Tuesday, January 18, 2022 7:08 AM

Transseries, Model Theory, and Hardy Fields

(mostly Hardy fields)

§1 Germs and Hausdorff Fields

Original Sources for today:

P. du Bois-Reymond, Ueber asymptotische Werthe, infinitäre Approximationen, und infinitäre Auflösung von Gleichungen, MA 8 (1875), 362–414.

F. Hausdorff, Die Graduierung nach der Endverlauf, Abh. Sächs. Akad. ... Leipzig (1909) 295–334.

E. Artin & O. Schreier, Algebraische Konstruktion reeller Körper, Abh. Math. Sem. Hamburg 5 (1926), 83–99.

M. Boshernitzan, An extension of Hardy's class L of "orders of infinity", J. Analyse Math. (1981), 235–255.

(1.1) $\mathcal{C}$ = ring of germs at $+\infty$ of the real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ on which the fct. is continuous. So $\mathbb{R} \subseteq \mathcal{C}$

Conventions: for $f \in \mathcal{C}$ we also let f denote any representative of it. So we can use expressions like "$f(t) \to 0$ as $t \to +\infty$"

s, t range over $\mathbb{R}$; $f, g, h \in \mathcal{C}$

We partially order $\mathcal{C}$:

$f \leq g :\Longleftrightarrow f(t) \leq g(t)$, eventually

$\mathcal{C}^\times$: multiplicative group of units of $\mathcal{C}$

$f <_{ev} g :\Longleftrightarrow f(t) < g(t)$, eventually.

Note: $f \in \mathcal{C}^\times \Longleftrightarrow f <_{ev} 0$ or $f >_{ev} 0$.

$|f| \in \mathcal{C}$ given by $|f|(t) = |f(t)|$.

(1.2) Asymptotic Relations on $\mathcal{C}$

$f \preceq g :\Longleftrightarrow \exists c \in \mathbb{R}^{>} \ |f| \leq c|g|$

$\Updownarrow$

$g \succeq f$

$f \prec g :\Longleftrightarrow \forall c \in \mathbb{R}^{>} \ |f| <_{ev} c|g|$

$\Updownarrow \qquad \Longleftrightarrow g \in \mathcal{C}^\times$ and $\frac{f(t)}{g(t)} \to 0$ as $t \to \infty$

$g \succ f$

$f \asymp g :\Longleftrightarrow f \preceq g$ and $g \preceq f$

$f \sim g :\Longleftrightarrow f - g \prec g$ (eq. rel. on $\mathcal{C}^\times$)

$\Longleftrightarrow g \in \mathcal{C}^\times$ and $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$

Remark All this (except for $\leq$ and $<_{ev}$) extends in the obvious way to the complexification

$$\mathcal{C}[i] = \mathcal{C} + \mathcal{C}i$$

$\mathcal{C} \subseteq \mathcal{C}[i]$, $\mathbb{C} \subseteq \mathcal{C}[i]$.

For $f = g + hi \in \mathcal{C}[i]$, $g, h \in \mathcal{C}$,

$$|f| = \sqrt{g^2 + h^2} \in \mathcal{C}$$

So $f \asymp |f|$ (not $f \sim |f|$, in general)

(1.3) Hausdorff Fields

A Hausdorff field is a subfield of $\mathcal{C}$.

Examples:

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}(x)$, $\mathbb{R}(x)$

where x = germ of the identity fct. on $\mathbb{R}$,

for any o-minimal expansion $\tilde{\mathbb{R}}$ of the real ordered field,

$H(\tilde{\mathbb{R}}) := \{$germs of definable fct's $\mathbb{R} \to \mathbb{R}\}$

Let H be a Hausdorff field till further notice.

H is an ordered field: if $f \in H^{\neq}$, then $f \in \mathcal{C}^\times$, so $f >_{ev} 0$, or $f <_{ev} 0$.

We'll just use $>$ and $<$ for germs in a Hausdorff fld instead of $>_{ev} 0$ and $<_{ev} 0$.

Exercise $f \in H \Rightarrow \lim_{t \to +\infty} f(t)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$.

Lemma R subring of $\mathcal{C}$, $R^{\neq} \subseteq \mathcal{C}^\times$.

$\Rightarrow$ R generates a Hausdorff field $\operatorname{Frac}(R) \subseteq \mathcal{C}$

Special Case: $f \in H^{>} \Rightarrow$

$H[\sqrt{f}]$ is a Hausdorff fld

(1.3.2) Lemma Suppose $P(Y) \in H[Y]$ is irreducible and $y \in \mathcal{C}$, $P(y) = 0$.

Then $H[y]$ is a Hausdorff fld.

Pf The kernel of the evaluation map

$H[Y] \to H[y]$, $Q(Y) \mapsto Q(y)$

contains $P(Y)$, so equals $P(Y)H[Y]$, so

$H[Y]/PH[Y] \cong H[y]$, so $H[y]$ a field. ☒

Proposition 1.3.3. Let $P(Y) \in H[Y]$ be irreducible. Say

$P(Y) = P(t, Y) = a_0(t)Y^n + a_1(t)Y^{n-1} + \dots + a_n(t)$,

where $a_0, a_1, \dots, a_n \in H$, $a_0 \neq 0$, $n \geq 1$.

Then:

(i) the zeros of $P(t, Y) = a_0(t)Y^n + \dots + a_n(t)$

$\in \mathbb{R}[Y]$ (for large t)

in $\mathbb{C}$ are nonsingular (i.e. not zeros of $\frac{\partial P}{\partial Y}(t, Y)$.)

(ii) the number of real zeros of $P(t, Y)$ is eventually constant, and are given by germs $y_1 <_{ev} y_2 < \dots <_{ev} y_d$ in $\mathcal{C}$.