial order relation defined by:

$$\alpha \leq \beta \quad \text{if and only if} \quad \alpha_j \leq \beta_j \,, \, j \in \{1, 2, \ldots, m\} \,, \quad \alpha, \beta \in \mathbb{R}^m_{\geq 0} \,.$$

Remark

We stress that $\alpha \leq \beta$ if and only if X^α divides X^β.

Thus, given any generalized power series $s \in \mathbb{R}[[X^*]]$ ($s \in \mathbb{R}[[Y]][[X^*]]$), we define its *minimal support*,

$$\mathrm{Supp}_{\mathrm{min}}(s)\,,$$

as the finite subset of its support conformed by the minimal elements with respect to the division order.
So, each formal generalized power series, $s \in \mathbb{R}[[X^*]]$ ($s \in \mathbb{R}[[Y]][[X^*]]$), can be rewritten as a finite sum of monomials:

$$s = \sum_{\alpha \in \mathrm{Supp}_{\mathrm{min}}(s)} u_\alpha X^\alpha\,, \quad u_\alpha(0) \neq 0\,,\ \text{for each}\ \ \alpha \in \mathrm{Supp}_{\mathrm{min}}(s)\,,$$

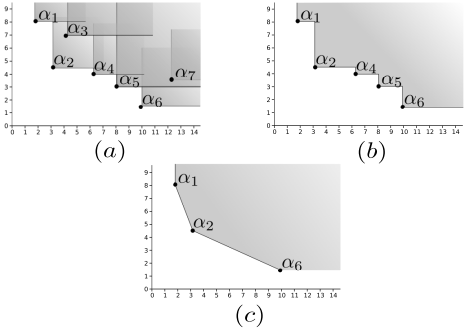
called the *monomial representation* of s.

Remark

Notice that a formal generalized power series is of *monomial-type* if and only if its minimal support is a singleton.

(b) Newton Polyhedron

Let $s \in \mathbb{R}[[Y]][[X^*]]$ be a mixed power series. The *Newton polyhedron* of s is the boundary of the convex hull of $\mathrm{Supp}_{\min}(s) + \mathbb{R}^m_{\geq 0}$.



(a) (b)

(c)

Remark

A formal generalized power series is of monomial-type if and only if the vertices set of its Newton polyhedron is a singleton.

Generalized analytic functions

The sum of every convergent generalized power series $s \in \mathbb{R}\{X^*\}$ defines a continuous function, $f_s \colon U \to \mathbb{R}$, in its definition domain; namely, a hyper-rectangle,

$$U = [0, r_1] \times [0, r_2] \times \cdots \times [0, r_m] \subset \mathbb{R}^m_{\geq 0}\,,$$

which is real-analytic in the interior of it. We call f_s a *generalized analytic function*.

Note

We say that a generalized analytic function $f_s \colon (\mathbb{R}^n_{\geq 0}, 0) \to (\mathbb{R}, 0)$ is of monomial-type at the origin if the power series $s \in \mathbb{R}\{X^*\}$ that defines it is so.

Standard and generalized analytic manifolds

Definition

A *standard analytic manifold* of dimension $m \in \mathbb{Z}_{>0}$, is a topological manifold with boundary and corners, $|A|$, endowed with a real-analytic structure, $A = (|A|, \mathcal{O}_A)$.

Similarly,

Definition

A *generalized analytic manifold* of dimension $m \in \mathbb{Z}_{>0}$ is a topological manifold with boundary and corners, $|M|$, endowed with a generalized analytic structure, $M = (|M|, \mathcal{G}_M)$.

Natural stratification

Any topological manifold with boundary and corners, $X = (|X|, \mathcal{E}_X)$, endowed with a (generalized) analytic structure, $\mathcal{E}_X \in \{\mathcal{O}_X, \mathcal{G}_X\}$, admits a stratification induced by its boundary, called its *natural stratification*.

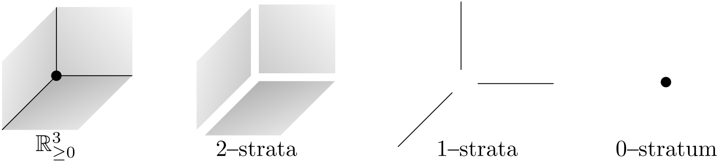


Figure: Natural stratification of the orthant $\mathbb{R}^3_{\geq 0}$ induced by its boundary.

How can one define morphisms between generalized analytic manifolds (*e.g.*, blowing-ups)?

There are two technical notions to introduce first, which allow us to carry out, in the generalized analytic setting, similar constructions to those done in the standard and classical one:

1. Enrichment of a standard manifold, and
2. A special subclass of generalized manifolds: Standardizable ones.

Enrichment of a standard manifold

Let $A = (|A|, \mathcal{O}_A)$ be a standard manifold, described by the analytic structure $\mathcal{U}_A = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$. We can endow $|A|$ with a generalized analytic structure in the following natural way:

Considering the subsheaf $\mathcal{G}_A$ of the sheaf of continuous functions on $|A|$ whose sections over any open subset of $|A|$, $U \subset |A|$, are given by

$$(f \colon U \to \mathbb{R}) \in \mathcal{G}_A\,(U) \quad \text{iff} \quad f\big|_{U \cap U_\alpha} \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U \cap U_\alpha)} \in \mathcal{G}_{\mathbb{R}^n_{\geq 0}}\,(U \cap U_\alpha)\,,$$

for all $\alpha \in \Lambda\,, U \cap U_\alpha \neq \emptyset$.

We denote by $A^{\mathfrak{e}} := (|A|, \mathcal{G}_A)$ the generalized analytic manifold gotten by this means, and it is called the *enrichment* of the standard manifold A.

Standardizable manifolds

Definition

A *standardization* of a generalized analytic manifold $M = (|M|, \mathcal{G}_M)$ is a subsheaf $\mathcal{O}$ of $\mathcal{G}_M$ such that $M_{\mathcal{O}} := (|M|, \mathcal{O})$ is a standard analytic manifold with $(M_{\mathcal{O}})^{\mathfrak{e}} = M$.

We say that M is *standardizable* if there exists a standardization of it.

Remark

Not every generalized analytic manifold is standardizable; *cf.*, *exotic cylinder* example given in [Martín-Villaverde et al., 2013].

Local description of morphisms between standardizable manifolds

In general, morphisms between generalized analytic manifolds are locally of monomial-type at points on the boundary, so:

Remark

The class of morphisms between standard analytic manifolds that can be lifted to morphisms between their enrichments are those locally described by monomial-type transformations at the points on their boundaries.

$$\begin{tikzcd}
A^{\mathfrak{e}} \arrow[r, "f^{\mathfrak{e}}"] \arrow[d, "\mathrm{id}_A"'] & B^{\mathfrak{e}} \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

An important instance: blowing-up morphisms

Definition

Let $M = (|M|, \mathcal{G}_M)$ be a generalized analytic manifold, $\mathcal{O}$ a standardization of M, and $Y \subset |M|$ a subset of $|M|$. We say that the pair $\mathcal{Y} = (Y, \mathcal{O})$ is an *center of blowing-up for M* if Y is an admissible center of blowing-up (in the ordinary analytic sense) for $M_{\mathcal{O}} = (|M|, \mathcal{O})$.

Definition

Let $M = (|M|, \mathcal{G}_M)$ be a generalized analytic manifold, and $\mathcal{Y} = (Y, \mathcal{O})$ a center of blowing-up for M. The *blowing-up morphism* of M centered at $\mathcal{Y}$, $\pi_{\mathcal{Y}} \colon \tilde{M} \to M$, is the enrichment of the usual blowing-up of $M_{\mathcal{O}}$ centered at Y:

$$\begin{tikzcd} M \arrow[d] & \tilde{M} \arrow[l, "\pi_{\mathcal{Y}}"'] \arrow[d] \\ M_{\mathcal{O}} & \tilde{M}_{\mathcal{O}} \arrow[l, "\pi_Y"] \end{tikzcd} , \quad \pi_{\mathcal{Y}} := \pi_Y^{\mathfrak{e}} , \quad \tilde{M} := (\tilde{M}_{\mathcal{O}})^{\mathfrak{e}} .$$

Remark

The morphism $\pi_{\mathcal{Y}}\colon \tilde{M} \to M$ is actually proper, surjective and its restriction to $\tilde{M} \setminus E_Y$ is an isomorphism that maps $\tilde{M} \setminus E_Y$ onto $M \setminus Y$, where $E_Y := (\pi_Y^M)^{-1}(Y)$ is called the *exceptional divisor*.

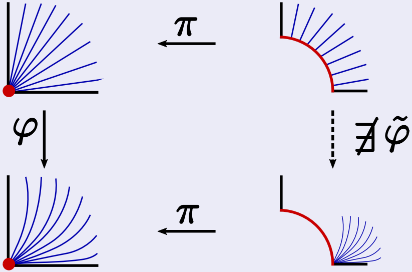
In what follows we will focus on the, so-called, combinatorial blowing-up morphisms:

Definition

We say that a blowing-up morphism $\pi_{\mathcal{Y}}\colon \tilde{M} \to M$ is *combinatorial* if Y is the closure of a stratum of the natural stratification of M.

Remark

There are non-isomorphic blowing-ups: Let us consider $\lambda > 0$, $\lambda \neq 1$, and let $\varphi\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ be the generalized analytic automorphism given by $(x_1, x_2) \mapsto (x_1^\lambda, x_2)$. Considering the standard blow-up of $\mathbb{R}^2_{\geq 0}$ centered at the origin, does not exist a lifting $\tilde{\varphi}$ of φ.



Resolution of Singularities problem

Local setting:

1. The general local monomialization theorem in any dimension was proved in [Martín-Villaverde et al., 2013], and it took inspiration from the methods in [Bierstone and Milman, 1988, Rolin et al., 2003].
2. The simplest monomialization result at generalized variables can be derived by arguments given in [van den Dries and Speissegger, 1998].
3. In the papers [Rolin and Servi, 2015, Servi, 2015] are also given different local monomialization algorithms that work for generalized analytic functions.

Global setting:

There was only partial answers for small dimensional cases ($n \leq 3$), *cf.*, [Palma-Márquez, 2022].

Stratified Resolution of Singularities

Definition

Let $M = (|M|, \mathcal{G}_M)$ be a generalized analytic manifold, and let $f \in \mathcal{G}(M)$ be a generalized analytic function on M. We say that f is *stratified monomial* on M if it is locally of monomial-type at every corner point on M.

Theorem (Stratified Resolution of Singularities – B. Molina Samper, F. Sanz Sánchez, -.)

Let $M = (|M|, \mathcal{G})$ be a generalized analytic manifold, and let $f \in \mathcal{G}_M$ be a generalized analytic function on M. Then, for any point $p \in |M|$, there exists an open neighbourhood of p in M, $V \subset M$, and a finite sequence of combinatorial blowing-up morphisms,

$$V = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} M_n\,, \quad \pi := \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n\,, \tag{1}$$

such that the total transform of f, $f \circ \pi$, is stratified monomial on M_n. Moreover, the centers in the sequence (1) *can be chosen to be of codimension two.*

Remark

Stratified resolution of singularities is the first step towards a general resolution of singularities process for generalized analytic functions, till now unexplored (as far as we know).

Sketch of the proof

Notice that this result is a consequence of the following more general situation: Given

$$M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} M_n\,, \quad \pi := \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n\,,$$

any finite sequence of combinatorial blowing-up morphisms over M_0, then it is enough to prove that there exists a finite sequence of combinatorial blowing-up morphisms,

$$M_n \xleftarrow{\tilde{\pi}_1} \tilde{M}_1 \xleftarrow{\tilde{\pi}_2} \cdots \xleftarrow{\tilde{\pi}_k} \tilde{M}_k\,, \quad \tilde{\pi} := \tilde{\pi}_1 \circ \tilde{\pi}_2 \circ \cdots \circ \tilde{\pi}_k\,,$$

over M_n, such that $F \circ \tilde{\pi}$ is stratified monomial on $\tilde{M}_k$, where $F := f \circ \pi$.

Remark

This strategy leads us to analyze the space of all possible finite sequences of combinatorial blowing-ups over M_0; which, inspired by H. Hironaka [Hironaka, 1973], we have called the *monomial voûte étoilée* of M_0.

Combinatorics of $\mathfrak{m}$-stars and Linear Algebra

Given a $\mathfrak{m}$-*star* of M_0; *i.e.*, a finite sequence of combinatorial blowing-up morphisms,

$$M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} M_n \,,$$

as before, we want to understand the generalized analytic structure of every topological space appearing in such sequence as well as the explicit formulas of the morphisms involved. We do that by means of Linear Algebra methods, using the so-called *matrices of exponents*; *cf.*, [Bruno, 1989, Berezovskaya and Medvedeva, 1992, Palma-Márquez, 2022].

Looking at the inner structure of the previous sequence, we find out a finite quiver richer than just a simple path:

$$\begin{tikzcd}
 & U_{1_{m_1}} \arrow[ddl, "\pi_1^{m_1}"'] & \cdots \arrow[l, "\pi_2^{m_2}"'] & U_{n_{m_n}} \arrow[l, "\pi_n^{m_n}"'] \\
 & \vdots \arrow[u, "\varphi^1_{m_1-1\,m_1}"'] & \cdots \arrow[l, dotted] \arrow[u, dotted] & \vdots \arrow[l, dotted] \arrow[u, "\varphi^n_{m_n-1\,m_n}"'] \\
M_0 & U_{1_2} \arrow[l, "\pi_1^2"'] \arrow[u, "\varphi^1_{2\,3}"'] & \cdots \arrow[l, "\pi_2^2"'] \arrow[u, dotted] & U_{n_2} \arrow[l, "\pi_n^2"'] \arrow[u, "\varphi^n_{2\,3}"'] \\
 & U_{1_1} \arrow[ul, "\pi_1^1"] \arrow[u, "\varphi^1_{1\,2}"'] & \cdots \arrow[l, "\pi_2^1"] \arrow[u, dotted] & U_{n_1} \arrow[l, "\pi_n^1"] \arrow[u, "\varphi^n_{1\,2}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & U_{1_{m_1}} \arrow[ddl, "\pi_1^{m_1}"'] & \cdots \arrow[l, "\pi_2^{m_2}"'] & U_{n_{m_n}} \arrow[l, "\pi_n^{m_n}"'] & & & \mathbb{R}^n \arrow[ddl, "L(\pi_1^{m_1})"'] & \cdots \arrow[l, "L(\pi_2^{m_2})"'] & \mathbb{R}^n \arrow[l, "L(\pi_n^{m_n})"'] \\
 & \vdots \arrow[u, "\varphi^1_{m_1-1\,m_1}"'] & \cdots \arrow[l, dotted] \arrow[u, dotted] & \vdots \arrow[l, dotted] \arrow[u, "\varphi^n_{m_n-1\,m_n}"'] & & & \vdots \arrow[u, "L(\varphi^1_{m_1-1\,m_1})"'] & \cdots \arrow[l, dotted] \arrow[u, dotted] & \vdots \arrow[l, dotted] \arrow[u, "L(\varphi^n_{m_n-1\,m_n})"'] \\
M_0 & & & & \Leftrightarrow & \mathbb{R}^n & & & \\
 & U_{1_2} \arrow[ul, "\pi_1^2"'] \arrow[uu, "\varphi^1_{2\,3}"'] & \cdots \arrow[l, "\pi_2^2"'] \arrow[uu, dotted] & U_{n_2} \arrow[l, "\pi_n^2"'] \arrow[uu, "\varphi^n_{2\,3}"'] & & & \mathbb{R}^n \arrow[ul, "L(\pi_1^2)"'] \arrow[uu, "L(\varphi^1_{2\,3})"'] & \cdots \arrow[l, "L(\pi_2^2)"'] \arrow[uu, dotted] & \mathbb{R}^n \arrow[l, "L(\pi_n^2)"'] \arrow[uu, "L(\varphi^n_{2\,3})"'] \\
 & U_{1_1} \arrow[uul, "\pi_1^1"] \arrow[u, "\varphi^1_{1\,2}"'] & \cdots \arrow[l, "\pi_2^1"] \arrow[u, dotted] & U_{n_1} \arrow[l, "\pi_n^1"] \arrow[u, "\varphi^n_{1\,2}"'] & & & \mathbb{R}^n \arrow[uul, "L(\pi_1^1)"] \arrow[u, "L(\varphi^1_{1\,2})"'] & \cdots \arrow[l, "L(\pi_2^1)"] \arrow[u, dotted] & \mathbb{R}^n \arrow[l, "L(\pi_n^1)"] \arrow[u, "L(\varphi^n_{1\,2})"']
\end{tikzcd}$$

Example

$$\begin{tikzcd}
 & U_2 \arrow[dl, "\pi^2_\lambda"'] & & & \mathbb{R}^2 \arrow[dl, "L(\pi^2_\lambda)"'] \\
\mathbb{R}^2_{\geq 0} & & \Leftrightarrow & \mathbb{R}^2 & \\
 & U_1 \arrow[ul, "\pi^1_\lambda"] \arrow[uu, "\varphi"'] & & & \mathbb{R}^2 \arrow[ul, "L(\pi^1_\lambda)"] \arrow[uu, "L(\varphi)"']
\end{tikzcd}$$

where

$$\pi^1_\lambda(x,u) = (x, x^\lambda u)\,, \quad \pi^2_\lambda(v,y) = (vy^{\frac{1}{\lambda}}, y)\,, \quad \varphi(x,u) = (u^{-\frac{1}{\lambda}}, x^\lambda u)\,;$$

$$L(\pi^1_\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\,, \quad L(\pi^2_\lambda) = \begin{pmatrix} 1 & \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix}\,, \quad L(\varphi) = \begin{pmatrix} 0 & -\frac{1}{\lambda} \\ \lambda & 1 \end{pmatrix}\,.$$

Remark

This formalism, in particular, allows us to perform computations relative to standardizations and blowing-up morphisms in an easy and diaphanous way, working at each moment with the associated matrices to such morphisms.

Summing-up, the main steps of the proof are:

1. We study in detail all possible finite sequences of combinatorial blowing-ups over M_0; *i.e.*, the so-called *monomial voûte étoilée*.
2. Besides, we prove that local monomial standardizations around corner points give rise to global standardizations.
3. Finally, we present a resolution of singularities algorithm based on ideas that work in the local counterpart, such as the van den Dries-Speissegger strategy, using only centers of codimension two. In fact, the kind of algorithms, meaningful for our purposes, can be thought of as *shortcuts* of standard (stratified) monomialization algorithms, or solutions to *Hironaka's polyhedra game (see [Hironaka, 1972])*; *cf.*, [Spivakovsky, 1983, Zeillinger, 2006, Goward, 2005, Fernández Duque, 2015, Molina-Samper, 2019, de Moraes and Novacoski, 2020].

P.S.

Do not miss Beatriz's talk in June ;)

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Thank you!