Lecture 4

Thursday, January 27, 2022 7:55 AM

H : Hardy field

(2.15) Consequences of Consequences:

(iv) $H(\mathbb{R})$ is a Hardy field

(v) $H(x)$ " " " "

(vi) $H(\log f)$ " " " for $f \in H^{>}$

(vii) for $f \in H$, e^f generates a Hardy field $H(e^f)$

Pf of (vii): can arrange H is real closed and closed under log (by (i), (vi)).

If $e^f \in H$, we are done, so assume $e^f \notin H$. Then e^f transcendental over H

Enough to show (for $H(e^f)$ being a Hausdorff fld: given $h \in H$, either $e^f <_{ev} h$, or $e^f >_{ev} h$.

Of course, $e^f >_{ev} h$ if $h \leq 0$ in H, so assume $h>0$. Then $f <_{ev} \log h$ or $f >_{ev} \log h$, hence $e^f <_{ev} h$ or $e^f >_{ev} h$. So $H(e^f)$ a Hausdorff fld.

From $(e^f)' = f'e^f$ it follows easily that $\partial H(e^f) \subseteq H(e^f)$

$(H(e^f) \subseteq \mathcal{C}').$ $\boxtimes$

(2.1.6) Def H Liouville closed

$$:\Longleftrightarrow$$

H real closed and every $h \in H$ has an antiderivative in H, and $e^h \in H$.

$$\underset{\text{exercise}}{\Longleftrightarrow}$$

H real closed and every ODE $y'+fy=g$ $(f,g \in H)$ has a sol'n $y \in H^{\neq}$.

(2.1.7) Cor For every $H \supseteq \mathbb{R}$ there is a smallest Liouville closed Hardy field extension of H; we denote it by $Li(H)$

"the Hardy-Liouville closure of $\mathcal{H}$".

$Li(\mathbb{R})$ contains all logarithmico-exponential functions considered by Hardy

General Remarks

Call a germ $\in \mathcal{C}$ (and any representative of it) hardian if it lies in a Hardy field.

Can think of "hardian" as "has very regular growth".

2.2 Remarks on differential rings

For us a differential ring is a commutative ring (containing $\mathbb{Q}$ as a subring), equipped with a derivation $a \mapsto a' : R \to R$ s.t. $(a+b)' = a' + b'$, $(ab)' = a'b + ab'$ $(a, b \in R)$

Hence $\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2}$ for $a \in R$, $b \in R^{\times}$.

Let R be a diff.-ring with derivation ∂. Then $C_R := \{a \in R : a' = 0\}$ is a subring of R, called the ring of constants of R.

Almost as important for us as ∂ is the associated logarithmic-derivative map $a \mapsto a^\dagger := \frac{a'}{a} : R^\times \to R$,

satisfying $(ab)^\dagger = a^\dagger + b^\dagger$, $(a^{-1})^\dagger = -a^\dagger$

$$a^\dagger = b^\dagger \iff a \in C_R^\times b \qquad (a, b \in R^\times)$$
$$a^\dagger = 0 \iff a \in C_R^\times$$

Note: $C_H = H \cap \mathbb{R}$

$e^{<\infty}$, e^∞, e^ω are useful differential rings for us.

For $\phi \in R^\times$ we also have the derivation

$\phi^{-1}\partial : a \mapsto \phi^{-1}a' : R \to R$

This is like a change of variables and R^ϕ is R with the derivation $\phi^{-1}\partial$ instead of ∂.

Note: $R^{\phi_1\phi_2} = (R^{\phi_1})^{\phi_2}$

for $\phi_1, \phi_2 \in R^{\times}$.

(2.3) Some Hardy field asymptotics

(2.3.1) Asymptotic Rules for $f, g \in H^{\neq}$:

(i) $f \prec g \Longrightarrow f^{\dagger} < g^{\dagger}$

(ii) $f \prec g \not\asymp 1 \Longrightarrow f' \prec g'$ (To be checked)

(iii) $f \prec g \prec 1 \Longrightarrow f^{\dagger} \succcurlyeq g^{\dagger}$

$f \succ g \succ 1 \Longrightarrow f^{\dagger} \succcurlyeq g^{\dagger}$

(iv) $f \preccurlyeq 1,\ g \not\asymp 1 \Longrightarrow f' \prec g^{\dagger}$

Pf (i) let $f \prec g$; to show $f^{\dagger} < g^{\dagger}$, can assume $f, g > 0$, since $(-f)^{\dagger} = f^{\dagger}$. Then $\frac{g}{f} \succ 1$, and $\frac{g}{f} > 0$, so $\frac{g}{f} > \mathbb{N}$, so $\left(\frac{g}{f}\right)' > 0$, so $g'f - gf' > 0$, and thus $g'f > gf'$, so $g^{\dagger} > f^{\dagger}$.

(ii) let $f \prec g \notin \mathbb{R}$; to prove $f' \prec g'$, we arrange $f, g > 0$. Have $f^{\dagger} < g^{\dagger}$

by (i), so $\frac{f'}{f} < \frac{g'}{g}$, i.e. $f' < \left(\frac{f}{g}\right) g' \prec g'$
since $\frac{f}{g} \prec 1$ and $g' \neq 0$

(iii) $f \prec g \prec 1$ gives by (i):
$f^\dagger < g^\dagger < 1^\dagger = 0$, so $f^\dagger \succcurlyeq g^\dagger$.

(iv) let $f \preccurlyeq 1$, $g \not\asymp 1$; replacing f by $f+1$ if necesary, arrange $f \asymp 1 \prec g$.
and g by g^{-1}
Then $f^k \prec g$ for all $k \in \mathbb{Z}$, hence
$(f^k)^\dagger < g^\dagger$, i.e. $kf^\dagger < g^\dagger$ (all $k \in \mathbb{Z}$),
so $f^\dagger \prec g^\dagger$, so $f' = f f^\dagger \prec g^\dagger$. $\boxtimes$.

Exercise: For $f, g \in H^{\neq}$, $f, g \not\asymp 1$:

$$f \preccurlyeq g \iff f' \preccurlyeq g'$$

There is much more to say about Hardy asymptotics. For general orientation: any asymptotic rule valid in all Hardy fields can be deduced from the axioms for

H-fields with small derivation.

(2.4) Composition

Let $\ell \in \mathcal{C}'$, $\ell >_{ev} \mathbb{N}$, $\ell' = \phi \in H$.

Note: $\phi > 0$, and we have the Hardy field $H(\ell)$. It turns out that composition on the right with ℓ^{inv} is what we should focus on.

Note: $(\ell^{inv})' = \phi^{-1} \circ \ell^{inv}$ (recall: $\phi = \ell'$)

This composition is a kind of change of variable. We have the ring isomorphism $f \mapsto f^{\circ} := f \circ \ell^{inv}$ of $\mathcal{C}'$ with a useful identity

$$\boxed{(f^{\circ})' = (\phi^{-1} \underline{f'})^{\circ}}$$

For $f \in H$ this gives $(f^{\circ})' \in H^{\circ} := H \circ \ell^{\text{inv}}$,

so H° is a Hardy field

The ordered and valued field iso

$$h \mapsto h^{\circ} : H \to H^{\circ}$$ is not a differential field iso (unless $\ell = x + \text{constant}$)

but it is a differential field iso

$$H^{\phi} \longrightarrow H^{\circ}$$

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§3 Solving Firs-Order ODE in Hardy Fields

Motivation: $\sin(x)$ is not hardian, but what about $\sin\left(\frac{1}{x}\right)$? This is hardian, but is not a consequence of our extension results so far

($\sin\left(\frac{1}{x}\right) \notin \mathrm{Li}(\mathbb{R})$, I guess.)

The reason ~~is~~ that $\sin\left(\frac{1}{x}\right)$ is hardian is that it satisfies the first-order ODE $y'(t) = -\frac{1}{t^2}\sqrt{1-y(t)^2}$ and RHS is given by a semialgebraic $\mathcal{C}^1$-function ~~t,y~~ $-\frac{1}{t^2}\sqrt{1-y^2}$.

To make this precise, some notation: for open $U \subseteq \mathbb{R}^n$,

$\mathcal{C}^r(U)$ = ring of r times continuously differentiable functions $U \to \mathbb{R}$

($r \in \mathbb{N} \cup \{\infty\}$; $\mathcal{C}^\omega(U)$ also defined likewise)

(3.1) The explicit case

(3.1.1) Let $U \subseteq \mathbb{R}^{n+1}$ be open and semialgebraic, let $\underline{\Phi} : U \to \mathbb{R}$ be semialgebraic and in $\mathcal{C}^1(U)$. Suppose $h_1, \dots, h_n \in H$ and $\eta \in \mathcal{C}^1$ be such that eventually

(*) $(h(t), \eta(t)) \in \mathcal{U}$, $\eta'(t) = \underline{\Phi}(h(t), \eta(t))$

$$\underset{\displaystyle (h_1(t), \dots, h_n(t))}{\overset{\displaystyle h(t)}{\parallel}}$$

. Then η lies in a Hardy field extension of H.

(If $\mathbb{R} \subseteq H$, H real closed, $H(\eta)^{rc}$ is a Hardy field.)

Cor Let $h \in H$, $h \preceq 1$. Then $\sin(h)$, $\cos(h)$ lie in a common Hardy field extension of H

Pf Arrange $H \supseteq \mathbb{R}$. Take $c \in \mathbb{R}$ s.t. $h = c + \varepsilon$, $\varepsilon \in H$. Then $\cos(h) = \cos(c+\varepsilon) =$ $a\cos(\varepsilon) + b\sin(\varepsilon)$ $(a, b \in \mathbb{R})$

Now $\sin(\varepsilon)$ satisfies the ODE

$$y'(t) = \varepsilon'(t)\sqrt{1 - y(t)^2},$$

so we can apply Prop. 3.1.) with $n = 1$,

$$\underline{\Phi}(s_1, s_2) = s_1\sqrt{1 - s_2^2}$$

$(-\infty < s_1 < \infty, -1 < s_2 < 1)$, $h = \varepsilon'$.

$\boxtimes$

Pf Passing to an extension, arrange $H \supseteq \mathbb{R}$, H is real closed

Claim 1: $\eta(t) < 0$, ev., or $\eta(t) = 0$, ev., or $\eta(t) > 0$, ev.

Towards this claim, suppose $\eta(t) = 0$ for arbitrarily large t; enough to show: $\eta(t) = 0$, ~~for~~ eventually. Take $a \in \mathbb{R}$ and representatives $h_1, \dots, h_n, \eta \in \mathcal{C}^1(a, +\infty)$ of their germs such that $(\ast)$ holds for all $t > a$. Now if $t > a$ and $\eta(t) = 0$, then $(h(t), 0) \in \mathcal{U}$.

$\therefore$ $(h(t), 0) \in \mathcal{U}$, eventually, by Proposition (1.8.1).

Will finish proof next time.

Claim 3 $H(\eta)$ is a Hausdorff fld and $H(\eta)^{rc}$ is a Hardy field.