Lecture 5

Tuesday, February 1, 2022 7:09 AM

§3 Solving First-Order ODE's in Hardy Fields

We started this section last time, but I want to back up a bit.

Original Sources for this section:

1. M. Boshernitzan, An extension of Hardy's class of "orders of infinity", J. Analyse Math. 39 (1981), 235–255
2. M. Rosenlicht, Hardy Fields, J. Math. Anal. 93 (1983), 297–311
3. L. van den Dries, A. Macintyre, D. Marker, The elementary theory of restricted analytic fields with exponentiation, Ann. Math. 140 (1994), 183–205
4. L. van den Dries, An intermediate value property for first-order differential polynomials, Quaderni di Mat. 6 (2000), 95–105

(3.1) The semialgebraic explicit case

Mickaël's question last time prompts me to review a uniqueness fact about solutions of 1^{st} order ODE's.

Let $U \subseteq \mathbb{R}^2$ open, $F \in \mathcal{C}(U)$. A solution of

$$(*) \quad Y' = F(X, Y)$$

is a function $y \in \mathcal{C}^1(I)$, $I \subseteq \mathbb{R}$ an open interval, such that $(t, y(t)) \in U$, $y'(t) = F(t, y(t))$ for all $t \in I$.

Uniqueness of solutions with given initial condition: If $F \in \mathcal{C}^1(U)$ and $y \in \mathcal{C}^1(I)$, $z \in \mathcal{C}^1(J)$ are solutions of $(*)$ with $y(a) = z(a)$ for some $a \in I \cap J$, then $y = z$ on $I \cap J$.

The assumption "$F \in \mathcal{C}^1(U)$" is important, it cannot be replaced by "$F \in \mathcal{C}(U)$"

Till further notice: H a Hardy field

"germ": germ at $+\infty$.

3.1.1 Proposition Let $U \subseteq \mathbb{R}^{n+1}$ be open and semialgebraic and let $\Phi \in \mathcal{C}^1(U)$ be semialgebraic. Suppose $h_1, \dots, h_n \in H$ and $\eta \in \mathcal{C}^1$ are such that, eventually,

$$(**) \quad (h(t), \eta(t)) \in U, \quad \eta'(t) = \Phi(h(t), \eta(t)),$$

where $h(t) := (h_1(t), \dots, h_n(t))$. Then just η lies in a Hardy field extension of H.

Pf Passing to an extension we arrange $H \supseteq \mathbb{R}$ and H is real closed.

Claim 1: $\eta(t) < 0$, evt., or $\eta(t) = 0$, evt., or $\eta(t) > 0$, evt.

Towards this, suppose $\eta(t) = 0$ for arbitrarily large t; enough to show that then $\eta(t) = 0$, evt.

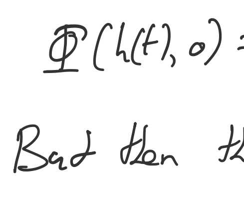
Take $a \in \mathbb{R}$ and representatives $h_1, \dots, h_n, \eta \in \mathcal{C}^1(a, +\infty)$ of their germs such that $(**)$ holds for all $t > a$. Last time we showed already: $(h(t), 0) \in U$, eventually, so by increasing a and restricting $h_1, \dots, h_n, \eta$ accordingly we arrange $(h(t), 0) \in U$ for all $t > a$.

Subclaim: let $a < t_1 < t_2$, $\eta(t_1) = \eta(t_2) = 0$; then $\Phi(h(t), 0) = 0$ for some $t \in [t_1, t_2]$.

Subclaim holds trivially for $t = t_1$ if $\eta'(t_1) = 0$, since $\eta'(t) = \Phi(h(t), \eta(t))$. So assume $\eta'(t_1) \neq 0$, say $\eta'(t_1) > 0$ (case $\eta'(t_1) < 0$ is similar)

Then $\eta(t) > 0$, $\Phi(h(t), 0) > 0$ for all $t > t_1$ suff. close to t_1. Decreasing t_2 if necessary we arrange $\eta(t) > 0$ for all $t \in (t_1, t_2)$. So

$$\eta'(t_2) = \Phi(h(t_2), 0) \leq 0.$$



$\therefore \exists t \in [t_1, t_2]$ $\Phi(h(t), \eta(t)) = 0$.

The subclaim gives arbitrarily large $t > a$ with $\Phi(h(t), 0) = 0$, so $\Phi(h(t), 0) = 0$, eventually.

But then the ODE

$$y'(t) = \Phi(h(t), y(t))$$ has the germ sol'n η and the germ sol'n 0. So by uniqueness of sol'ns, get $\eta = 0$.

This proves the claim 1.

Claim 2 Given any $f \in H$, either $\eta(t) < f(t)$, evt., or $\eta(t) = f(t)$, evt., or $\eta(t) > f(t)$ evt.

To see this, just apply claim 1 to $\zeta = \eta - f$ and the ODE

$$z'(t) = \Phi(h(t), f(t) + z(t)) - f'(t)$$

satisfied by ζ, with $h_1, \dots, h_n$ augmented to $h_{n+1} := f$, $h_{n+2} := f'$.

Claim 2 already gives a Hausdorff fld $H(\eta) \subseteq \mathcal{C}^1$, and its real closure $H(\eta)^{rc} \subseteq \mathcal{C}^1$.

Claim 3 $H(\eta)^{rc}$ is a Hardy field (establishing the proposition)

First, $\eta'(t) = \Phi(h(t), \eta(t))$, evt., and Φ semialgebraic, so by Prop 1.8.1, $\eta' \in H(\eta)^{rc}$. $\therefore \partial H(\eta) \subseteq H(\eta)^{rc}$.

Let $g \in H(\eta)^{rc}$. To get $g' \in H(\eta)^{rc}$, take the min. poly. $P(Y) \in H(\eta)[Y]$ of g over $H(\eta)$. Then by earlier equality

$$g' = \frac{-P^{\partial}(g)}{(\partial P/\partial Y)(g)}, \quad P^{\partial}(g) \in H(\eta)^{rc}, \ (\partial P/\partial Y)(g) \in H(\eta)^{rc},$$

so $g' \in H(\eta)^{rc}$. ☒

Why only semialgebraic Φ? Would like to allow exp, log, etc. be involved in Φ. In fact, we can allow "o-minimal" Φ.

(3.2) An o-minimal version (of much of what we did)

L: first-order language extending the language $\{<, 0, 1, -, +, \cdot\}$ of ordered rings with possibly new fct. symbols but no new rel'n symbols.

$\tilde{\mathbb{R}} = (\mathbb{R}; <, 0, 1, -, +, \cdot, \dots)$: an L-expansion of $\mathbb{R}_{alg} := (\mathbb{R}; <, 0, 1, -, +, \cdot)$

$T := Th(\tilde{\mathbb{R}})$, a complete L-theory

Assumptions $\tilde{\mathbb{R}}$ is o-minimal, T has QE.

Examples: $\mathbb{R}_{alg}$, $\mathbb{R}_{an,D}$, $\mathbb{R}_{an,exp,log}$