Lecture 3 (mostly on Hardy fields)

Tuesday, January 25, 2022 7:41 AM

Last time we constructed the real closure H^{rc} of a Hausdorff field H.

By the way, we go into Hausdorff fields partly as a kind of baby version of Hardy fields but also because they will appear as intermediate stages in certain Hardy field constructions.

There is one more item we cover in the Hausdorff field setting:

1.8 Eventual behaviour of H with respect to semialgebraic sets and functions

Proposition Suppose the Hausdorff field H contains $\mathbb{R}$. Let $h_1, \ldots, h_n \in H$, let $S \subseteq \mathbb{R}^n$ be semialgebraic, and set, for big t: $h(t) := (h_1(t), \ldots, h_n(t)) \in \mathbb{R}^n$.

$(i)_n$ either $h(t) \in S$, eventually, or $h(t) \notin S$, eventually.

$(ii)_n$ if $h(t) \in S$ and $\phi: S \to \mathbb{R}$ is semialgebraic, then $\phi \circ h \in H^{rc}$ where $(\phi \circ h)(t) = \phi(h(t))$, big t.

Proof Sketch: induction on n; case $n=0$ is trivial. Assume $(i)_n$. Then obtain $(ii)_n$ by first proving a purely semialgebraic **Lemma** (Exercise in QE for $\mathbb{R}_{alg}$) For any semialgebraic set $S \subseteq \mathbb{R}^n$ and semialgebraic $\phi: S \to \mathbb{R}$ there are semialgebraic sets $S_1, \ldots, S_m$ and polynomials $P_1, \ldots, P_m \in \mathbb{R}[X, Y]$, $X = (X_1, \ldots, X_n)$ such that $S = S_1 \cup \cdots \cup S_m$ and for all $x \in S_i$, $P_i(x, Y) \neq 0$, $P_i(x, \phi(x)) = 0$, $i = 1, \ldots, m$.

Next prove $(i)_{n+1}$ using $(i)_n$ and $(ii)_n$ and reduction to the case where S is a cell. ☒

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§2 Hardy Fields (just the beginnings)

Original Sources

G. H. Hardy, Properties of logarithmic-exponential functions, PLMS 10 (1911) 54–90

G. H. Hardy, Orders of Infinity (CUP, 1924)

N. Bourbaki "Appendice" to "Fonctions d'une Variable Réelle" (Hermann, Paris, 1976)

Many papers by Rosenlicht and Boshernitzan (1980–1996)

(2.1) Definition & Some Consequences

(2.1.1) We let $\partial: \mathcal{C}^1 \to \mathcal{C}^0 = \mathcal{C}$ be the derivation $f \mapsto f'$.

A **Hardy field** is a subfield H of $\mathcal{C}^1$ such that $\partial H \subseteq H$. Then $H \subseteq \mathcal{C}^n$ for all n, i.e. $H \subseteq \mathcal{C}^{<\infty} = \bigcap_n \mathcal{C}^n$. The subring $\mathcal{C}^{<\infty}$ is closed under ∂, and so we consider $\mathcal{C}^{<\infty}$ as a diff. ring, and H accordingly as a differential subfield of $\mathcal{C}^{<\infty}$.

(2.1.2) Examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, $H(\tilde{\mathbb{R}})$ for any o-minimal expansion $\tilde{\mathbb{R}}$ of $\mathbb{R}_{alg}$

Till further notice H is a Hardy field

(2.1.3) for $f \in H$, one of three things:

(i) $f' < 0$: f eventually strictly decreasing

(ii) $f' = 0$: " " constant

(iii) $f' > 0$: " " strictly increasing

- $f > \mathbb{N} \Rightarrow f' > 0$
- $f \preceq 1 \Rightarrow f - c \prec 1$ for some $c \in \mathbb{R}$

($\therefore \mathbb{R} \subseteq H \Rightarrow \mathcal{O}_H = \mathbb{R} + o_H$)

- $f \prec 1 \Rightarrow f' \prec 1$

(2.1.4) Consequences (of facts about Hausdorff fields)

(i) H^{rc} is a Hardy field

(ii) any $r \in \mathbb{R}$ generates a Hardy field $H(r)$ over H

(iii) any $f \in \mathcal{C}^1$ with $f' \in H$ generates a Hardy field $H(f)$ over H

Pf (i): $H \subseteq \mathcal{C}^{<\infty}$, so $H^{rc} \subseteq \mathcal{C}^{<\infty}$; for $y \in H^{rc}$ with minimum polynomial $P(Y) \in H[Y]$ we saw last time:

$$y' = \frac{-P^{\partial}(y)}{\frac{\partial P}{\partial Y}(y)} \in H(y) \subseteq H^{rc}$$

where for $P = \sum a_i Y^i$, $P^{\partial} := \sum a_i' Y^i$

(ii) Can assume H real closed. If $r \in H$, we are done. Suppose $r \notin H$. Then r transcendental over H, and for every $h \in H$, either $r <_{ev} h$ or $r >_{ev} h$. Now use Lemma 1.6.1.

(iii) For $f \in \mathcal{C}^1$ with $f' \in H$, 1.5.2 gives a Hausdorff field $H(f)$. Clearly $H(f) \subseteq \mathcal{C}^1$, and easy to check that $\partial H(f) \subseteq H(f)$, so $H(f)$ a Hardy field. ☒

(2.1.5) Consequences of those consequences

(iv) $\mathbb{R}$ generates a Hardy field $H(\mathbb{R})$ over H.

(v) x " " " " $H(x)$ over H

(vi) for $f \in H^{>}$, $\log f$ generates a Hardy field $H(\log f)$ over H

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use $(\log f)' = \frac{f'}{f} \in H$.