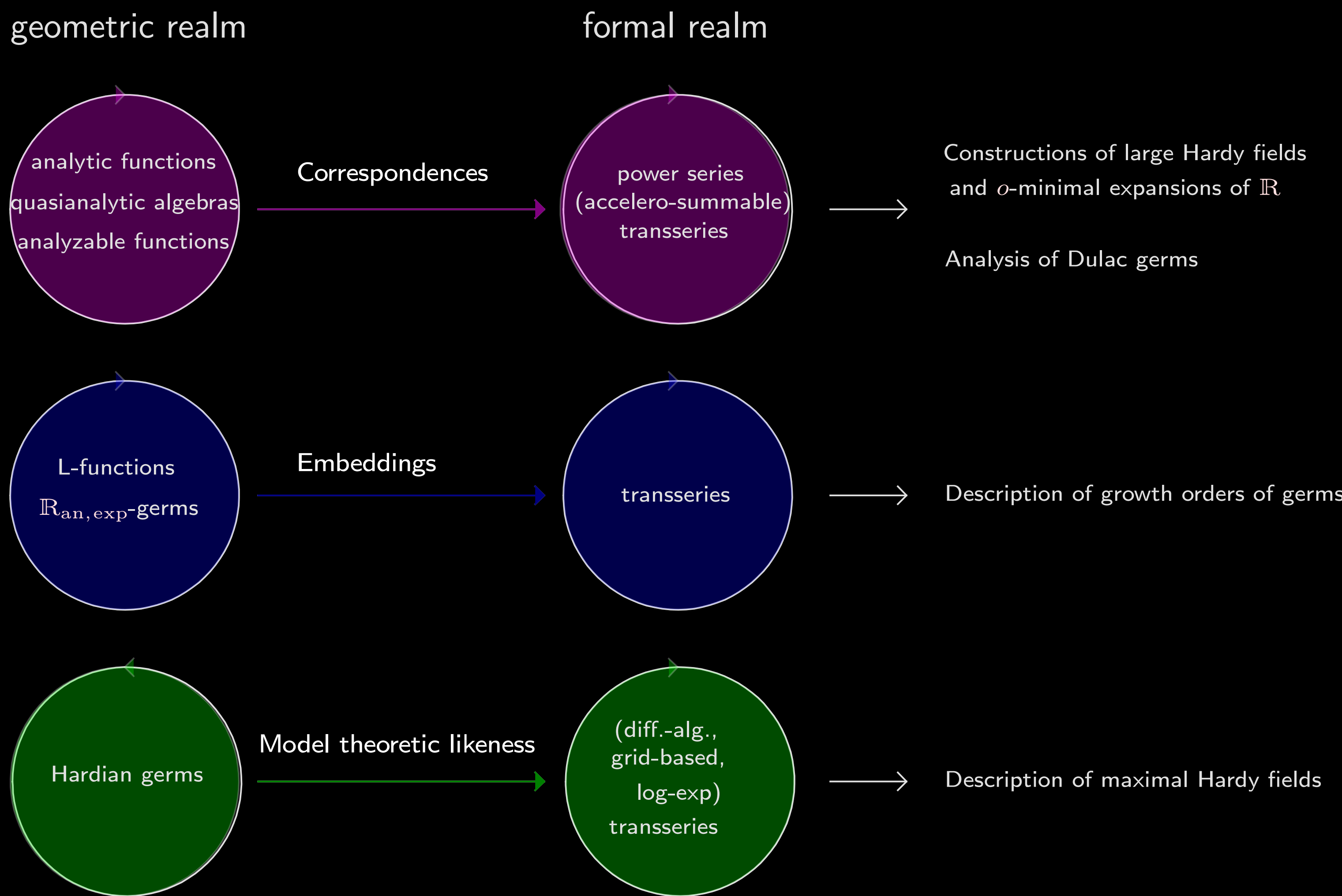
Embedding Hardy fields with composition into transseries

March 15, 2022

including joint work with Elliot Kaplan and Joris van der Hoeven.

geometric / formal



Operations on germs

Germs (at $+\infty$)

Identify two functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ if for $t \gg 1$, we have $f(t) = g(t)$. Equivalence classes are called **germs**. *$\mathcal{G}$ is the ring of germs with pointwise sum and product.*

Operations on germs

Germs (at $+\infty$)

Identify two functions $f, g \colon \mathbb{R} \longrightarrow \mathbb{R}$ if for $t \gg 1$, we have $f(t) = g(t)$. Equivalence classes are called **germs**. *$\mathcal{G}$ is the ring of germs with pointwise sum and product.*

Further structure for $f, g \in \mathcal{G}$.

Ordering. $f < g$ if $f(t) < g(t)$ for $t \gg 1$. So $g > \mathbb{R}$ if $\lim_{t \to +\infty} g(t) = +\infty$.

Operations on germs

Germs (at $+\infty$)

Identify two functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ if for $t \gg 1$, we have $f(t) = g(t)$. Equivalence classes are called **germs**. *$\mathcal{G}$ is the ring of germs with pointwise sum and product.*

Further structure for $f, g \in \mathcal{G}$.

Ordering. $f < g$ if $f(t) < g(t)$ for $t \gg 1$. So $g > \mathbb{R}$ if $\lim_{t \to +\infty} g(t) = +\infty$.

Dominance relation. $f \prec g$ if $r\,|f| < |g|$ for all $r \in \mathbb{R}^{>0}$.

Operations on germs

Germs (at $+\infty$)

Identify two functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ if for $t \gg 1$, we have $f(t) = g(t)$. Equivalence classes are called **germs**. *$\mathcal{G}$ is the ring of germs with pointwise sum and product.*

Further structure for $f, g \in \mathcal{G}$.

Ordering. $f < g$ if $f(t) < g(t)$ for $t \gg 1$. So $g > \mathbb{R}$ if $\lim_{t \to +\infty} g(t) = +\infty$.

Dominance relation. $f \prec g$ if $r\,|f| < |g|$ for all $r \in \mathbb{R}^{>0}$.

Derivation. If $t \mapsto f(t)$ is differentiable on $(r, +\infty)$ for $r \gg 1$, then f' is the germ of the derivative.

Operations on germs

Germs (at $+\infty$)

Identify two functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ if for $t \gg 1$, we have $f(t) = g(t)$. Equivalence classes are called **germs**. *$\mathcal{G}$ is the ring of germs with pointwise sum and product.*

Further structure for $f, g \in \mathcal{G}$.

Ordering. $f < g$ if $f(t) < g(t)$ for $t \gg 1$. So $g > \mathbb{R}$ if $\lim_{t\to+\infty} g(t) = +\infty$.

Dominance relation. $f \prec g$ if $r\,|f| < |g|$ for all $r \in \mathbb{R}^{>0}$.

Derivation. If $t \mapsto f(t)$ is differentiable on $(r, +\infty)$ for $r \gg 1$, then f' is the germ of the derivative.

Composition. If $g > \mathbb{R}$, then the germ $f \circ g$ of $t \mapsto f(g(t))$ only depends on f and g. We have a *composition law*

$$\circ\colon \mathcal{G} \times \mathcal{G}^{>\mathbb{R}} \longrightarrow \mathcal{G}.$$

For fixed $g \in \mathcal{G}^{>\mathbb{R}}$, the function $\mathcal{G} \longrightarrow \mathcal{G};\ f \mapsto f \circ g$ is a strictly increasing morphism of rings.

Hardy fields with composition

Hardy fields

A **Hardy field** *is a subfield H of $\mathcal{G}$ containing $\mathbb{R}$ and closed under derivation of germs.*

Hardy fields with composition

Hardy fields

A **Hardy field** *is a subfield H of $\mathcal{G}$ containing $\mathbb{R}$ and closed under derivation of germs.*

- Each $f \in H$ is either constant or strictly monotonous
- $\forall f \in H, f > \mathbb{R} \Longrightarrow f' > 0.$
- $\forall f \in H, f \preccurlyeq 1 \Longrightarrow \exists r = \lim_{+\infty} f \in \mathbb{R}, f - r \prec 1.$

Hardy fields with composition

Hardy fields

A **Hardy field** *is a subfield H of $\mathcal{G}$ containing $\mathbb{R}$ and closed under derivation of germs.*

- Each $f \in H$ is either constant or strictly monotonous
- $\forall f \in H, f > \mathbb{R} \Longrightarrow f' > 0.$
- $\forall f \in H, f \preccurlyeq 1 \Longrightarrow \exists r = \lim_{+\infty} f \in \mathbb{R}, f - r \prec 1.$

Hardy fields with composition

A **Hardy field with composition** *is a Hardy field $\mathcal{H}$ which contains id and which is closed under compositions $(f, g) \mapsto f \circ g$ with $f \in \mathcal{H}$ and $g \in \mathcal{H}^{>\mathbb{R}}$.*

Hardy fields

A **Hardy field** *is a subfield H of $\mathcal{G}$ containing $\mathbb{R}$ and closed under derivation of germs.*

- Each $f \in H$ is either constant or strictly monotonous
- $\forall f \in H, f > \mathbb{R} \Longrightarrow f' > 0.$
- $\forall f \in H, f \preccurlyeq 1 \Longrightarrow \exists r = \lim_{+\infty} f \in \mathbb{R}, f - r \prec 1.$

Hardy fields with composition

A **Hardy field with composition** *is a Hardy field $\mathcal{H}$ which contains id and which is closed under compositions $(f, g) \mapsto f \circ g$ with $f \in \mathcal{H}$ and $g \in \mathcal{H}^{>\mathbb{R}}$.*

Examples:

- $\mathbb{R}(\mathrm{id})$: germs of rational functions.
- Hardy's field of L-functions: closure of $\mathbb{R}(\mathrm{id})$ under semialgebraic functions, $\exp$ and $\log$.
- Boshernitzan's field $\mathcal{B} := \cap\{M : M \text{ is a } \mathit{maximal} \text{ Hardy field}\}$.

A short-lived conjecture?

Conjecture on $(\mathcal{H}, \circ)$

Let $\mathcal{H}$ be a Hardy field with composition and let $f, g \in \mathcal{H}^{>\mathbb{R}}$.

Conjecture 1. *For all $\delta \in \mathcal{H}$ with $\delta \prec g$ and $(f' \circ g)\,\delta \prec (f \circ g)$, we have*

$$f \circ (g + \delta) \sim f \circ g.$$

Conjecture 2. *The centralizer $\mathcal{C}(f) := \{h \in \mathcal{H}^{>\mathbb{R}} : h \circ f = f \circ h\}$ is commutative.*

Conjecture 3. *If $f > \underset{\text{(iterates)}}{g^{[\mathbb{N}]}} > \mathrm{id}$, then $f \circ g > g \circ f$.*

A short-lived conjecture?

Conjecture on $(\mathcal{H}, \circ)$

Let $\mathcal{H}$ be a Hardy field with composition and let $f, g \in \mathcal{H}^{>\mathbb{R}}$.

Conjecture 1. *For all $\delta \in \mathcal{H}$ with $\delta \prec g$ and $(f' \circ g)\,\delta \prec (f \circ g)$, we have*

$$f \circ (g + \delta) \sim f \circ g.$$

Conjecture 2. *The centralizer $\mathcal{C}(f) := \{h \in \mathcal{H}^{>\mathbb{R}} : h \circ f = f \circ h\}$ is commutative.*

Conjecture 3. *If $f > \underset{\text{(iterates)}}{g^{[\mathbb{N}]}} > \mathrm{id}$, then $f \circ g > g \circ f$.*

Conjecture 1 holds whenever $\mathcal{H}$ contains $\exp$, but has no transexponential germ.

Let $\mathcal{R} = (\mathbb{R}, +, -, \times, <, \dots)$ be a functional expansion of the real ordered ring in a first-order language $\mathcal{L}$.

Terms and definable functions

Let $\mathcal{R} = (\mathbb{R}, +, -, \times, <, \ldots)$ be a functional expansion of the real ordered ring in a first-order language $\mathcal{L}$.

Define:

- $\mathcal{H}_{\mathcal{R}}$ as the set of germs of functions $\mathbb{R} \longrightarrow \mathbb{R}$ that are *definable* with parameters in $\mathcal{R}$.
- $\mathcal{T}_{\mathcal{R}}$ as the subset of $\mathcal{H}$ of *germs of unary functions* $r \mapsto t(r)$ for all arity $\leqslant 1$ *terms* $t[u]$.

Terms and definable functions

Let $\mathcal{R} = (\mathbb{R}, +, -, \times, <, \ldots)$ be a functional expansion of the real ordered ring in a first-order language $\mathcal{L}$.

Define:

- $\mathcal{H}_{\mathcal{R}}$ as the set of germs of functions $\mathbb{R} \longrightarrow \mathbb{R}$ that are *definable* with parameters in $\mathcal{R}$.
- $\mathcal{T}_{\mathcal{R}}$ as the subset of $\mathcal{H}$ of *germs of unary functions* $r \mapsto t(r)$ for all arity $\leqslant 1$ *terms* $t[u]$.

Those are partially ordered rings, and even $\mathcal{L}$-structures under pointwise operations.

Moreover $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{H}_{\mathcal{R}}$ are closed under composition.

Terms and definable functions

Let $\mathcal{R} = (\mathbb{R}, +, -, \times, <, \ldots)$ be a functional expansion of the real ordered ring in a first-order language $\mathcal{L}$.

Define:

- $\mathcal{H}_{\mathcal{R}}$ as the set of germs of functions $\mathbb{R} \longrightarrow \mathbb{R}$ that are *definable* with parameters in $\mathcal{R}$.
- $\mathcal{T}_{\mathcal{R}}$ as the subset of $\mathcal{H}$ of *germs of unary functions* $r \mapsto t(r)$ for all arity $\leqslant 1$ *terms* $t[u]$.

Those are partially ordered rings, and even $\mathcal{L}$-structures under pointwise operations.

Moreover $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{H}_{\mathcal{R}}$ are closed under composition.

We will consider in particular $\mathcal{R} = \mathbb{R}_{\mathrm{an,exp}}$, and we write $\mathcal{H}_{\mathrm{an,exp}} = \mathcal{H}_{\mathbb{R}_{\mathrm{an,exp}}}$.

Model theoretic perks

If $\mathrm{Th}(\mathcal{R})$ is o-minimal

Then $\mathcal{H}_{\mathcal{R}} \succcurlyeq \mathcal{R}$ is a Hardy field with composition.

Model theoretic perks

If $\mathrm{Th}(\mathcal{R})$ is o-minimal

Then $\mathcal{H}_{\mathcal{R}} \succcurlyeq \mathcal{R}$ is a Hardy field with composition.

$\mathcal{M} \succ \mathcal{R}$: non-standard model

ξ: element of $\mathcal{M}$ with $\xi > \mathbb{R}$.

Model theoretic perks

If $\mathrm{Th}(\mathcal{R})$ is o-minimal

Then $\mathcal{H}_{\mathcal{R}} \succcurlyeq \mathcal{R}$ is a Hardy field with composition.

$\mathcal{M} \succ \mathcal{R}$: non-standard model

ξ: element of $\mathcal{M}$ with $\xi > \mathbb{R}$.

There is a natural $\mathcal{L}$-embedding $\Psi : \mathcal{H}_{\mathcal{R}} \longrightarrow \mathcal{M}$ which sends id to ξ. This map commutes with definable functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

Model theoretic perks

If $\mathrm{Th}(\mathcal{R})$ is o-minimal

Then $\mathcal{H}_{\mathcal{R}} \succcurlyeq \mathcal{R}$ is a Hardy field with composition.

$\mathcal{M} \succ \mathcal{R}$: non-standard model

ξ: element of $\mathcal{M}$ with $\xi > \mathbb{R}$.

There is a natural $\mathcal{L}$-embedding $\Psi: \mathcal{H}_{\mathcal{R}} \longrightarrow \mathcal{M}$ which sends id to ξ. This map commutes with definable functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

There is a unique $\mathcal{L}$-embedding $\Phi: \mathcal{T}_{\mathcal{R}} \longrightarrow \mathcal{M}$ with $\Phi(\mathrm{id}) = \xi$.

Model theoretic perks

If $\mathrm{Th}(\mathcal{R})$ is o-minimal

Then $\mathcal{H}_{\mathcal{R}} \succcurlyeq \mathcal{R}$ is a Hardy field with composition.

$\mathcal{M} \succ \mathcal{R}$: non-standard model

ξ: element of $\mathcal{M}$ with $\xi > \mathbb{R}$.

There is a natural $\mathcal{L}$-embedding $\Psi: \mathcal{H}_{\mathcal{R}} \longrightarrow \mathcal{M}$ which sends id to ξ. This map commutes with definable functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

There is a unique $\mathcal{L}$-embedding $\Phi: \mathcal{T}_{\mathcal{R}} \longrightarrow \mathcal{M}$ with $\Phi(\mathrm{id}) = \xi$.

If $\mathrm{Th}(\mathcal{R})$ has QE and a universal axiomatization

Then any definable function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is given piecewise by (a finite list of) terms. In particular for $n = 1$, the germ of f lies in $\mathcal{T}_{\mathcal{R}}$. So $\mathcal{T}_{\mathcal{R}} = \mathcal{H}_{\mathcal{R}}$.

Model theoretic perks

If $\mathrm{Th}(\mathcal{R})$ is o-minimal

Then $\mathcal{H}_{\mathcal{R}} \succcurlyeq \mathcal{R}$ is a Hardy field with composition.

$\mathcal{M} \succ \mathcal{R}$: non-standard model

ξ: element of $\mathcal{M}$ with $\xi > \mathbb{R}$.

There is a natural $\mathcal{L}$-embedding $\Psi: \mathcal{H}_{\mathcal{R}} \longrightarrow \mathcal{M}$ which sends id to ξ. This map commutes with definable functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

There is a unique $\mathcal{L}$-embedding $\Phi: \mathcal{T}_{\mathcal{R}} \longrightarrow \mathcal{M}$ with $\Phi(\mathrm{id}) = \xi$.

If $\mathrm{Th}(\mathcal{R})$ has QE and a universal axiomatization

Then any definable function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is given piecewise by (a finite list of) terms. In particular for $n = 1$, the germ of f lies in $\mathcal{T}_{\mathcal{R}}$. So $\mathcal{T}_{\mathcal{R}} = \mathcal{H}_{\mathcal{R}}$.

Both conditions are satisfied for $\mathcal{R} = \mathbb{R}_{\mathrm{an,exp}}$ (vdDMM, '94).

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ for $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering on $\mathfrak{L}$ is lexicographic.

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ for $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering on $\mathfrak{L}$ is lexicographic.

Logarithmic transseries

$\mathbb{T}_L$ is the field $\mathbb{R}[[\mathfrak{L}]]$ of Hahn series with real coefficients and monomial group $\mathfrak{L}$. E.g.

$$f_0 = x + \pi (\log x)^3 + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$
$$f_1 = x + \log x + \log \log x + \cdots$$

We have a logarithm $\log \colon \mathbb{T}_L^{>0} \longrightarrow \mathbb{T}_L$:

$$\log \left(\underbrace{\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}}_{\in \mathfrak{L}} \right) :=$$

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ for $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering on $\mathfrak{L}$ is lexicographic.

Logarithmic transseries

$\mathbb{T}_L$ is the field $\mathbb{R}[[\mathfrak{L}]]$ of Hahn series with real coefficients and monomial group $\mathfrak{L}$. E.g.

$$f_0 = x + \pi (\log x)^3 + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$

$$f_1 = x + \log x + \log \log x + \cdots$$

We have a logarithm $\log \colon \mathbb{T}_L^{>0} \longrightarrow \mathbb{T}_L$:

$$\log \left(\prod_{k<n} (\log_k x)^{\mathfrak{l}_k} \right) := \sum_{k<n} \mathfrak{l}_k \log_{k+1} x$$

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ for $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering on $\mathfrak{L}$ is lexicographic.

Logarithmic transseries

$\mathbb{T}_L$ is the field $\mathbb{R}[[\mathfrak{L}]]$ of Hahn series with real coefficients and monomial group $\mathfrak{L}$. E.g.

$$f_0 = x + \pi (\log x)^3 + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$
$$f_1 = x + \log x + \log \log x + \cdots$$

We have a logarithm $\log \colon \mathbb{T}_L^{>0} \longrightarrow \mathbb{T}_L$:

$$\log\Big(\underset{\in \mathfrak{L}}{\mathfrak{l}} \; \overset{\in \mathbb{R}}{r} \Big(1 + \underset{\prec 1}{\varepsilon}\Big)\Big) :=$$

Transseries

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ for $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering on $\mathfrak{L}$ is lexicographic.

Logarithmic transseries

$\mathbb{T}_L$ is the field $\mathbb{R}[[\mathfrak{L}]]$ of Hahn series with real coefficients and monomial group $\mathfrak{L}$. E.g.

$$f_0 = x + \pi (\log x)^3 + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$
$$f_1 = x + \log x + \log \log x + \cdots$$

$$\log(\mathfrak{l}\, r\, (1 + \varepsilon)) \quad := \quad \log(\mathfrak{l}) + \log r + \sum_{k>0} \frac{(-1)^{k+1}}{k} \varepsilon^k.$$

Transseries

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ for $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering on $\mathfrak{L}$ is lexicographic.

Logarithmic transseries

$\mathbb{T}_L$ is the field $\mathbb{R}[[\mathfrak{L}]]$ of Hahn series with real coefficients and monomial group $\mathfrak{L}$. E.g.

$$f_0 = x + \pi (\log x)^3 + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$
$$f_1 = x + \log x + \log \log x + \cdots$$

We have a morphism $\log : (\mathbb{T}_L^{>0}, \times, <) \longrightarrow (\mathbb{T}_L, +, <)$.

Transseries

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ for $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering on $\mathfrak{L}$ is lexicographic.

Logarithmic transseries

$\mathbb{T}_L$ is the field $\mathbb{R}[[\mathfrak{L}]]$ of Hahn series with real coefficients and monomial group $\mathfrak{L}$. E.g.

$$f_0 = x + \pi (\log x)^3 + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$
$$f_1 = x + \log x + \log \log x + \cdots$$

We have a morphism $\log\colon (\mathbb{T}_L^{>0}, \times, <) \longrightarrow (\mathbb{T}_L, +, <)$.

Exponential extensions

$\log\colon \mathbb{T}_L^{>0} \longrightarrow \mathbb{T}_L$ *is not surjective, but* $\mathbb{T}_L$ *can be closed under exponentials: iteratively adjoin formal monomials* e^{φ} *for certain transseries* φ *as long as* $\exp(\varphi)$ *is undefined.*

Transseries with derivation and composition

Transseries, introduced by Dahn-Göring and Écalle, can be declined in several forms: grid-based, log-exp ($\mathbb{T}_{\mathrm{LE}}$), exp-log, . . .

We consider *generalized transseries*, which form a class sized Hahn series field $\mathbb{R}\langle\langle x\rangle\rangle \supsetneq \mathbb{T}_{\mathrm{LE}}$ closed under $\exp$ and $\log$.

Transseries with derivation and composition

Transseries, introduced by Dahn-Göring and Écalle, can be declined in several forms: grid-based, log-exp ($\mathbb{T}_{\mathrm{LE}}$), exp-log, ...

We consider *generalized transseries*, which form a class sized Hahn series field $\mathbb{R}\langle\langle x\rangle\rangle \supsetneq \mathbb{T}_{\mathrm{LE}}$ closed under $\exp$ and $\log$.

Dervation and composition (Scmheling, Berarducci-Mantova)

The field $\mathbb{R}\langle\langle x\rangle\rangle$ is equipped with a derivation $\partial\colon \mathbb{R}\langle\langle x\rangle\rangle \longrightarrow \mathbb{R}\langle\langle x\rangle\rangle$ and a composition law $\circ\colon \mathbb{R}\langle\langle x\rangle\rangle \times (\mathbb{R}\langle\langle x\rangle\rangle)^{>\mathbb{R}} \longrightarrow \mathbb{R}\langle\langle x\rangle\rangle$ whose properties mimic those of Hardy fields with composition.

Transseries with derivation and composition

Transseries, introduced by Dahn-Göring and Écalle, can be declined in several forms: grid-based, log-exp ($\mathbb{T}_{\mathrm{LE}}$), exp-log, ...

We consider *generalized transseries*, which form a class sized Hahn series field $\mathbb{R}\langle\langle x\rangle\rangle \supsetneq \mathbb{T}_{\mathrm{LE}}$ closed under $\exp$ and $\log$.

Dervation and composition (Scmheling, Berarducci-Mantova)

The field $\mathbb{R}\langle\langle x\rangle\rangle$ is equipped with a derivation $\partial\colon \mathbb{R}\langle\langle x\rangle\rangle \longrightarrow \mathbb{R}\langle\langle x\rangle\rangle$ and a composition law $\circ\colon \mathbb{R}\langle\langle x\rangle\rangle \times (\mathbb{R}\langle\langle x\rangle\rangle)^{>\mathbb{R}} \longrightarrow \mathbb{R}\langle\langle x\rangle\rangle$ whose properties mimic those of Hardy fields with composition.

$$f_0 = x + \pi\,(\log x) + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{2\,(\log x)^2}{x}$$

$$f_0' = 1 + \frac{\pi}{x} - \frac{1}{x\,(\log x)^2} - \frac{2}{x\,(\log x)^3} - \cdots - \frac{4\,x\,(\log x) - 2\,x\,(\log x)^2}{x^2}$$

Transseries with derivation and composition

Transseries, introduced by Dahn-Göring and Écalle, can be declined in several forms: grid-based, log-exp ($\mathbb{T}_{\mathrm{LE}}$), exp-log, ...

We consider *generalized transseries*, which form a class sized Hahn series field $\mathbb{R}\langle\langle x\rangle\rangle \supsetneq \mathbb{T}_{\mathrm{LE}}$ closed under exp and log.

Dervation and composition (Scmheling, Berarducci-Mantova)

The field $\mathbb{R}\langle\langle x\rangle\rangle$ is equipped with a derivation $\partial \colon \mathbb{R}\langle\langle x\rangle\rangle \longrightarrow \mathbb{R}\langle\langle x\rangle\rangle$ and a composition law $\circ \colon \mathbb{R}\langle\langle x\rangle\rangle \times (\mathbb{R}\langle\langle x\rangle\rangle)^{>\mathbb{R}} \longrightarrow \mathbb{R}\langle\langle x\rangle\rangle$ whose properties mimic those of Hardy fields with composition.

$$f_1 = x + \log x + \log\log x + \cdots$$

$$f_1 \circ (\log x) = \log x + \log\log x + \log\log\log x + \cdots$$

Log-exp-analytic functions as transseries

Theorem (vdDries-Macintyre-Marker, 1997 and 2001)

There is a unique $\mathcal{L}_{\mathrm{an},\exp}$-embedding $\mathcal{H}_{\mathrm{an},\exp} \longrightarrow \mathbb{R}\langle\langle x \rangle\rangle$ which sends id to x. This embedding also preserves ∂ and $\circ$.

Theorem (vdDries-Macintyre-Marker, 1997 and 2001)

There is a unique $\mathcal{L}_{\mathrm{an,exp}}$-embedding $\mathcal{H}_{\mathrm{an,exp}} \longrightarrow \mathbb{R}\langle\langle x \rangle\rangle$ which sends id to x. This embedding also preserves ∂ and $\circ$.

This embedding can be used to describe growth orders of germs in $\mathcal{H}_{\mathrm{an,exp}}$.

Theorem (vdDries-Macintyre-Marker, 1997 and 2001)

There is a unique $\mathcal{L}_{\mathrm{an},\exp}$-embedding $\mathcal{H}_{\mathrm{an},\exp} \longrightarrow \mathbb{R}\langle\langle x \rangle\rangle$ which sends id to x. This embedding also preserves ∂ and $\circ$.

This embedding can be used to describe growth orders of germs in $\mathcal{H}_{\mathrm{an},\exp}$.

Using this, vdDMM showed that:

- the inverse of $t \mapsto (\log t)(\log\log t)$ is not an L-function (as conjectured by Hardy).
- no primitive $\int_a^t \mathrm{e}^s \,\mathrm{ds}$ of $t \mapsto \mathrm{e}^{t^2}$ is definable in $\mathbb{R}_{\mathrm{an},\exp}$.

Theorem (vdDries-Macintyre-Marker, 1997 and 2001)

There is a unique $\mathcal{L}_{\mathrm{an,exp}}$-embedding $\mathcal{H}_{\mathrm{an,exp}} \longrightarrow \mathbb{R}\langle\langle x \rangle\rangle$ which sends id to x. This embedding also preserves ∂ and $\circ$.

This embedding can be used to describe growth orders of germs in $\mathcal{H}_{\mathrm{an,exp}}$.

Using this, vdDMM showed that:

- the inverse of $t \mapsto (\log t)(\log\log t)$ is not an L-function (as conjectured by Hardy).
- no primitive $\int_a^t \mathrm{e}^s \,\mathrm{ds}$ of $t \mapsto \mathrm{e}^{t^2}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

We will see that the field $\mathcal{H}_{\mathrm{an,exp}}$ satisfies the conjecture.

Consider the functional, conjugation equation

$$E(t+1) = \mathrm{e}^{E(t)}, \qquad \text{for } t \gg 1. \tag{1}$$

in E, called **Abel's equation** for $\exp$. Continuous solutions of (1) are transexponential.

Consider the functional, conjugation equation

$$E(t+1) = \mathrm{e}^{E(t)}, \qquad \text{for } t \gg 1. \tag{2}$$

in E, called **Abel's equation** for $\exp$. Continuous solutions of (2) are transexponential.

Kneser showed in 1949 that (2) has an analytic solution $\exp_\omega$, say with $\exp_\omega(0) = 1$. The (germ of the) functional inverse $\log_\omega$ satisfies the dual equation

$$\log_\omega(\log t) = \log_\omega(t) - 1 \quad \text{for } t \gg 1.$$

Consider the functional, conjugation equation

$$E(t+1) = \mathrm{e}^{E(t)}, \qquad \text{for } t \gg 1. \tag{3}$$

in E, called **Abel's equation** for $\exp$. Continuous solutions of (3) are transexponential.

Kneser showed in 1949 that (3) has an analytic solution $\exp_\omega$, say with $\exp_\omega(0) = 1$. The (germ of the) functional inverse $\log_\omega$ satisfies the dual equation

$$\log_\omega(\log t) = \log_\omega(t) - 1 \quad \text{for } t \gg 1.$$

Those functions induce a flow of **real-iterates** of $\exp$, i.e. a strictly increasing morphism

$$\begin{aligned} \exp^{[\cdot]} \colon (\mathbb{R}, +, <) &\longrightarrow (\mathcal{G}, \circ, <) \\ r &\longmapsto \exp^{[r]} := \exp_\omega \circ (\log_\omega + r). \end{aligned}$$

Consider the functional, conjugation equation

$$E(t+1) = e^{E(t)}, \qquad \text{for } t \gg 1. \tag{4}$$

in E, called **Abel's equation** for $\exp$. Continuous solutions of (4) are transexponential.

KNESER showed in 1949 that (4) has an analytic solution $\exp_\omega$, say with $\exp_\omega(0) = 1$. The (germ of the) functional inverse $\log_\omega$ satisfies the dual equation

$$\log_\omega(\log t) = \log_\omega(t) - 1 \quad \text{for } t \gg 1.$$

Those functions induce a flow of **real-iterates** of $\exp$, i.e. a strictly increasing morphism

$$\begin{aligned} \exp^{[\cdot]} \colon (\mathbb{R}, +, <) &\longrightarrow (\mathcal{G}, \circ, <) \\ r &\longmapsto \exp^{[r]} := \exp_\omega \circ (\log_\omega + r). \end{aligned}$$

Theorem [PADGETT, '22]

There is a Hardy field $\mathcal{T}_\omega$ with composition which contains $\exp_\omega$ and $\log_\omega$.

Exp-log classes

Given an ordered exponential field F *and* $a \in F^{>\mathbb{R}}$, *the* **exp-log class** $\mathrm{EL}(a)$ *of* $a \in F$ *is its equivalence class for*

$$a \asymp^{L} b \qquad \textit{if and only if} \qquad \exists n \in \mathbb{N}, (\log^{[n]} a \sim \log^{[n]} b).$$

Exp-log classes

Given an ordered exponential field F *and* $a \in F^{>\mathbb{R}}$*, the* **exp-log class** $\mathrm{EL}(a)$ *of* $a \in F$ *is its equivalence class for*

$$a \asymp^L b \qquad \textit{if and only if} \qquad \exists n \in \mathbb{N}, (\log^{[n]} a \sim \log^{[n]} b).$$

Let $\mathcal{H}$ be a Hardy field with composition with $\exp, \log \in \mathcal{H}$. Set

$$\mathcal{E} := \{\exp^{[n]} \circ (\log^{[n]} \pm 1) : n \in \mathbb{N}\} \subseteq \mathcal{H}^{>\mathbb{R}}.$$

Then each $\mathrm{EL}(f)$ is the convex hull of $\mathcal{E} \circ f = \{g \circ f : g \in \mathcal{E}\}$.

Exp-log classes

Given an ordered exponential field F and $a \in F^{>\mathbb{R}}$, the **exp-log class** $\mathrm{EL}(a)$ *of* $a \in F$ *is its equivalence class for*

$$a \asymp^L b \qquad \textit{if and only if} \qquad \exists n \in \mathbb{N}, (\log^{[n]} a \sim \log^{[n]} b).$$

Let $\mathcal{H}$ be a Hardy field with composition with $\exp, \log \in \mathcal{H}$. Set

$$\mathcal{E} := \{\exp^{[n]} \circ (\log^{[n]} \pm 1) : n \in \mathbb{N}\} \subseteq \mathcal{H}^{>\mathbb{R}}.$$

Then each $\mathrm{EL}(f)$ is the convex hull of $\mathcal{E} \circ f = \{g \circ f : g \in \mathcal{E}\}$.

Marker-Miller: EL classes in $\mathcal{H}_{\mathrm{an,exp}}$ and $\mathbb{R}\langle\langle x \rangle\rangle$ are parametrized by integers. Each f lies in $\mathrm{EL}(\exp^{[n]})$ for a unique $n \in \mathbb{Z}$ called the **level** of f. Write $\lambda_n = \mathrm{EL}(f)$.

Exp-log classes

Given an ordered exponential field F and $a \in F^{>\mathbb{R}}$, the **exp-log class** $\mathrm{EL}(a)$ *of* $a \in F$ *is its equivalence class for*

$$a \asymp^L b \qquad \textit{if and only if} \qquad \exists n \in \mathbb{N}, (\log^{[n]} a \sim \log^{[n]} b).$$

Let $\mathcal{H}$ be a Hardy field with composition with $\exp, \log \in \mathcal{H}$. Set

$$\mathcal{E} := \{\exp^{[n]} \circ (\log^{[n]} \pm 1) : n \in \mathbb{N}\} \subseteq \mathcal{H}^{>\mathbb{R}}.$$

Then each $\mathrm{EL}(f)$ is the convex hull of $\mathcal{E} \circ f = \{g \circ f : g \in \mathcal{E}\}$.

Marker-Miller: EL classes in $\mathcal{H}_{\mathrm{an,exp}}$ and $\mathbb{R}\langle\langle x \rangle\rangle$ are parametrized by integers. Each f lies in $\mathrm{EL}(\exp^{[n]})$ for a unique $n \in \mathbb{Z}$ called the **level** of f. Write $\lambda_n = \mathrm{EL}(f)$.

Note that $\log \lambda_n = \lambda_{n-1}$ for all $n \in \mathbb{Z}$.

Let us come back to the field $\mathcal{T}_\omega$. Write

$$\lambda_\omega := \mathrm{EL}(\exp_\omega), \qquad \lambda_{-\omega} := \mathrm{EL}(\log_\omega), \qquad \text{and } \lambda_r := \mathrm{EL}(\exp^{[r]}) \text{ for all } r \in \mathbb{R}.$$

More levels

Let us come back to the field $\mathcal{T}_\omega$. Write

$$\lambda_\omega := \mathrm{EL}(\exp_\omega), \qquad \lambda_{-\omega} := \mathrm{EL}(\log_\omega), \qquad \text{and } \lambda_r := \mathrm{EL}(\exp^{[r]}) \text{ for all } r \in \mathbb{R}.$$

We have

$$\lambda_\omega > \lambda_{\mathbb{Z}}, \qquad \lambda_{-\omega} < \lambda_{\mathbb{Z}}, \qquad \text{and} \quad \forall r, s \in \mathbb{R}, (\lambda_r < \lambda_s \Longleftrightarrow r < s).$$

More levels

Let us come back to the field $\mathcal{T}_\omega$. Write

$$\lambda_\omega := \mathrm{EL}(\exp_\omega), \qquad \lambda_{-\omega} := \mathrm{EL}(\log_\omega), \qquad \text{and } \lambda_r := \mathrm{EL}(\exp^{[r]}) \text{ for all } r \in \mathbb{R}.$$

We have

$$\lambda_\omega > \lambda_{\mathbb{Z}}, \qquad \lambda_{-\omega} < \lambda_{\mathbb{Z}}, \qquad \text{and} \quad \forall r, s \in \mathbb{R}, (\lambda_r < \lambda_s \Longleftrightarrow r < s).$$

We also have levels $\omega - 1$, $\omega + 1, \ldots$ with

$$\lambda_{\mathbb{R}} < \lambda_{\omega-1} = \log(\lambda_\omega) < \lambda_\omega < \lambda_{\omega+1} = \exp(\lambda_\omega)$$

and so on…

Even more levels

For $\varphi, \psi \in \mathcal{T}_P^{>\mathbb{R}}$, we have

$$\begin{aligned} \mathrm{EL}(\exp_\omega \circ \varphi) < \mathrm{EL}(\exp_\omega \circ \psi) &\iff \mathcal{E} \circ (\exp_\omega \circ \varphi) < \mathcal{E} \circ (\exp_\omega \circ \psi) \\ &\iff (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \varphi < (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \psi. \end{aligned}$$

Even more levels

For $\varphi, \psi \in \mathcal{T}_\omega^{>\mathbb{R}}$, we have

$$\begin{aligned} \mathrm{EL}(\exp_\omega \circ \varphi) < \mathrm{EL}(\exp_\omega \circ \psi) \iff & \mathcal{E} \circ (\exp_\omega \circ \varphi) < \mathcal{E} \circ (\exp_\omega \circ \psi) \\ \iff & (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \varphi < (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \psi. \end{aligned}$$

Even more levels

For $\varphi, \psi \in \mathcal{T}_\omega^{>\mathbb{R}}$, we have

$$\begin{aligned} \mathrm{EL}(\exp_\omega \circ \varphi) < \mathrm{EL}(\exp_\omega \circ \psi) &\iff \mathcal{E} \circ (\exp_\omega \circ \varphi) < \mathcal{E} \circ (\exp_\omega \circ \psi) \\ &\iff (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \varphi < (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \psi. \end{aligned}$$

Set $g = \exp^{[n]} \circ (\log^{[n]} + 1) \in \mathcal{E}$. Since $\log'_\omega \approx {}^1\!/_{\mathrm{id}}$, the mean value theorem for $\log_\omega$ gives

$$\mathrm{id} + \frac{1}{\log^{[n-1]} \circ \exp_\omega} < \log_\omega \circ g \circ \exp_\omega < \mathrm{id} + \frac{1}{\log^{[n]} \circ \exp_\omega}.$$

Even more levels

For $\varphi, \psi \in \mathcal{T}_\omega^{>\mathbb{R}}$, we have

$$\mathrm{EL}(\exp_\omega \circ \varphi) < \mathrm{EL}(\exp_\omega \circ \psi) \iff \mathcal{E} \circ (\exp_\omega \circ \varphi) < \mathcal{E} \circ (\exp_\omega \circ \psi)$$
$$\iff (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \varphi < (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \psi.$$

Set $g = \exp^{[n]} \circ (\log^{[n]} + 1) \in \mathcal{E}$. Since $\log_\omega' \approx {}^1\!/_{\mathrm{id}}$, the mean value theorem for $\log_\omega$ gives

$$\mathrm{id} + \frac{1}{\log^{[n-1]} \circ \exp_\omega} < \log_\omega \circ g \circ \exp_\omega < \mathrm{id} + \frac{1}{\log^{[n]} \circ \exp_\omega}.$$

If $\varphi + \left(\frac{1}{\log^{[\mathbb{N}]} \circ \exp_\omega \circ \varphi}\right) < \psi$, then $\mathrm{EL}(\exp_\omega \circ \varphi) < \mathrm{EL}(\exp_\omega \circ \psi)$. The EL class $\lambda_{1/\omega}$ of

$$\exp_\omega \circ \left(\log_\omega + \frac{1}{\log_\omega}\right)$$

is “infinitesimal”, i.e. larger than λ_0 but smaller than each λ_r for $r \in \mathbb{R}^{>}$.

All levels

Tentative description of all possible levels in models of $\mathbb{R}_{\exp}$ using Conway's field $\mathbf{No}$ of surreal numbers:

Theorem (Berarducci-Mantova, 2015)

EL *classes in* $(\mathbf{No}, \exp)$ *are in canonical order isomorphism with* $(\mathbf{No}, <)$ *itself.*

There is an order embedding $\mathbf{No} \longrightarrow \mathbf{No}^{>\mathbb{R}}; z \mapsto \lambda_z$ *such that each surreal number lies in* $\mathrm{EL}(\lambda_z)$ *for a unique* $z \in \mathbf{No}$*.*

All levels

Tentative description of all possible levels in models of $\mathbb{R}_{\exp}$ using Conway's field $\mathbf{No}$ of surreal numbers:

Theorem (Berarducci-Mantova, 2015)

EL *classes in* $(\mathbf{No}, \exp)$ *are in canonical order isomorphism with* $(\mathbf{No}, <)$ *itself.*

There is an order embedding $\mathbf{No} \longrightarrow \mathbf{No}^{>\mathbb{R}}; z \mapsto \lambda_z$ *such that each surreal number lies in* $\mathrm{EL}(\lambda_z)$ *for a unique* $z \in \mathbf{No}$.

They defined a canonical derivation ∂ on $\mathbf{No}$ such that $(\mathbf{No}, \partial)$ is a Liouville-closed H-field. It is an elementary extension of $\mathbb{T}_{\mathrm{LE}}$ (Ashcenbrenner-vdDries-vdHoeven, '17).

All levels

Tentative description of all possible levels in models of $\mathbb{R}_{\exp}$ using Conway's field $\mathbf{No}$ of surreal numbers:

Theorem (Berarducci-Mantova, 2015)

EL *classes in* $(\mathbf{No}, \exp)$ *are in canonical order isomorphism with* $(\mathbf{No}, <)$ *itself.*

There is an order embedding $\mathbf{No} \longrightarrow \mathbf{No}^{>\mathbb{R}}; z \mapsto \lambda_z$ *such that each surreal number lies in* $\mathrm{EL}(\lambda_z)$ *for a unique* $z \in \mathbf{No}$*.*

They defined a canonical derivation ∂ on $\mathbf{No}$ such that $(\mathbf{No}, \partial)$ is a Liouville-closed H-field. It is an elementary extension of $\mathbb{T}_{\mathrm{LE}}$ (Ashcenbrenner-vdDries-vdHoeven, '17).

Every model of $\mathbb{R}_{\exp}$ embeds into $(\mathbf{No}, \exp)$.

All levels

Tentative description of all possible levels in models of $\mathbb{R}_{\exp}$ using Conway's field $\mathbf{No}$ of surreal numbers:

Theorem (Berarducci-Mantova, 2015)

EL *classes in* $(\mathbf{No}, \exp)$ *are in canonical order isomorphism with* $(\mathbf{No}, <)$ *itself.*

There is an order embedding $\mathbf{No} \longrightarrow \mathbf{No}^{>\mathbb{R}}; z \mapsto \lambda_z$ *such that each surreal number lies in* $\mathrm{EL}(\lambda_z)$ *for a unique* $z \in \mathbf{No}$*.*

They defined a canonical derivation ∂ on $\mathbf{No}$ such that $(\mathbf{No}, \partial)$ is a Liouville-closed H-field. It is an elementary extension of $\mathbb{T}_{\mathrm{LE}}$ (Ashcenbrenner-vdDries-vdHoeven, '17).

Every model of $\mathbb{R}_{\exp}$ embeds into $(\mathbf{No}, \exp)$.

Question. Is any real-closed Hardy field H closed under $\exp$ and $\log$ an elementary extension of $\mathbb{R}_{\exp}$?

Goal

Berarducci-Mantova (2017) defined a composition law

$$\circ \colon \mathbb{R}\langle\langle x\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}.$$

$\mathbb{R}\langle\langle x\rangle\rangle$ naturally embeds into $\mathbf{No}$ by sending f to $f \circ \omega$ for a certain $\omega \in \mathbf{No}^{>\mathbb{R}}$.

Goal

Berarducci-Mantova (2017) defined a composition law

$$\circ \colon \mathbb{R}\langle\langle x \rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}.$$

$\mathbb{R}\langle\langle x \rangle\rangle$ naturally embeds into $\mathbf{No}$ by sending f to $f \circ \omega$ for a certain $\omega \in \mathbf{No}^{>\mathbb{R}}$.

The composition law does not extend to $\mathbf{No} \times \mathbf{No}^{>\mathbb{R}}$ in a compatible way with respect to ∂. So let us leave the surreal realm.

Goal

BERARDUCCI-MANTOVA (2017) defined a composition law

$$\circ\colon \mathbb{R}\langle\langle x\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}.$$

$\mathbb{R}\langle\langle x\rangle\rangle$ naturally embeds into $\mathbf{No}$ by sending f to $f \circ \omega$ for a certain $\omega \in \mathbf{No}^{>\mathbb{R}}$.

The composition law does not extend to $\mathbf{No} \times \mathbf{No}^{>\mathbb{R}}$ in a compatible way with respect to ∂. So let us leave the surreal realm.

$\rightarrow$ Can we "directly" build a field of generalized power series involving transexponential and sublogarithmic behavior?

Goal

Berarducci-Mantova (2017) defined a composition law

$$\circ \colon \mathbb{R}\langle\langle x\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}.$$

$\mathbb{R}\langle\langle x\rangle\rangle$ naturally embeds into $\mathbf{No}$ by sending f to $f \circ \omega$ for a certain $\omega \in \mathbf{No}^{>\mathbb{R}}$.

The composition law does not extend to $\mathbf{No} \times \mathbf{No}^{>\mathbb{R}}$ in a compatible way with respect to ∂. So let us leave the surreal realm.

$\longrightarrow$ Can we "directly" build a field of generalized power series involving transexponential and sublogarithmic behavior?

$\longrightarrow$ Can we do so while solving the functional equations

$$E \circ (x+1) = \mathrm{e}^x \circ E \qquad \text{and} \qquad L - 1 = L \circ \log x \; ?$$

Berarducci-Mantova (2017) defined a composition law

$$\circ\colon \mathbb{R}\langle\langle x\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}.$$

$\mathbb{R}\langle\langle x\rangle\rangle$ naturally embeds into $\mathbf{No}$ by sending f to $f\circ\omega$ for a certain $\omega \in \mathbf{No}^{>\mathbb{R}}$.

The composition law does not extend to $\mathbf{No}\times\mathbf{No}^{>\mathbb{R}}$ in a compatible way with respect to ∂. So let us leave the surreal realm.

$\longrightarrow$ Can we "directly" build a field of generalized power series involving transexponential and sublogarithmic behavior?

$\longrightarrow$ Can we do so while solving the functional equations

$$E\circ(x+1) = \mathrm{e}^x \circ E \qquad \text{and} \qquad L-1 = L\circ \log x \ ?$$

$\longrightarrow$ Do we also get a formal version of the conjecture for those fields?

Formal hyperlogarithms

Let us build a structure $(\mathbb{L}, \partial, \circ)$ which contains a solution $\ell_\omega x$ to

$$\ell_\omega x - 1 = (\ell_\omega x) \circ (\ell_1 x). \tag{5}$$

Let us build a structure $(\mathbb{L}, \partial, \circ)$ which contains a solution $\ell_\omega x$ to

$$\ell_\omega x - 1 = (\ell_\omega x) \circ (\ell_1 x). \tag{6}$$

We gather symbols $\ell_\gamma x, \gamma < \omega^2$ with $\ell_0 x = x$ is the identity, $\ell_1 x$ is the logarithm, and

$$\forall m, n \in \mathbb{N}, \ell_{\omega m + n} x = (\ell_1 x)^{[n]} \circ (\ell_\omega x)^{[m]}.$$

Formal hyperlogarithms

Let us build a structure $(\mathbb{L}, \partial, \circ)$ which contains a solution $\ell_\omega x$ to

$$\ell_\omega x - 1 = (\ell_\omega x) \circ (\ell_1 x). \tag{7}$$

We gather symbols $\ell_\gamma x, \gamma < \omega^2$ with $\ell_0 x = x$ is the identity, $\ell_1 x$ is the logarithm, and

$$\forall m, n \in \mathbb{N}, \ell_{\omega m + n} x = (\ell_1 x)^{[n]} \circ (\ell_\omega x)^{[m]}.$$

Differentiating (7), we get

$$(\ell_\omega x)' \;=\; \frac{1}{x} \times (\ell_\omega x)' \circ (\ell_1 x)$$

Let us build a structure $(\mathbb{L}, \partial, \circ)$ which contains a solution $\ell_\omega x$ to

$$\ell_\omega x - 1 = (\ell_\omega x) \circ (\ell_1 x). \tag{8}$$

We gather symbols $\ell_\gamma x, \gamma < \omega^2$ with $\ell_0 x = x$ is the identity, $\ell_1 x$ is the logarithm, and

$$\forall m, n \in \mathbb{N}, \ell_{\omega m + n} x = (\ell_1 x)^{[n]} \circ (\ell_\omega x)^{[m]}.$$

Differentiating (8), we get

$$\begin{aligned} (\ell_\omega x)' &= \frac{1}{x} \times (\ell_\omega x)' \circ (\ell_1 x) \\ &= \frac{1}{x\,(\ell_1 x) \cdots (\ell_n x)} \times (\ell_\omega x)' \circ (\ell_{n+1} x) \end{aligned}$$

Formal hyperlogarithms

Let us build a structure $(\mathbb{L}, \partial, \circ)$ which contains a solution $\ell_\omega x$ to

$$\ell_\omega x - 1 = (\ell_\omega x) \circ (\ell_1 x). \tag{9}$$

We gather symbols $\ell_\gamma x, \gamma < \omega^2$ with $\ell_0 x = x$ is the identity, $\ell_1 x$ is the logarithm, and

$$\forall m, n \in \mathbb{N}, \ell_{\omega m + n} x = (\ell_1 x)^{[n]} \circ (\ell_\omega x)^{[m]}.$$

Differentiating (9), we get

$$\begin{aligned} (\ell_\omega x)' &= \frac{1}{x} \times (\ell_\omega x)' \circ (\ell_1 x) \\ &= \frac{1}{x\,(\ell_1 x) \cdots (\ell_n x)} \times o\left(\frac{1}{\ell_n x}\right) \end{aligned}$$

Let us build a structure $(\mathbb{L}, \partial, \circ)$ which contains a solution $\ell_\omega x$ to

$$\ell_\omega x - 1 = (\ell_\omega x) \circ (\ell_1 x). \tag{10}$$

We gather symbols $\ell_\gamma x, \gamma < \omega^2$ with $\ell_0 x = x$ is the identity, $\ell_1 x$ is the logarithm, and

$$\forall m, n \in \mathbb{N}, \ell_{\omega m + n} x = (\ell_1 x)^{[n]} \circ (\ell_\omega x)^{[m]}.$$

Differentiating (10), we get

$$(\ell_\omega x)' \overset{?}{=} \prod_{n<\omega} (\ell_n x)^{-1}.$$

So one needs to have, as basic symbols, formal products

$$\mathfrak{l} := \prod_{\gamma<\omega^2} (\ell_\gamma x)^{\mathfrak{l}_\gamma}, \quad \text{for } (\mathfrak{l}_\gamma)_{\gamma<\omega^2} \in \mathbb{R}^{\omega^2}.$$

So one needs to have, as basic symbols, formal products

$$\mathfrak{l} := \prod_{\gamma<\omega^2} (\ell_\gamma x)^{\mathfrak{l}_\gamma}, \quad \text{for } (\mathfrak{l}_\gamma)_{\gamma<\omega^2} \in \mathbb{R}^{\omega^2}.$$

Gathering those in a lexicographically ordered group $\mathfrak{L}_{<\omega^2}$ yields a Hahn series field

$$\mathbb{L}_{<\omega^2} = \mathbb{R}[[\mathfrak{L}_{<\omega^2}]].$$

So one needs to have, as basic symbols, formal products

$$\mathfrak{l} := \prod_{\gamma < \omega^2} (\ell_\gamma x)^{\mathfrak{l}_\gamma}, \quad \text{for } (\mathfrak{l}_\gamma)_{\gamma < \omega^2} \in \mathbb{R}^{\omega^2}.$$

Gathering those in a lexicographically ordered group $\mathfrak{L}_{<\omega^2}$ yields a Hahn series field

$$\mathbb{L}_{<\omega^2} = \mathbb{R}[[\mathfrak{L}_{<\omega^2}]].$$

The derivation is defined by extending the rule $(\ell_\omega x)' = \prod_{n<\omega} (\ell_n x)^{-1}$.

So one needs to have, as basic symbols, formal products

$$\mathfrak{l} := \prod_{\gamma<\omega^2} (\ell_\gamma x)^{\mathfrak{l}_\gamma}, \quad \text{for } (\mathfrak{l}_\gamma)_{\gamma<\omega^2} \in \mathbb{R}^{\omega^2}.$$

Gathering those in a lexicographically ordered group $\mathfrak{L}_{<\omega^2}$ yields a Hahn series field

$$\mathbb{L}_{<\omega^2} = \mathbb{R}[[\mathfrak{L}_{<\omega^2}]].$$

The derivation is defined by extending the rule $(\ell_\omega x)' = \prod_{n<\omega} (\ell_n x)^{-1}$.

For any ordinal α, we similarly have a field $\mathbb{L}_{<\alpha}$, and a class sized field

$$\mathbb{L} := \bigcup_{\alpha \in \mathbf{On}} \mathbb{L}_{<\alpha}$$

called the field of **logarithmic hyperseries** (vdD-vdH-Kaplan).

Main properties of $\mathbb{L}$

Theorem [vdDries-vdHoeven-Kaplan - 2018]

There is a composition law $\circ\colon \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ *with* $\ell_{\omega^{\mu+1}} x - 1 = (\ell_{\omega^{\mu+1}} x) \circ (\ell_{\omega^{\mu}} x)$ *for all ordinals* μ*.* $(\mathbb{L}, \partial)$ *is an H-field with small derivation and surjective derivation. We have the chain rule for* $(\circ, \partial)$*.*

Main properties of $\mathbb{L}$

Theorem [vdDries-vdHoeven-Kaplan - 2018]

There is a composition law $\circ\colon \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ *with* $\ell_{\omega^{\mu+1}} x - 1 = (\ell_{\omega^{\mu+1}} x) \circ (\ell_{\omega^{\mu}} x)$ *for all ordinals* μ*.* $(\mathbb{L}, \partial)$ *is an H-field with small derivation and surjective derivation. We have the chain rule for* $(\circ, \partial)$*.*

How is $(\ell_\omega x) \circ g$ defined at say $g = x + 1$? Using Taylor series:

Main properties of $\mathbb{L}$

Theorem [vdDries-vdHoeven-Kaplan - 2018]

There is a composition law $\circ\colon \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ *with* $\ell_{\omega^{\mu+1}} x - 1 = (\ell_{\omega^{\mu+1}} x) \circ (\ell_{\omega^\mu} x)$ *for all ordinals* μ*.* $(\mathbb{L}, \partial)$ *is an H-field with small derivation and surjective derivation. We have the chain rule for* $(\circ, \partial)$*.*

How is $(\ell_\omega x) \circ g$ defined at say $g = x + 1$? Using Taylor series:

Theorem [vdDries-vdHoeven-Kaplan - 2018]

For $f, \delta \in \mathbb{L}$ *and* $g \in \mathbb{L}^{>\mathbb{R}}$ *with* $\delta \prec g$*, we have the following Taylor expansion around* g*:*

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!} \delta^k.$$

Main properties of $\mathbb{L}$

Theorem [vdDries-vdHoeven-Kaplan - 2018]

There is a composition law $\circ\colon \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ *with* $\ell_{\omega^{\mu+1}} x - 1 = (\ell_{\omega^{\mu+1}} x) \circ (\ell_{\omega^\mu} x)$ *for all ordinals* μ*.* $(\mathbb{L}, \partial)$ *is an H-field with small derivation and surjective derivation. We have the chain rule for* $(\circ, \partial)$*.*

How is $(\ell_\omega x) \circ g$ defined at say $g = x + 1$? Using Taylor series:

Theorem [vdDries-vdHoeven-Kaplan - 2018]

For $f, \delta \in \mathbb{L}$ *and* $g \in \mathbb{L}^{>\mathbb{R}}$ *with* $\delta \prec g$*, we have the following Taylor expansion around* g*:*

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!} \delta^k.$$

In particular

$$f \circ (x + 1) = \sum_{k \in \mathbb{N}} \frac{f^{(k)}}{k!}.$$

Hyperexponential closure

Write L_ω for the strictly increasing function $\mathbb{L}_{<\omega^2}^{>\mathbb{R}} \longrightarrow \mathbb{L}_{<\omega^2}^{>\mathbb{R}}; f \mapsto (\ell_\omega x) \circ f$.

Hyperexponential closure

Write L_ω for the strictly increasing function $\mathbb{L}_{<\omega^2}^{>\mathbb{R}} \longrightarrow \mathbb{L}_{<\omega^2}^{>\mathbb{R}}; f \mapsto (\ell_\omega x) \circ f$.

Its right inverse E_ω is partially defined. E.g. $E_\omega(x)$ is undefined. We close $\mathbb{L}_{<\omega^2}^{>\mathbb{R}}$ under E_ω by adjoining formal monomials $\mathrm{e}_\omega^\varphi$ to $\mathfrak{L}_{<\omega^2}$, for certain series φ.

Write L_ω for the strictly increasing function $\mathbb{L}_{<\omega^2}^{>\mathbb{R}} \longrightarrow \mathbb{L}_{<\omega^2}^{>\mathbb{R}}; f \mapsto (\ell_\omega x) \circ f$.

Its right inverse E_ω is partially defined. E.g. $E_\omega(x)$ is undefined. We close $\mathbb{L}_{<\omega^2}^{>\mathbb{R}}$ under E_ω by adjoining formal monomials $\mathrm{e}_\omega^\varphi$ to $\mathfrak{L}_{<\omega^2}$, for certain series φ.

When should $\boldsymbol{E_\omega(\varphi)}$ be a new monomial $\mathbf{e}_{\boldsymbol{\omega}}^{\boldsymbol{\varphi}}$? If $E_\omega(\varphi)$ is defined and

$$\varepsilon \prec \frac{1}{(\ell_n x) \circ E_\omega(\varphi)}$$

for some $n \in \mathbb{N}$, then $E_\omega(\varphi + \varepsilon)$ is given by Taylor expansions around φ.

Write L_ω for the strictly increasing function $\mathbb{L}_{<\omega^2}^{>\mathbb{R}} \longrightarrow \mathbb{L}_{<\omega^2}^{>\mathbb{R}}; f \mapsto (\ell_\omega x) \circ f$.

Its right inverse E_ω is partially defined. E.g. $E_\omega(x)$ is undefined. We close $\mathbb{L}_{<\omega^2}^{>\mathbb{R}}$ under E_ω by adjoining formal monomials $\mathrm{e}_\omega^\varphi$ to $\mathfrak{L}_{<\omega^2}$, for certain series φ.

When should $\boldsymbol{E_\omega(\varphi)}$ be a new monomial $\mathbf{e}_\omega^{\boldsymbol{\varphi}}$? If $E_\omega(\varphi)$ is defined and

$$\varepsilon \prec \frac{1}{(\ell_n x) \circ E_\omega(\varphi)}$$

for some $n \in \mathbb{N}$, then $E_\omega(\varphi + \varepsilon)$ is given by Taylor expansions around φ.

$\longrightarrow$ It is enough to add E_ω^φ for representatives φ in each convex hull

$$\mathcal{L}(g) := \mathrm{Conv}\left(\left\{ g \pm \frac{1}{(L_n x) \circ E_\omega^\varphi} : n \in \mathbb{N} \right\} \right).$$

Hyperexponential closure

Write L_ω for the strictly increasing function $\mathbb{L}_{<\omega^2}^{>\mathbb{R}} \longrightarrow \mathbb{L}_{<\omega^2}^{>\mathbb{R}}; f \mapsto (\ell_\omega x) \circ f$.

Its right inverse E_ω is partially defined. E.g. $E_\omega(x)$ is undefined. We close $\mathbb{L}_{<\omega^2}^{>\mathbb{R}}$ under E_ω by adjoining formal monomials $\mathrm{e}_\omega^\varphi$ to $\mathfrak{L}_{<\omega^2}$, for certain series φ.

When should $E_\omega(\varphi)$ be a new monomial $\mathrm{e}_\omega^\varphi$? If $E_\omega(\varphi)$ is defined and

$$\varepsilon \prec \frac{1}{(\ell_n x) \circ E_\omega(\varphi)}$$

for some $n \in \mathbb{N}$, then $E_\omega(\varphi + \varepsilon)$ is given by Taylor expansions around φ.

$\longrightarrow$ It is enough to add E_ω^φ for representatives φ in each convex hull

$$\mathcal{L}(g) := \mathrm{Conv}\left(\left\{g \pm \frac{1}{(L_n x) \circ E_\omega^\varphi} : n \in \mathbb{N}\right\}\right).$$

For any two distinct representatives φ, ψ, the EL-classes of $\mathrm{e}_\omega^\varphi$ and e_ω^ψ should be disjoint. This determines an ordering of the extension of $\mathfrak{L}_{<\omega^2}$ by monomials $\mathrm{e}_\omega^\varphi$.

Extending derivations and compositions

Theorem [B.-vdHoeven-Kaplan]

There are a minimal extension $\tilde{\mathbb{L}}$ of $\mathbb{L}$, and an extension $\circ \colon \mathbb{L} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ of the composition law on $\mathbb{L}$, for which each $L_{\omega^\mu} \colon \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}$ for ordinals μ is bijective.

Theorem [B.-vdHoeven-Kaplan]

There are a minimal extension $\tilde{\mathbb{L}}$ *of* $\mathbb{L}$*, and an extension* $\circ\colon \mathbb{L} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ *of the composition law on* $\mathbb{L}$*, for which each* $L_{\omega^\mu}\colon \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}$ *for ordinals* μ *is bijective.*

$\tilde{\mathbb{L}}$ is obtained by iteratively adjoining hyperexponentials $E_{\omega^\mu}^{\varphi}$ of hyperseries φ. Any $f \in \tilde{\mathbb{L}}$ has a concrete expression involving $\ell_\gamma x$'s, e_γ's, real numbers, and transfinite sums.

We have everything we need to define:

Extending derivations and compositions

Theorem [B.-vdHoeven-Kaplan]

There are a minimal extension $\tilde{\mathbb{L}}$ *of* $\mathbb{L}$*, and an extension* $\circ \colon \mathbb{L} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ *of the composition law on* $\mathbb{L}$*, for which each* $L_{\omega^\mu} \colon \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}$ *for ordinals* μ *is bijective.*

$\tilde{\mathbb{L}}$ is obtained by iteratively adjoining hyperexponentials $E^{\varphi}_{\omega^\mu}$ of hyperseries φ. Any $f \in \tilde{\mathbb{L}}$ has a concrete expression involving $\ell_\gamma x$'s, e_γ's, real numbers, and transfinite sums.

We have everything we need to define:

Work in progress [B.]

There is a derivation $\tilde{\partial} \colon \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ *such that* $(\tilde{\mathbb{L}}, \tilde{\partial})$ *is an elementary extension of the ordered, valued, differential field* $\mathbb{T}_{\mathrm{LE}}$ *of* $\log\text{-}\exp$ *transseries.*

Extending derivations and compositions

Theorem [B.-vdHoeven-Kaplan]

There are a minimal extension $\tilde{\mathbb{L}}$ *of* $\mathbb{L}$, *and an extension* $\circ\colon \mathbb{L} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ *of the composition law on* $\mathbb{L}$, *for which each* $L_{\omega^\mu}\colon \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}$ *for ordinals* μ *is bijective.*

$\tilde{\mathbb{L}}$ is obtained by iteratively adjoining hyperexponentials $E_{\omega^\mu}^{\varphi}$ of hyperseries φ. Any $f \in \tilde{\mathbb{L}}$ has a concrete expression involving $\ell_\gamma x$'s, e_γ's, real numbers, and transfinite sums.

We have everything we need to define:

Work in progress [B.]

There is a derivation $\tilde{\partial}\colon \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ *such that* $(\tilde{\mathbb{L}}, \tilde{\partial})$ *is an elementary extension of the ordered, valued, differential field* $\mathbb{T}_{\mathrm{LE}}$ *of* $\log$-$\exp$ *transseries.*

Work in progress [B.]

There is a composition law $\tilde{\circ}\colon \tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ *such that* $(\tilde{\partial}, \tilde{\circ})$ *satisfies the chain rule.*

The derivation and composition (on the right) should be strongly linear, i.e. commute with transfinite sums. For instance

$$\left(\sum_{n\in\mathbb{N}} n!\,\mathrm{e}_{\omega}^{\sum_{k\geqslant n}\ell_k x}\right)\circ\left(x+\frac{1}{\mathrm{e}_{\omega}^{x}}\right)=\sum_{n\in\mathbb{N}} n!\,E_{\omega}\!\left(\sum_{k\geqslant n}(\ell_k\,x)\circ\left(x+\frac{1}{\mathrm{e}_{\omega}^{x}}\right)\right).$$

Strongly linear algebra

The derivation and composition (on the right) should be strongly linear, i.e. commute with transfinite sums. For instance

$$\left(\sum_{n\in\mathbb{N}} n!\,\mathrm{e}_\omega^{\sum_{k\geqslant n}\ell_k x}\right)\circ\left(x+\frac{1}{\mathrm{e}_\omega^x}\right)=\sum_{n\in\mathbb{N}} n!\,E_\omega\left(\sum_{k\geqslant n}\underbrace{(\ell_k\,x)\circ\left(x+\frac{1}{\mathrm{e}_\omega^x}\right)}_{\text{Taylor}}\right).$$

Strongly linear algebra

The derivation and composition (on the right) should be strongly linear, i.e. commute with transfinite sums. For instance

$$\left(\sum_{n\in\mathbb{N}} n!\, \mathrm{e}_{\omega}^{\sum_{k\geqslant n}\ell_k x}\right)\circ\left(x+\frac{1}{\mathrm{e}_{\omega}^{x}}\right)=\sum_{n\in\mathbb{N}} n!\, E_{\omega}\left(\sum_{k\geqslant n} \underbrace{(\ell_k\, x)\circ\left(x+\frac{1}{\mathrm{e}_{\omega}^{x}}\right)}_{\text{Taylor}}\right).$$

Problem: making sense of those transfinite sums. We use *strongly linear algebra*: a set of order theoretic results regarding a formal notion of summability.

Strongly linear algebra

The derivation and composition (on the right) should be strongly linear, i.e. commute with transfinite sums. For instance

$$\left(\sum_{n\in\mathbb{N}} n!\, \mathrm{e}_\omega^{\sum_{k\geqslant n}\ell_k x}\right)\circ\left(x+\frac{1}{\mathrm{e}_\omega^x}\right)=\sum_{n\in\mathbb{N}} n!\, E_\omega\left(\sum_{k\geqslant n} \underbrace{(\ell_k x)\circ\left(x+\frac{1}{\mathrm{e}_\omega^x}\right)}_{\text{Taylor}}\right).$$

Problem: making sense of those transfinite sums. We use *strongly linear algebra*: a set of order theoretic results regarding a formal notion of summability.

Idea: Hahn series fields are "formal" Banach spaces. Two results of van der Hoeven:

- If $\Psi\colon \mathbb{R}[[\mathfrak{M}]]\longrightarrow\mathbb{R}[[\mathfrak{M}]]$ is strongly linear with $\Psi(s)\prec s$ for all $s\neq 0$, then $\mathrm{Id}+\Psi$ has a strongly linear functional inverse

$$(\mathrm{Id}+\Psi)^{[-1]}(s)=\sum_{k\in\mathbb{N}}(-1)^k\,\Psi^{[k]}(s).$$

- We have a strongly linear implicit function theorem.

The ordered group $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$

Work in progress [B.]: bi-ordered group of hyperseries

The class $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$ *is a linearly bi-ordered group:* $f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ *has an inverse in* $\tilde{\mathbb{L}}^{>\mathbb{R}}$ *and each function* $\tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}; g \mapsto f \circ g$ *is strictly increasing.*

Work in progress [B.]: bi-ordered group of hyperseries

The class $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$ *is a linearly bi-ordered group:* $f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ *has an inverse in* $\tilde{\mathbb{L}}^{>\mathbb{R}}$ *and each function* $\tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}; g \mapsto f \circ g$ *is strictly increasing.*

This reflects in particular the monotonicity properties of germs in Hardy fields.

The ordered group $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$

Work in progress [B.]: bi-ordered group of hyperseries

The class $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$ is a linearly bi-ordered group: $f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ has an inverse in $\tilde{\mathbb{L}}^{>\mathbb{R}}$ and each function $\tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}; g \mapsto f \circ g$ is strictly increasing.

This reflects in particular the monotonicity properties of germs in Hardy fields.

Work in progress [B.]: conjugacy

Any two series $f, g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ with $f, g > x$ are conjugate, i.e. satisfy

$$V \circ f = g \circ V$$

for a certain $V \in \tilde{\mathbb{L}}^{>\mathbb{R}}$.

For instance, the series e^x and $x + 1$ are conjugate via $V = \ell_\omega x$:

$$(\ell_\omega x) \circ \mathrm{e}^x = \ell_\omega x + 1.$$

Work in progress [B.]: Taylor expansions

For all $f, g, \delta \in \tilde{\mathbb{L}}$ *with* $g > \mathbb{R}$, *if* $\delta \prec g$ *and* $(\mathfrak{m}' \circ g)\,\delta \prec \mathfrak{m} \circ g$ *for all* $\mathfrak{m} \in \operatorname{supp} f$, *then*

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!} \delta^k.$$

The conjecture: formal / geometric

Work in progress [B.]: Taylor expansions

For all $f, g, \delta \in \tilde{\mathbb{L}}$ *with* $g > \mathbb{R}$, *if* $\delta \prec g$ *and* $(\mathfrak{m}' \circ g)\,\delta \prec \mathfrak{m} \circ g$ *for all* $\mathfrak{m} \in \operatorname{supp} f$, *then*

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!}\,\delta^k.$$

Work in progress [B.]: real iterates

For each $x \neq f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$, *there is a unique isomorphism* $(\mathbb{R}, +, <) \longrightarrow (\mathcal{C}(f), \circ, <)$ *sending* 1 *to* f. *This is defined by conjugating* f *with* $x \pm 1$: *indeed* $\mathcal{C}(x+1) = x + \mathbb{R}$.

The conjecture: formal / geometric

Work in progress [B.]: Taylor expansions

For all $f, g, \delta \in \tilde{\mathbb{L}}$ *with* $g > \mathbb{R}$*, if* $\delta \prec g$ *and* $(\mathfrak{m}' \circ g)\, \delta \prec \mathfrak{m} \circ g$ *for all* $\mathfrak{m} \in \operatorname{supp} f$*, then*

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!} \delta^k.$$

Work in progress [B.]: real iterates

For each $x \neq f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$*, there is a unique isomorphism* $(\mathbb{R}, +, <) \longrightarrow (\mathcal{C}(f), \circ, <)$ *sending* 1 *to* f*. This is defined by conjugating* f *with* $x \pm 1$*: indeed* $\mathcal{C}(x+1) = x + \mathbb{R}$*.*

Work in progress [B.]: solving inequations

For all $f, g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ $f > g^{[\mathbb{N}]} > x$*, we have* $f \circ g > g \circ f$*.*

The conjecture: formal / geometric

Work in progress [B.]: Taylor expansions

For all $f, g, \delta \in \tilde{\mathbb{L}}$ *with* $g > \mathbb{R}$*, if* $\delta \prec g$ *and* $(\mathfrak{m}' \circ g)\,\delta \prec \mathfrak{m} \circ g$ *for all* $\mathfrak{m} \in \operatorname{supp} f$*, then*

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!}\,\delta^k.$$

Work in progress [B.]: real iterates

For each $x \neq f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$*, there is a unique isomorphism* $(\mathbb{R}, +, <) \longrightarrow (\mathcal{C}(f), \circ, <)$ *sending* 1 *to* f*. This is defined by conjugating* f *with* $x \pm 1$*: indeed* $\mathcal{C}(x+1) = x + \mathbb{R}$*.*

Work in progress [B.]: solving inequations

For all $f, g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ $f > g^{[\mathbb{N}]} > x$*, we have* $f \circ g > g \circ f$*.*

Any Hardy field with composition which embeds into $\tilde{\mathbb{L}}$ satisfies the conjecture.

Thanks to these:

[1] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. The surreal numbers as a universal H-field. *Journal of the European Mathematical Society*, 21(4), 2019.

[2] A. Berarducci and V. Mantova. Surreal numbers, derivations and transseries. *JEMS*, 20(2):339–390, 2018.

[3] A. Berarducci and V. Mantova. Transseries as germs of surreal functions. *Transactions of the American Mathematical Society*, 371:3549–3592, 2019.

[4] M. Boshernitzan. New "orders of infinity". *J. d.'Analyse Math.*, 41:130–167, 1981.

[5] B. Dahn and P. Göring. Notes on exponential-logarithmic terms. *Fundamenta Mathematicae*, 127(1):45–50, 1987.

[6] L. van den Dries, J. van der Hoeven, and E. Kaplan. Logarithmic hyperseries. *Transactions of the American Mathematical Society*, 372, 2019.

[7] L. van den Dries, A. Macintyre, and D. Marker. The elementary theory of restricted analytic fields with exponentiation. *Annals of Mathematics*, 140(1):183–205, 1994.

[8] L. van den Dries, A. Macintyre, and D. Marker. Logarithmic-exponential power series. *Journal of the London Mathematical Society*, 56(3):417–434, 1997.

[9] L. van den Dries, A. Macintyre, and D. Marker. Logarithmic-exponential series. *Annals of Pure and Applied Logic*, 111:61–113, 07 2001.

[10] J. Ecalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Actualités Mathématiques. Hermann, 1992.

[11] G. H. Hardy. *Orders of infinity, the 'Infinitärcalcül' of Paul du Bois-Reymond*. Cambridge University Press edition, 1910.

[12] H. Kneser. Reelle analytische Lösung der Gleichung $\phi(\phi(x)) = \mathrm{e}^x$ und verwandter Funktionalgleichungen. *Journal Für Die Reine Und Angewandte Mathematik*, 1950:56–67, 01 1950.

[13] D. Marker and C. Miller. Levelled o-minimal structures. *Revista Matemática de la Universidad Complutense de Madrid*, 10(Special Issue suppl.):241–249, 1997.

[14] A. Padgett. *TBA*. PhD thesis, UC Berkeley, 2022.

[15] M. C. Schmeling. *Corps de transséries*. PhD thesis, Université Paris-VII, 2001.

and thank you!