Lecture 6

Thursday, February 3, 2022 6:58 AM

I decided it is better to start a new section with the material at the end of Lecture 5

§4 O-minimal Generalization

$\mathcal{L}$: a 1st order (1-sorted) language extending the language $\{<, 0, 1, -, +, \cdot\}$ of ordered rings with possible new fct. symbols but no new rel'n symbols

$\tilde{\mathbb{R}} = (\mathbb{R}; <, 0, 1, -, +, \cdot, \ldots)$: an $\mathcal{L}$-expansion of

$\mathbb{R}_{alg} = (\mathbb{R}; <, 0, 1, -, +, \cdot)$

$T := Th(\tilde{\mathbb{R}})$, a complete $\mathcal{L}$-theory.

Assumptions : $\tilde{\mathbb{R}}$ is o-minimal, T has QE.

(Unlike in the papers 3, 4 mentioned in Lecture 5 we do not assume T has a universal axiomatization.)

Examples : $\mathbb{R}_{alg}$, $\mathbb{R}_{an,D}$, $\mathbb{R}_{an,exp,log}$

Instead of $\mathcal{C}$ we now prefer to work in the larger ambient ring

$\mathcal{G} :=$ ring of germs at $+\infty$ of real-valued fct's whose domain is a subset of $\mathbb{R}$ containing some interval $(a, +\infty)$.

For $F : \mathbb{R}^n \to \mathbb{R}$ we define

$$F^{\mathcal{G}} : \mathcal{G}^n \to \mathcal{G} \text{ by}$$

$$F^{\mathcal{G}}(g_1, \dots, g_n)(t) = F(g_1(t), \dots, g_n(t)), \text{ eventually.}$$

Likewise any n-ary fct. symbol $F \in \mathcal{L}$ yields $F^{\mathcal{G}} : \mathcal{G}^n \to \mathcal{G}$. To simplify notation we often drop the superscript $\mathcal{G}$.

Terminology: call $(f, g) \in \mathcal{G}^2$ total

if either $f(t) < g(t)$, eventually,

or $f(t) = g(t)$, 〃 $(\text{so } f = g)$,

or $f(t) > g(t)$, 〃 .

(4.1) $\tilde{\mathbb{R}}$-domains

Def. An $\tilde{\mathbb{R}}$-domain is a set $D \subseteq \mathcal{G}$ such that for all $(f, g) \in D^2$ are total and $F^{\mathcal{G}}(g_1, \dots, g_n) \in D$ for all n-ary fct. symbols $F \in \mathcal{L}$ and $g_1, \dots, g_n \in D$.

Till further notice D is an $\tilde{\mathbb{R}}$-domain.

We construe D as an $\mathcal{L}$-structure:

$$f < g \;:\iff\; f(t) < g(t), \text{ eot, and}$$

for n-ary $F \in \mathcal{L}$, $F^{D}(g_1,\dots,g_n) = F^{\mathcal{G}}(g_1,\dots,g_n)$.

In particular, D is a subring of $\mathcal{G}$ and is totally ordered by $<$, with

$$D \subseteq \mathcal{G}^{\times} \cup \{0\},$$

so D has a fraction field $\mathrm{Frac}(D) \subseteq \mathcal{G}$. (If $D \subseteq \mathcal{C}$, then $\mathrm{Frac}(D) \subseteq \mathcal{C}$ is a Hausdorff field.)

Example: $H(\tilde{\mathbb{R}}) \subseteq \mathcal{C}$ is an $\tilde{\mathbb{R}}$-domain.

Remark $\quad D \models T_{\forall}$

Pf Let $\forall y\, \phi(y)$ belong to T, $y = (y_1,\dots,y_n)$, $\phi(y)$ a q-free $\mathcal{L}$-fmla. Then for all $g_1,\dots,g_n \in D$ we have

$$\tilde{\mathbb{R}} \models \phi(g_1(t),\dots,g_n(t)), \text{ eventually,}$$

$$\therefore\ D \models \phi(g_1,\dots,g_n).$$

(4.1.2) **Lemma** Let $S \subseteq \mathbb{R}^n$ be o-definable (in $\mathbb{R}$). Then for all $g_1, \dots, g_n \in D$, either $g(t) \in S$ eventually, or $g(t) \notin S$ eventually, where $g(t) = (g_1(t), \dots, g_n(t))$.

Pf If true for o-definable $S, S' \subseteq \mathbb{R}^n$, then also for $S \cup S'$, $S \cap S'$, $\mathbb{R}^n \setminus S$. This gives a reduction to the case

$$S = \{ s \in \mathbb{R}^n \mid \tau(s) = 0 \}$$ and the case

$$S = \{ s \in \mathbb{R}^n \mid \tau(s) > 0 \}, \ \tau(y_1, \dots, y_n) \text{ an } \mathcal{L}\text{-term.}$$

For such τ, and $g_1, \dots, g_n \in D$, we have $\tau(g_1, \dots, g_n) \in D$, so now we can use that $(\tau(g_1, \dots, g_n), 0)$ is total. $\boxtimes$

(4.1.3) **Cor** Let $f : \mathbb{R}^n \to \mathbb{R}$ be o-definable and $g_1, \dots, g_n \in D$. Then $(f(g_1, \dots, g_n), 0)$ is total.

Pf Apply previous lemma to the o-definable sets

$$S_< = \{ s \in \mathbb{R}^n \mid f(s) < 0 \}$$

$S_= = \{ s \in \mathbb{R}^n \mid f(s) = 0 \}$

$S_> = \{ s \in \mathbb{R}^n \mid f(s) > 0 \}$. $\boxtimes$.

Def The T-closure of D is

$$\begin{array}{c} D^T \\ \rotatebox{90}{$\subseteq$} \\ D \end{array} = \bigcup_n \left\{ f(g_1,\dots,g_n) : \begin{array}{l} f: \mathbb{R}^n \to \mathbb{R} \text{ is } 0\text{-definable}, \\ g_1,\dots,g_n \in D \end{array} \right\}$$

(4.1.4) Cor D^T is an $\tilde{\mathbb{R}}$-domain and $D^T \models T$.

Pf D^T an $\hat{\mathbb{R}}$-domain by previous lemma.

To get $D^T \models T$ we use a cheap trick: we extend $\mathcal{L}$ to $\hat{\mathcal{L}}$ by adding for each 0-definable $f: \mathbb{R}^n \to \mathbb{R}$ an n-ary fct. symbol (also written as f), and extend T to $\hat{T}$ by adding for each such f a defining axiom $\forall y, z \, (f(y) = z \leftrightarrow \phi_f(y, z))$

where $\phi_f(y,z)$, $y=(y_1,\dots,y_n)$, is a qn free $\mathcal{L}$-fmla defining graph(f) in $\tilde{\mathbb{R}}$.

Then the $\hat{\mathcal{L}}$-theory axiomatized by $\hat{T}$ is "universal" (has an axiomatization by universal $\hat{\mathcal{L}}$-sentences).

$$\hat{\tilde{\mathbb{R}}} = \tilde{\mathbb{R}} \text{ expanded to a model of } \hat{T}.$$

Now D^T is not just an $\tilde{\mathbb{R}}$-domain, but even an $\hat{\tilde{\mathbb{R}}}$-domain; so by earlier remark, $D^T \models \hat{T}_\forall$. But $\hat{T}_\forall$ axiomatizes the same $\mathcal{L}$-theory as $\hat{T} \supseteq T$, so $D^T \models T$. ⊠

(4.1.5) Cor $D \models T \iff f(g_1,\dots,g_n) \in D$ for all o-definable $f : \mathbb{R}^n \to \mathbb{R}$ and $g_1,\dots,g_n \in D$.

Pf $\Leftarrow$: use cheap trick.

$\Rightarrow$: exercise. ⊠

(4.1.6) D^T is the smallest $\tilde{\mathbb{R}}$-domain (w.r.t. inclusion) that contains D and is a model of T.

Example: Hausdorff fields are $\mathbb{R}_{alg}$-domains and for a Hausdorff field H, H^{rc} is the T-closure of H, for $T = Th(\mathbb{R}_{alg})$

Exercise If $f : \mathbb{R}^n \to \mathbb{R}$ is 0-definable and $g_1, \dots, g_n \in D \cap \mathcal{C}^r$ $(r \in \mathbb{N})$, then $f(g_1, \dots, g_n) \in D^T \cap \mathcal{C}^r$.

Cor If $r \in \mathbb{N}$, $D \subseteq \mathcal{C}^r$, then $D^T \subseteq \mathcal{C}^r$.

(4.2) Adjoining a germ to an $\tilde{\mathbb{R}}$-domain (using model theory)

Let $\mathcal{M} = (M; <, \dots)$ be a $|D|^+$-saturated elementary extension of

$(\tilde{\mathbb{R}}, (g)_{g:\mathbb{R}\to\mathbb{R}})$, and let $a \in M$, $a > \mathbb{R}$ in $\mathcal{M}$.

Define $\iota_a : \mathcal{G} \to M$ by

$\iota_a(g) = g^{\mathcal{M}}(a)$ where $g^{\mathcal{M}} : M \to M$ is the function named in $\mathcal{M}$ by some $g : \mathbb{R} \to \mathbb{R}$ representing the germ $g \in \mathcal{G}$.

Routine verification :

$$\iota_a\left(F^{\mathcal{G}}(g_1, \dots, g_n)\right) = F^{\mathcal{M}}\left(\iota_a(g_1), \dots, \iota_a(g_n)\right)$$

for any n-ary fct. symbol $F \in \mathcal{L}$ and $g_1, \dots, g_n \in \mathcal{G}$

$\therefore$ ι_a restricts to an embedding $D \to \mathcal{M}$, since $g > 0$ in D yields $\iota_a(g) > 0$ in $\mathcal{M}$, so

$$D \underset{\iota_a}{\simeq} \iota_a(D) \underset{\mathcal{L}}{\subseteq} \mathcal{M}.$$

If $D \models T$, this yields

$\mathcal{D} \underset{\iota_a}{\simeq} \iota_a(\mathcal{D}) \underset{\mathcal{L}}{\preceq} \mathfrak{M}$, since T has QE.

We use this to generalize an earlier fact about germs to Hausdorff fields.

(4.2.1) Suppose $\mathcal{D} \models T$, $g \in \mathcal{G}$, and (g,h) is total for all $h \in D$. Then

$$D[g]_{\mathcal{L}} := \bigcup_{n} \{\tau(g_1,\dots,g_n,g) : \tau(y_1,\dots,y_n,y) \text{ an } \mathcal{L}\text{-term}, g_1,\dots,g_n \in D\}$$

is an $\tilde{\mathbb{R}}$-domain.

Pf If $g \in D$, then $D[g]_{\mathcal{L}} = D$, and we are done. So assume $g \notin D$. Let $a,b \in M$, $a,b > \mathbb{R}$ in $\mathfrak{M}$. Then we have a commuting diagram of $\mathcal{L}$-isomorphisms.

$$\begin{tikzcd} & \iota_a(D) \arrow[d, "\iota_{ab}"] \\ D \arrow[ur, "\iota_a"] \arrow[r, "\iota_b"'] & \iota_b(D) \end{tikzcd}$$

For all $f \in D$:

$$f <_{ev} g \iff \iota_a f < \iota_a g \text{ in } \mathfrak{M}$$

$\iff \iota_b f < \iota_b g$ in $\mathfrak{M}$.

So $\iota_a g$ and $\iota_b g$ realize cuts in $\iota_a D$ and $\iota_b D$ corresponding under ι_{ab}.

$\therefore$ (o-minimality) ι_{ab} extends to an $\mathcal{L}$-isomorphism $\iota_a D[\iota_a g]_{\hat{\mathcal{L}}} \xrightarrow{\sim} \iota_b D[\iota_b g]_{\hat{\mathcal{L}}}$ between the elementary substructures of $\mathfrak{M}$ generated over $\iota_a D$ and $\iota_b D$ by $\iota_a g$ and $\iota_b g$, respectively.

$\therefore$ for all $\mathcal{L}$-terms $\tau(y_1, \dots, y_n, y)$ and $g_1, \dots, g_n \in D$

$$\operatorname{sign}(\iota_a \tau(g_1, \dots, g_n, g)) = \operatorname{sign}(\iota_b(\tau(g_1, \dots, g_n, g))),$$

so this sign is independent of the choice of a. If this sign is -1, we must have $\tau(g_1, \dots, g_n, g)(t) < 0$, eot. If this sign is 0, then $\tau(g_1, \dots, g_n)(t) = 0$, eot. Likewise if the sign is $+1$. So $(\tau(g_1, \dots, g_n, g), 0)$ is total. $\boxtimes$.

(4.3 | $\tilde{\mathbb{R}}$ - Hardy fields

An **$\tilde{\mathbb{R}}$ - Hardy domain** is an $\tilde{\mathbb{R}}$ - domain $D \subseteq \mathcal{C}^1$ such that $\partial D \subseteq D$ (so $D \subseteq e^{<\infty}$)

An $\tilde{\mathbb{R}}$ - Hardy field is an $\tilde{\mathbb{R}}$ - Hardy domain whose underlying ring is a field.

Example $H(\tilde{\mathbb{R}})$ is an $\tilde{\mathbb{R}}$ - Hardy field. (the smallest $\tilde{\mathbb{R}}$ - Hardy field containing $\mathbb{R}$, x, that is a model of T.

(4.3.1) **Lemma** Suppose D is an $\tilde{\mathbb{R}}$ - Hardy domain. Then D^T is an $\tilde{\mathbb{R}}$-Hardy field.

Pf. Let $f : \mathbb{R}^n \to \mathbb{R}$ be o-definable and $g_1, \dots, g_n \in D$. An earlier exercise already claimed $f(g_1, \dots, g_n) \in \mathcal{C}^1$. In fact, the argument we are about to give for $f(g_1, \dots, g_n)' \in D^T$ is basically the same as that giving $f(g_1, \dots, g_n) \in \mathcal{C}^1$ in the first place.

By o-minimality we have o-definable $S_1, \dots, S_m \subseteq \mathbb{R}^n$ covering $\mathbb{R}^n$, o-definable open $U_1, \dots, U_m \subseteq \mathbb{R}^n$ with $S_1 \subseteq U_1, \dots, S_m \subseteq U_m$, and o-definable $f_1 \in \mathcal{C}^1(U_1), \dots, f_m \in \mathcal{C}^1(U_m)$ s.t. $f = f_i$ on S_i. Take $i \in \{1, \dots, m\}$ s.t. $(g_1(t), \dots, g_n(t)) \in S_i$, eventually. Then

$$f(g_1(t), \dots, g_n(t)) = f_i(g_1(t), \dots, g_n(t)) \text{ ev } t.$$

$$\therefore\ f(g_1, \dots, g_n)' = \underbrace{\sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}(g_1, \dots, g_n)}_{\in D^T} \underset{\in D}{g_j'}$$

$$\therefore\ f(g_1, \dots, g_n)' \in D^T. \qquad \boxtimes.$$