Lecture 2

Thursday, January 20, 2022 7:20 AM

Transseries, Model Theory, Hardy Fields

Lecture 2

Last time introduced $\mathcal{C}$ and its subfields, called Hausdorff fields, which are naturally ordered fields.

H: Hausdorff field till further notice

Proposition 1.3.3 Let

$P(Y) = a_n Y^n + a_{n-1} Y^{n-1} + \cdots + a_0 \in H[Y]$ be irreducible ($n \geq 1$, $a_0, \ldots, a_n \in H$, $a_n \neq 0$).

Then, eventually (i.e. for large enough t):

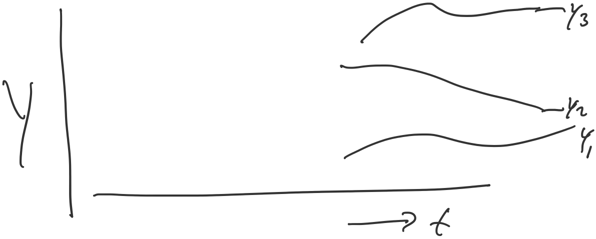
(i) $P(t, Y) = a_n(t) Y^n + a_{n-1}(t) Y^{n-1} + \cdots + a_0(t) \in \mathbb{R}[Y]$

has exactly n distinct zeros in $\mathbb{C}$, and each is nonsingular (i.e., not a zero of

$$\frac{\partial P}{\partial Y}(t, Y) = n a_n(t) Y^{n-1} + \cdots \text{)}$$

(ii) there are $y_1 <_{ev} y_2 <_{ev} \cdots <_{ev} y_d$ in $\mathcal{C}$

such that $y_1(t) < y_2(t) < \cdots < y_d(t)$ are the real zeros of $P(t,Y)$



Pf (i) $P \in H[Y]$ irr, so $P, \frac{\partial P}{\partial Y}$ are coprime, so $AP + B\frac{\partial P}{\partial Y} = 1$, $A, B \in H[Y]$, so for big t,

$$A(t,Y)\,P(t,Y) + B(t,Y)\frac{\partial P}{\partial Y}(t,Y) = 1,$$

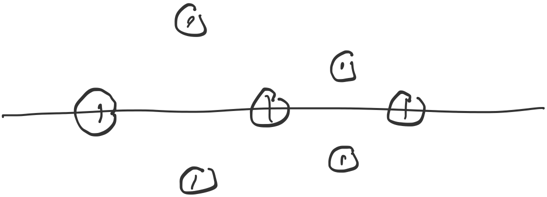
so $P(t,Y)$, $\frac{\partial P}{\partial Y}(t,Y)$ have no common zero in $\mathbb{C}$.

(ii) For big t, let $y_1(t) < \dots < y_d(t)$ be the real zeros; have to show d does not depend on t, and $y_1(t), \dots, y_d(t)$ are continuous at t. Let

$u_1(t) + i v_1(t), \quad \dots, \quad u_m(t) + i v_m(t)$

$u_1(t) - i v_1(t) \quad \dots, \quad u_m(t) - i v_m(t)$

be the other zeros, $d+2m=n$



Take little balls $B_1, \dots, B_d$, $B_{d+1}, \overline{B_{d+1}}, \dots, B_{d+m}, \overline{B_{d+m}}$ centered at these zeros, disjoint, with $B_{d+1}, \dots$ not meeting $\mathbb{R}$. Then for t' close enough to t, each ball will still contain a zero of $P(t', y)$ ("Continuity of roots", see p. 32 of "Tame Top. ..."). So $B_1, \dots, B_d$ contain exactly one zero of $P(t', y)$, and so these zeros must be real. So the number of real zeros remains d, and these zeros $y_1(t') < \dots < y_d(t')$ are continuous of t as functions of t'. Now use connectedness to go from local behaviour of big enough t to the global behaviour described in (ii). $\boxtimes$

Cor 1.3.4 If n odd, then $d \geq 1$ (since $d+2m=n$)

Def An ordered fld K is said to be real closed if every $a \in K^{>}$ is a square in K and every odd degree $P(Y) \in K[Y]$ has a zero in K; equivalently, $K[i]$ with $i^2=-1$ is algebraically closed.

Th 1.3.5 (Hausdorff, Boshernitzan)

H has an algebraic real closed Hausdorff field extension H^{rc}. For any such H^{rc} we have

(*) $H^{rc} = \{y \in \mathcal{C} : P(y)=0$ for some $P \in H[Y]^{\neq}\}$, so H^{rc} is uniquely determined by H.

Pf First part is clear from what we said about adjoining $\mathcal{U}f$ for $f \in H$? and from Prop. 1.3.3. Let H^{rc} be as in first part. We show (*)

$\subseteq$ holds because H^{rc} algebraic over H

$\supseteq$: H^{rc} is real closed; let $y \in \mathcal{C}$ and $P(y) = 0$, $P \in H[Y]^{\neq}$;

to show: $y \in H^{rc}$. Have

$$P(Y) = a(Y-y_1)^{e_1} \cdots (Y-y_d)^{e_d} \prod_j (Y+b_j)^2 + c_j^2$$

$a \neq 0$, $b_j, c_i \in H$, $c_j \neq 0$, $y_1 < \cdots < y_d$ in H^{rc}

Then for big t, $y(t) = y_i(t)$, with i independent of t.

so $y = y_i \in H^{rc}$.

$\boxed{X}$

Cor 1.3.6 H^{rc} is the unique real closed algebraic Hausdorff fld extension of H. For every $y \in H^{rc}$ there is $h \in H$ s.t. $|y| \leq |h|$.

(For the second part, use that for any ordered fld K and any y in any ordered fld extension with $y^n + a_{n-1}y^{n-1} + \dots + a_0 = 0$, all $a_i \in K$: $|y| \le 1 + \max\limits_{0 \le i \le n-1} |a_i|$.)

Cor 1.3.7 Maximal Hausdorff flds are real closed (Hausdorff)

Exercise 1.3.8 Let $\mathbb{R}_{alg} := (\mathbb{R}; <, 0, 1, +, -, \cdot)$.
Then $H(\mathbb{R}_{alg}) = \mathbb{R}(x)^{rc} \subseteq \mathcal{C}$.

1.4 Smoothness

Let $r \in \mathbb{N} \cup \{\infty\}$. Then

$\mathcal{C}^r :=$ subring of $\mathcal{C}$ consisting of the germs of r-times continuously differentiable real valued fct's on intervals $(a, +\infty)$.
So $\mathcal{C}^0 = \mathcal{C}$. For $r \in \mathbb{N}^{\ge 1}$, have the derivation $f \mapsto f' : \mathcal{C}^r \to \mathcal{C}^{r-1}$;

which restricts to the usual derivation of $\mathcal{C}^\infty$ and on $\mathcal{C}^\omega$ = same as above with "real analytic" instead of "r-times ...".

Prop. 1.4.1 Let $r \in \mathbb{N} \cup \{\infty, \omega\}$, and suppose $P(Y) = a_n Y^n + \dots + a_0 \in H(Y)$ is irreducible with all $a_i \in \mathcal{C}^r$. Then the $y_1, \dots, y_d$ from Prop 1.3.3(ii) are also in $\mathcal{C}^r$.

Pf For $y = y_i$ we have $P(t, y(t)) = 0$,

$$\frac{\partial P}{\partial Y}(t, y(t)) \neq 0 \quad \text{(big enough } t\text{)}.$$

Then IFT gives $y \in \mathcal{C}^r$. $\boxtimes$

Cor 1.4.2 For r as above:

$$H \subseteq \mathcal{C}^r \Longrightarrow H^{rc} \subseteq \mathcal{C}^r$$

For later use we note that for $P(Y)$ and $y = y_i$ as in 1.4.1 and $r \geq 1$ we have an explicit fmla for y':

$$a_n(t)\, y(t)^n + a_{n-1}(t)\, y(t)^{n-1} + \dots + a_0(t) = 0;$$ take

derivatives to get

$$\sum_{i=0}^{n} a_i'(t)\, y(t)^n + \underbrace{\left(\sum_{i=1}^{n} i\, a_i(t)\, y(t)^{n-1}\right)}_{\frac{\partial P}{\partial Y}(t,\, y(t)) \neq 0,\ \text{so}} y'(t) = 0$$

$$\boxed{y'(t) = -\frac{\sum_{i=0}^{n} a_i'(t)\, y(t)^i}{\frac{\partial P}{\partial Y}(t,\, y(t))}}$$

1.5 Adjoining transcendental germs

Lemma 1.5.1 Suppose H is real closed and $y \in \mathcal{C}$ is such that for all $h \in H$, $y - h \in \mathcal{C}^\times$.

Then y generates a Hausdorff fld extension $H(y) \subseteq \mathcal{C}$ of H.

Pf Let $P(Y) \in H[Y]^{\neq}$, so

$$P(Y) = a(Y-h_1)\cdots(Y-h_d)\prod_j (Y+b_j)^2 + c_j^2$$

with $a \in H^{\neq}$, $h_1, \ldots, h_d, b_j, c_j \in H$, $c_j \neq 0$.

Then $P(y) = a(y-h_1)\cdots(y-h_d)\prod (y+b_j)^2 + c_j^2$

$\in \mathcal{C}^{\times}$, so $H[y]^{\neq} \subseteq \mathcal{C}^{\times}$, so

get Hausdorff fld $H(y) \subseteq \mathcal{C}$.

$\boxtimes$

Cor 1.5.2 Suppose $\mathbb{R} \subseteq H \subseteq \mathcal{C}'$ and $\partial H \subseteq H$
$y \in \mathcal{C}'$ satisfies $y' = f \in H$. Then y generates a Hausdorff fld $H(y)$.

Pf Can reduce to H real closed, $y \notin H$.
Let $h \in H$; enough to show $y - h \in \mathcal{C}^{\times}$.
Suppose $y - h \notin \mathcal{C}^{\times}$. Then $y(t) = h(t)$ for arbitrarily large t, so $y'(t) = h'(t)$ for
" " t, so $f(t) = h'(t)$ "
" ... so $f = h'$, so $y' = h'$, so
$y = h +$ real constant $\in H$. ↯.

Exercise

(1) If H is maximal, then

$$\mathbb{R} = \{ \lim_{t \to +\infty} h(t) : h \precsim 1 \}.$$

Note : $H \cap \mathbb{R}$ is real closed.

(2) For any real closed subfield E of $\mathbb{R}$

There is a maximal H s.t. $H \cap \mathbb{R} = E$

[Just do case $E = \{$real algebraic numbers$\}$.)

(3) $\bigcap$ maximal Hausdorff flds $=$

$\{$real algebraic numbers$\}$

(4) There are $2^{\text{continuum}}$ many

maximal Hausdorff flds

(5) Any two maximal Hausdorff fields

are back-and-forth equivalent

($\therefore$ isomorphic under CH)

1.6 The canonical valuation on H

$$\mathcal{O}_H := \{h \in H : h \preccurlyeq 1\} = \{h \in H : |h| \leq n \text{ for some } n\}$$

is a convex subring of H (the smallest such). $\therefore$ $\mathcal{O}_H$ is a valuation ring of H, i.e. if $f \in H$, $f \notin \mathcal{O}_H$, then $f^{-1} \in \mathcal{O}_H$.

$\therefore$ $\mathcal{O}_H$ is a local ring, with maximal ideal

$o_H = \{h \in H : h \prec 1\} = \{h \in H : |h| \le \frac{1}{n}$ for all $n \geq 1\}$

For $f, g \in H$:

$$f \preceq g \iff f \in \mathcal{O}_H\, g \iff vf \geq vg$$

$$f \prec g \iff f \in o_H\, g \ \& \ g \neq 0 \iff vf > vg$$

1.7 Composition

Let $\ell \in \mathcal{C}$ be eventually strictly increasing and $\ell >_{\mathrm{ev}} \mathbb{R}$. Then for $f \in \mathcal{C}[i]$ the composition $f \circ \ell \in \mathcal{C}[i]$ is given by $(f \circ \ell)(t) = f(\ell(t))$ ~~for~~ eventually.

The map $f \mapsto f \circ \ell : \mathcal{C}[i] \to \mathcal{C}[i]$ is an automorphism of the $\mathbb{C}$-algebra $\mathcal{C}[i]$, maps $\mathcal{C}$ onto $\mathcal{C}$, and respect $\preceq, \prec, \sim$ in both directions:

$$f \preceq g \iff f \circ \ell \preceq g \circ \ell.$$

The inverse is $f \mapsto f \circ \ell^{\mathrm{inv}}$, where

$\ell^{inv} \in \mathcal{C}$ (the compositional inverse of ℓ)

is given by $\ell^{inv} \circ \ell = x$. Then

ℓ^{inv} is also evt. strictly increasing

and $\ell^{inv} >_{ev} \mathbb{R}$.

Note: $f \mapsto f \circ \ell : \mathcal{C} \to \mathcal{C}$ also

respects $\leqslant$ in both directions and

$H \circ \ell$ is again a Hausdorff field.

$H \subseteq \mathcal{C}^{r}, \ell \in \mathcal{C}^{r} \Longrightarrow H \circ \ell \subseteq \mathcal{C}^{r}$

$\ell^{inv} \in \mathcal{C}^{r}$.
