On the Pila-Wilkie Theorem

Geometry and Model Theory Seminar

Thematic Program on Tame Geometry, Transseries and Applications to Analysis and Geometry

Neer Bhardwaj

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University of Illinois at Urbana-Champaign

Putnam 1971

Let $\alpha \geq 0$ be a real number such that $1^\alpha, 2^\alpha, 3^\alpha, \ldots$ are all integers. Show that α is an integer.

Let's rephrase:
Suppose $f(x) = x^\alpha$, for some $\alpha \in \mathbb{R}$. If $f(\mathbb{N}) \subseteq \mathbb{N}$, then $\alpha \in \mathbb{N}$.

The function f above is transcendental if α is irrational, and ought not to contain too many rational points. This is the overarching guiding principle with Pila-Wilkie type results.

Preliminaries

We want to show that "transcendental" sets contain "few" rational points.

To every rational number we assign a height,

$$\mathrm{H}(\frac{a}{b}) \ := \ \max(|a|,\ |b|) \in \mathbb{N}^{\geq 1}$$

for coprime $a, b \in \mathbb{Z}$, $b \neq 0$.

Throughout $n \geq 1$, and for $a = (a_1, ..., a_n) \in \mathbb{Q}^n$,

$$\mathrm{H}(a) \ := \ \max\{\mathrm{H}(a_i) \ : \ 1 \ \leq i \ \leq n\} \in \mathbb{N}^{\geq 1}.$$

Preliminaries cont'd

For $X \subseteq \mathbb{R}^n$, set $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$.

Also set,

$$X(\mathbb{Q}, T) := \{a \in X(\mathbb{Q}) : \ \mathrm{H}(a) \leq T\},$$

$$\mathrm{N}(X, T) := \#X(\mathbb{Q}, T) \ (\in \mathbb{N}).$$

Throughout $T \in \mathbb{R}^{\geq 1}$, $n \in \mathbb{N}^{\geq 1}$.

Our focus will be on asymptotic upper bounds on $\mathrm{N}(X, T)$ under natural geometric conditions on X. More precisely, we will aim for sub-polynomial upper bounds in T for apposite X.

Pila's counting for curves

Theorem (Pila, 1991)

Let $f : [0,1] \to \mathbb{R}$ be a transcendental analytic function with graph X. Then for all ϵ there is a c such that for all T,

$$\mathrm{N}(X, T) \leq c(X, \epsilon) T^{\epsilon}.$$

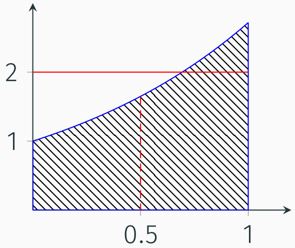
($\epsilon, c \in \mathbb{R}^{>}$ throughout)

Higher dimensions?

What about surfaces?

Consider the two dimensional set

$$X = \{(x, y) \in \mathbb{R}^2 : \; 0 < x < 1, 0 < y < e^x\}.$$



This set is not semialgebraic, but it is easy to see that $\mathrm{N}(X, T) = \Omega(T^4)$.

The transcendental part

To address this obstruction, we remove the "algebraic part" of X, denoted by X^{alg}, and set to be the union of all connected infinite semialgebraic subsets of X.

We set the transcendental part of X, $X^{\mathrm{tr}} := X \setminus X^{\mathrm{alg}}$.

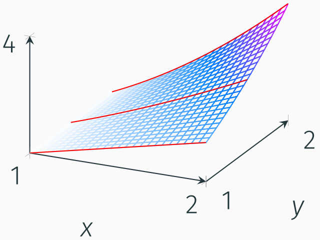
Theorem (Pila, 2005)

Let $X \subseteq \mathbb{R}^n$ be a compact subanalytic set of dimension 2. Then for all ϵ there is a c such that for all T,

$$\mathrm{N}(X^{\mathrm{tr}}, T) \leq c(X, \epsilon) T^{\epsilon}.$$

Non-trivial X^{tr}

$X = \{(x, y, z) \in \mathbb{R}^3 : \; z = x^y \text{ for } x, y \in [1, 2]\}$



The semialgebraic curves given by $Z(z - x), Z(z - x^{3/2})$, and $Z(z - x^2)$ highlighted in red are all part of X^{alg}.

$$X^{\mathrm{tr}} = \bigcup_{y \in [1,2] \setminus \mathbb{Q}} Z(z - x^y)$$

So X^{tr} has infinitely many connected components, and is not subanalytic.

Pila-Wilkie counting

All compact subanalytic sets are definable in a first-order structure $\mathbb{R}_{\mathrm{an}}$- the expansion of the real field with all *restricted analytic functions*.

$\mathbb{R}_{\mathrm{an}}$ is an o-minimal structure, and o-minimality turns out to be a natural setting for generalization.

Theorem (Pila-Wilkie, 2006)

Let $\tilde{\mathbb{R}}$ be an o-minimal expansion of the real field and let $X \subseteq \mathbb{R}^n$ be definable in $\tilde{\mathbb{R}}$. Then for all ϵ there is a c such that for all T,

$$\mathrm{N}(X^{\mathrm{tr}}, T) \ \leq \ c(X, \epsilon)T^{\epsilon}.$$

O-minimal expansions of the real field

An expansion of the real field is a sequence R=$(S_n)_n$ such that for all n,

- S_n is a boolean algebra of subsets of $\mathbb{R}^n$,
- R is closed under finite cartesian products and co-ordinate projections,
- S_n contains all semialgebraic subsets of $\mathbb{R}^n$.

A set X is said to be definable in R if $X \in S_n$ for some n.

R is an *o-minimal* expansion of the real field if moreover,

- S_1 contains only semialgebraic sets, in other words, each element of S_1 is a finite union of points and open intervals.

Proof schema

In what follows, we fix an o-minimal expansion $\tilde{\mathbb{R}}$ of the real field, and definable is with respect to $\tilde{\mathbb{R}}$.

We follow Pila and Wilkie's proof which relies on two intermediate results.

I state a combined consequence of these ingredients below, and then go on to describe our subsequent simplifications in some detail.

Proposition

Suppose $X \subseteq [-1,1]^n$ is definable with empty interior. Then for all ϵ there exists a $d = d(n,\epsilon)$ and a $K := K(X,\epsilon)$ such that for all T, at most KT^ϵ many hypersurfaces in $\mathbb{R}^n$ of degree $\leq d$ are enough to cover the set $X(\mathbb{Q},T)$.

($K \in \mathbb{R}^>$, $d \in \mathbb{N}$ throughout)

A rough sketch

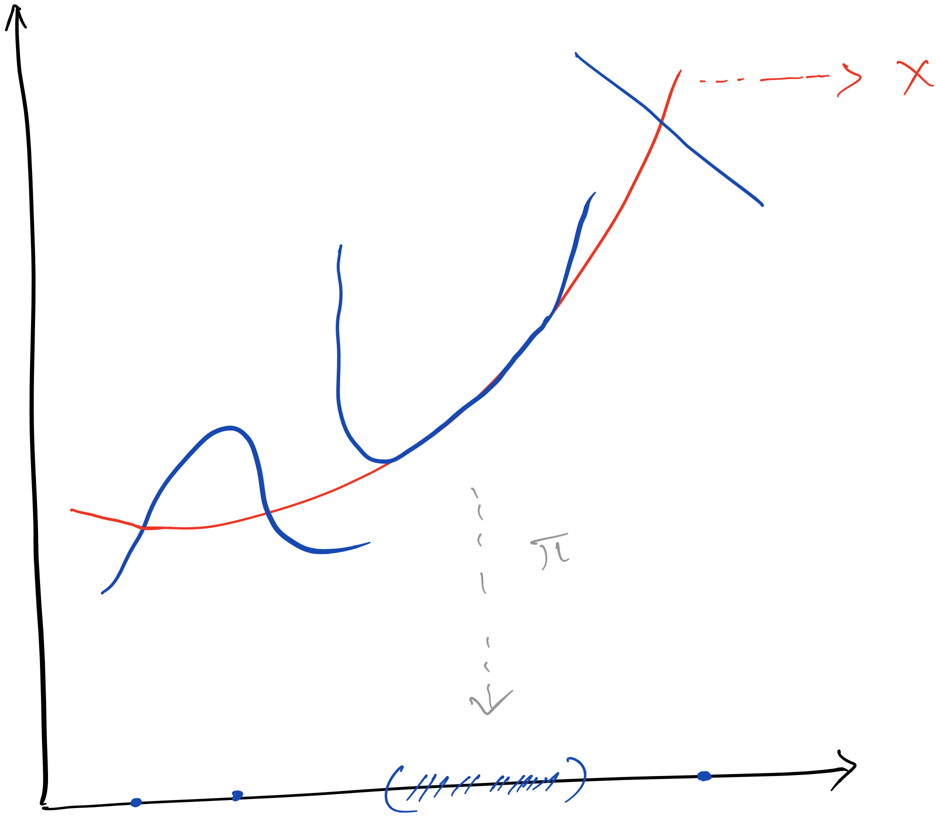
Theorem (Pila-Wilkie, 2006)

Let $X \subseteq \mathbb{R}^n$ be definable. Then for all ϵ there is a c such that for all T,

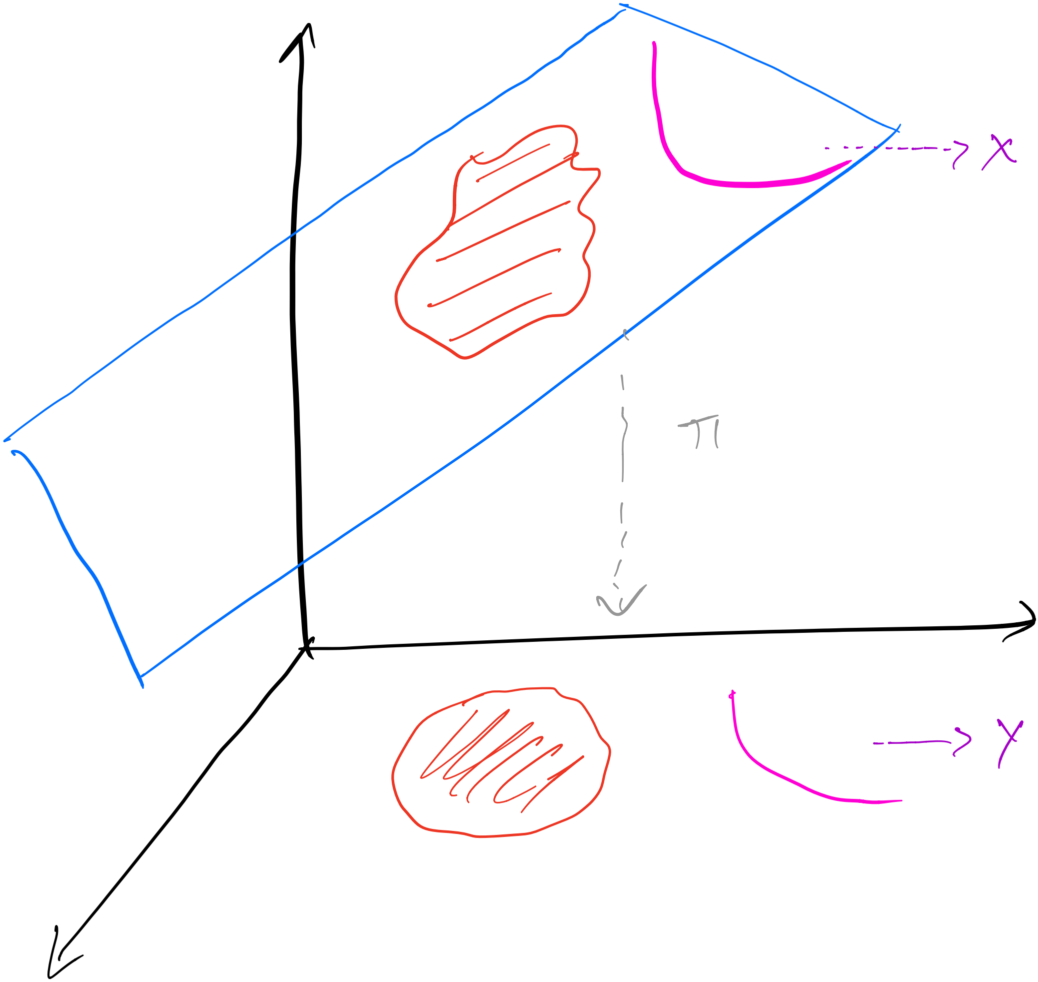
$$\mathrm{N}(X^{\mathrm{tr}}, T) \ \leq \ c(X, \epsilon) T^{\epsilon}.$$

Let's see how our proof goes for a couple of simple cases.

Fix some ϵ and T, and assume that $X \subseteq [-1, 1]^n$ for these cases.



$$N(X^{tr}, T) \leq M \cdot K T^{\varepsilon}$$



$$N(X^{tr}, T) \leq N(Y^{tr}, T)$$

Key Lemma

Lemma

Suppose $S \subseteq \mathbb{R}^n$ is semialgebraic, $X \subseteq S$, and $f : S \to \mathbb{R}^m$ is semialgebraic and injective, and maps X homeomorphically onto $Y = f(X) \subseteq \mathbb{R}^m$. Then $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$ and thus $f(X^{\mathrm{tr}}) = Y^{\mathrm{tr}}$.

Proof: It is clear that $f(X^{\mathrm{alg}}) \subseteq Y^{\mathrm{alg}}$.

Also, for any connected infinite semialgebraic set $C \subseteq Y$, the map $f^{-1}|_C$ is a semialgebraic homeomorphism; since C and f are semialgebraic, and f is injective. So the set $f^{-1}(C)$ is semialgebraic, contained in X, and connected and infinite, and thus $f^{-1}(C) \subseteq X^{\mathrm{alg}}$.

This shows $f^{-1}(Y^{\mathrm{alg}}) \subseteq X^{\mathrm{alg}}$, and thus $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$.

Proof sketch

Theorem (Pila-Wilkie, 2006)

Let $X \subseteq \mathbb{R}^n$ be definable. Then for all ϵ there is a c such that for all T,

$$\mathrm{N}(X^{\mathrm{tr}}, T) \ \leq \ c(X, \epsilon)T^{\epsilon}.$$

Proof: Fix a definable set $X \subseteq \mathbb{R}^n$. By induction on n, clear for $n = 1$.

Since the interior of X is contained in the X^{alg}, we may assume that X has empty interior.

A standard trick allows us to also reduce to the case where $X \subseteq [-1, 1]^n$.

Proof cont'd

Our basic reductions allow us to apply our preparatory proposition to X, and we get a d and K such that for all T, $X(\mathbb{Q}, T)$ can be covered by at most KT^{ϵ} hypersurfaces in $\mathbb{R}^n$ of degree $\leq d$.

<u>Claim</u>: Let H denote a hypersurface in $\mathbb{R}^n$ of degree $\leq d$. Then

$$\mathrm{N}\left((X \cap H)^{\mathrm{tr}}, T\right) \leq c_1 T^{\epsilon},$$

for some $c_1 \in \mathbb{R}^{>}$ which is independent of H and T.

It is clear that it is sufficient to prove the claim, as we then have that

$$\mathrm{N}\left(X^{\mathrm{tr}}, T\right) \leq K c_1 T^{2\epsilon}.$$

$\left(\text{Note } (X_1 \cup X_2)^{\mathrm{tr}} \subseteq X_1^{\mathrm{tr}} \cup X_2^{\mathrm{tr}}\right)$

Proof of the claim

Assume for the purposes of simplicity that H is a (semialgebraic) cell C of dimension $m(< n)$.

We have then a semialgebraic homeomorphism

$$p = p_C : C \to p(C)$$

onto an open cell $p(C)$ in $\mathbb{R}^m$, and so p maps $X \cap C$ homeomorphically onto its image $Y \subseteq p(C) \subseteq \mathbb{R}^m$.

Since p is actually given by omitting some coordinates, we have $\mathrm{H}\left(p(a)\right) \leq \mathrm{H}(a)$ for $a \in C \cap \mathbb{Q}^n$.

Proof of the claim cont'd

Since the $m < n$ we have by our inductive assumption that for all T,

$$\mathrm{N}(Y^{\mathrm{tr}}, T) \ \leq \ bT^{\epsilon},$$

with $b \in \mathbb{R}^{>}$ independent of T.

Hence for all T,

$$\mathrm{N}\left((X \cap H)^{\mathrm{tr}}, T\right) = \mathrm{N}\left((X \cap C)^{\mathrm{tr}}, T\right) \ \leq \ bT^{\epsilon}$$

by our key Lemma applied to the map $p = p_C$.

$c_1 \ := b$ is a positive real number as we were aiming for, assuming b *can be taken to depend only on* X, ϵ, *not on* H, Y.

The Theorems

In what follows, let $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ be definable.

Theorem

For all ϵ there exists a c such that for all $s \in E$ and all T we have $N(X(s)^{\mathrm{tr}}, T) \leq cT^\epsilon$.

Theorem

Let ϵ be given. Then there is a definable set $V = V(X, \epsilon) \subseteq X$ and a c such that for all $s \in E$ and T,

$$V(s) \subseteq X(s)^{\mathrm{alg}} \quad \text{and} \quad N(X(s) \setminus V(s), T) \leq cT^\epsilon.$$

Blocks

We saw in the proof sketch earlier that we deal with sets of the form $Z \cap H$, where Z is definable and H is a hypersurface.

In the case when $\dim(Z) = \dim(H)$, the top dimensional component $Z \cap H$ is part of Z^{alg}.

We want to capture this phenomenon: A *block* of dimension d in $\mathbb{R}^n$ is a definable connected open subset of a semialgebraic set $A \subseteq \mathbb{R}^n$ for which $\dim_a A = d$ for all $a \in A$. Notice, a block inside a definable set Z is always part of Z^{alg}.

Contrast this with Pila's version: A (basic) block of dimension d is a definable connected open subset of a semialgebraic C^1- submanifold of dimension d. *(equivalent formulation)*.

One has to do some work to arrive at this formulation, and to show that these blocks are contained inside the algebraic part...

Block family version of the Counting Theorem

Theorem

Let ϵ be given. Then there are a natural number $N = N(X, \epsilon) \geq 1$, a block family $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ in $\mathbb{R}^n$ of dimension $d_j \leq n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \ldots, N$, and a constant $c = c(X, \epsilon)$, such that:

(i) $V_j(s,t) \subseteq X(s)$ *for* $j = 1, \ldots, N$ *and* $(s,t) \in E \times F_j$;

(ii) *for all T and all $s \in E$, $X(s)(\mathbb{Q}, T)$ is covered by at most cT^ϵ blocks $V_j(s,t)$, $(1 \leq j \leq N,\ t \in F_j)$.*

Let $V_1, \ldots, V_N$ and c be as in the Theorem above. Then for all $s \in E$ the definable set $V(s) \subseteq \mathbb{R}^n$ given by

$$V(s) := \bigcup_{d_j \geq 1, t \in F_j} V_j(s,t)$$

is contained in $X(s)^{\mathrm{alg}}$ and $\mathrm{N}(X(s) \setminus V(s), T) \leq cT^\epsilon$ for all T.

The end

Thank you!