Lecture 7

Tuesday, February 8, 2022 8:45 AM

Last time we introduced $\tilde{\mathbb{R}}$-domains and $\tilde{\mathbb{R}}$-Hardy fields and generalized to this setting earlier results on Hausdorff fields and Hardy fields.

Ayhan and Elliot commented to me about the so-called "cheap trick", and in fact I swept under the rug an important fact, namely that $T = \mathrm{Th}(\tilde{\mathbb{R}})$ has definable Skolem functions. I will briefly return to this in the Question session on Thursday.

Improved Terminology:

$g \in \mathcal{G}$ oscillates :$\iff$ $g \neq 0$ and $g(t) = 0$ for arbitrarily large t

(In our terminology of last time: g oscillates $\iff$ $(g, 0)$ is not total.)

Some consequences of what we did last time:

(4.3.2) **Corollary** Let H be an $\tilde{\mathbb{R}}$-Hardy field, $H \models T$, and $g \in \mathcal{C}$. In each of the following situations $g - h$ does not oscillate, for all $h \in H$, and $H[g]^T_{\mathcal{L}}$ is again an $\tilde{\mathbb{R}}$-Hardy field, and

$$H[g]^T_{\mathcal{L}} = H[g]_{\hat{\mathcal{L}}} = \bigcup_n \left\{ f(g_1, \dots, g_n, g) \mid f : \mathbb{R}^{n+1} \to \mathbb{R} \text{ is } \emptyset\text{-definable}, \; g_1, \dots, g_n \in H \right\}:$$

(i) $g \in \mathbb{R}$

(ii) $g \in \mathcal{C}^1$ and $g' \in H$

(iii) $g = e^f$ with $f \in H$

(iv) $g = \log f$ with $f \in H^>$.

Pf The nonoscillation claims were already established for real closed Hardy fields. The rest is an exercise along the lines of the proof of Lemma 4.3.1. ☒

(4.4) First-Order ODE's over $\tilde{\mathbb{R}}$-Hardy Fields

(4.4.1) Let H be an $\tilde{\mathbb{R}}$-Hardy field, $U \subseteq \mathbb{R}^{n+1}$ open definable, $\Phi \in \mathcal{C}^1(U)$ definable. Suppose $h_1, \dots, h_n \in H$, $y \in \mathcal{C}^1$ are such that eventually

$$(h(t), y(t)) \in U, \quad y'(t) = \Phi(h(t), y(t)).$$

Then y lies in an $\tilde{\mathbb{R}}$-Hardy field extension of H.

Pf Much like that of the semialgebraic case (3.1.1): first extend H so that $\mathbb{R} \subseteq H \models T$. Then show: $y - h$ does not oscillate, for every $h \in H$. Then we have the $\tilde{\mathbb{R}}$-domain $H[y]^T_{\mathcal{L}} = H[y]_{\hat{\mathcal{L}}} \subseteq \mathcal{C}^1$. Now show $H[y]^T_{\mathcal{L}}$ is an $\tilde{\mathbb{R}}$-Hardy field, by an argument different (a bit) from the semialgebraic case. ☒

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§5 The ODE $y'' + y = h$ over Hardy Fields

Here we meet an interesting phenomenon: Let H be a Hardy field. Then for some $h \in H$ there is exactly one sol'n $y \in \mathcal{C}^2$ (of $y'' + y = h$) that lies in a Hardy fld extension of H, and for all other $h \in H$ there are infinitely many such solutions.

The reason for us to focus on 1st and 2nd order linear ODE's is that they control to a large extent the entire story about solving algebraic ODE's over Hardy fields.

Original Source for §5

M. Boshernitzan, Second Order Differential Equations over Hardy Fields, J. London Math. Soc. 35 (1987) 109–120.

Note: $g \in \mathcal{C}^1$ oscillates, then so does $g' \in \mathcal{C}$.

(5.1) The ODE $y'' = \Phi(y)$ with Φ semialgebraic over H

(5.1.1) **Lemma** Suppose $h \in \mathcal{C}^1$ oscillates. Then (i) or (ii) below:

(i) there are arbitrarily large s with $h'(s) = 0$, $h(s) > 0$

(ii) " " " large s with $h'(s) = 0$, $h(s) < 0$

In case (i) there are also arbitrarily large s with $h'(s) = 0$ and $h(s) \leq 0$.

In case (ii) there are also arbitrarily large s with $h'(s) = 0$ and $h(s) \geq 0$.

Pf Exercise. ☒

(5.1.2) **Lemma** Let H be a Hardy field and $f, g \in \mathcal{C}^1$ such that f generates a Hausdorff field $H(f)$ over H and $g \in H(f)^{rc}$. Then for $K = H(f, g)$:

$$\partial K \subseteq K + Kf'$$

Pf Enough to show $g' \in K + Kf'$ since we can then use this for any elt. of K instead of g. Usual rules: $\partial H(f) \subseteq H(f) + H(f)f'$. Let $P(Y) \in H(f)[Y]$ be the minimum poly. of g over $H(f)$. Then $P(g) = 0$ gives $g' = -\dfrac{P^{\partial}(g)}{P'(g)}$. Now

$$P^{\partial}(g) \in \sum_n \partial H(f) \cdot g^n \subseteq K + Kf'.$$

Also $P'(g) \in K^{\times}$. So $g' \in K + Kf'$. ☒

(5.1.3) **Theorem** Let H be a real closed Hardy field and suppose $g \in \mathcal{C}^2$ generates a Hausdorff field $H(g)$ over H with $g'' \in H(g)^{rc}$.

Then g generates a Hardy field $H(g, g', g'')$ over H.