Sublogarithmic-Transexponential Series

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Background

1991 - Wilkie proves $(\mathbb{R}, <, +, \cdot, 0, 1, \exp)$ is model complete, hence o-minimal

1994 - Miller proves "Exponentiation is hard to avoid"

Growth Dichotomy

Let $\mathcal{R} = (\mathbb{R}, <, +, \cdot, 0, 1, \ldots)$

$\mathcal{R}$ is polynomially bounded if for every definable function f, $\exists m, n \in \mathbb{N}$ such that $\forall x > m$, $|f(x)| \leq x^n$

Thm (Miller): If $\mathcal{R}$ is o-minimal and not polynomially bounded, then the exponential function is definable in $\mathcal{R}$.

$\mathcal{R}$ is exponentially bounded if for every definable function f, $\exists m, n \in \mathbb{N}$ such that $\forall x > m$, $|f(x)| \leq e^{\cdot^{\cdot^{e^x}}}$ n times

Question: Are there o-minimal structures that are $\underbrace{\text{not exponentially bounded}}_{\text{"Transexponential"}}$?

Kneser (1943): There is a real-analytic function E satisfying

$E(x+1) = \exp E(x)$ and $E(0)=1$

Long-term goal: Show that $(\mathbb{R}_{an,exp}, E)$ is o-minimal.

Lemma (van den Dries, Macintyre, Marker): Suppose $T = Th(\mathbb{R}, <, +, \cdot, 0, 1, \ldots)$ has quantifier elimination ①. Then T is o-minimal iff for each term $t(x)$, $\exists m \in \mathbb{R}$ such that

- $t(x) > 0$ for all $x > m$
- $t(x) = 0$ for all $x > m$
- $t(x) < 0$ for all $x > m$

②

① Find a language $\mathcal{L}_E$ in which $Th(\mathbb{R}_{an,exp}, E)$ has Q.E.

- $<, +, -, \cdot, 0, 1$
- Restricted analytic functions: a symbol $\tilde{f}$ for each $f \in \mathbb{R}\{\bar{x}\}$ = power series in $\bar{x}$ that converge in a nbd of $[-1,1]^{\ell(\bar{x})}$
  Interpret $\tilde{f}$ by $\tilde{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \bar{x} \in [-1,1]^{\ell(\bar{x})} \\ 0 & \text{else} \end{cases}$
- $\exp$, $\log$
- E, $\underset{E^{-1}}{\overset{\shortparallel}{L}}$, $E', E'', \ldots, (E')^{-1}, (E'')^{-1}, \ldots$

② Show the germs at $+\infty$ of $\mathcal{L}_E$-terms are ordered (thesis work)

- Build a field of formal sublogarithmic-transexponential transseries, into which the field of germs of $\mathcal{L}_E$-terms embeds

- Transseries field induces an order on field of germs

Other fields of transexponential transseries

"Corps de Transséries" - Schmeling PhD thesis (2001)

- Build fields of transseries closed under some countable ordinal iterates of log, exp

  ex. $e_1 = \exp$, $e_2 = \exp \circ \exp$, $e_\omega = E$

"Hyperserial Fields" - Bagayoko, van der Hoeven, Kaplan (2021)

- Build a field of transseries closed under all ordinal iterates of exp, log

High level overview

- Derive some basic info from difference eq. $E(x+1) = \exp E(x)$

- Formal transseries construction:

① Show that relatively simple finite formal sums can be ordered in a way that matches true germs of $\mathcal{L}_E$-terms, under some assumptions

(2) Adapt the logarithmic-exponential transseries construction of van den Dries, Macintyre, Marker, to show that given "$E(t)$-monomials" satisfying some assumptions, we can build an ordered field of series closed under exp, log, restr. an. func.
↳ via (1)

(3) Starting from the log-exp transseries, inductively close off under $E, E', \dots, L$.
At stage n: Having built F_n, run (2) on elements of F_n. Then assemble the many fields (2) gives you into a new field F_{n+1} with more closure under $E, E', \dots, L$

What can we deduce from $E(x+1) = \overset{E}{\exp}\, E(x)$?

- $E(x+1) > E(x)^n$ all $n \in \mathbb{N}$
- $E'(x+1) = \exp E(x) \cdot E'(x) = E(x+1)\, E'(x)$

$$\rightarrow E^{(d)}(x+1) = E(x+1)\left(\underbrace{E'(x)^d + \binom{d}{2} E'(x)^{d-2} E''(x) + \cdots + E^{(d)}(x)}_{\text{Bell polynomial in } E'(x), \dots, E^{(d)}(x)}\right)$$

- $E(x)^{1+r} > E^{(d)}(x)$ all $r \in \mathbb{R}_{>0}$, $d \in \mathbb{N}$
- $E(x+c) > E^{(d)}(x)^r$ all $c \in \mathbb{R}_{>0}$, $d \in \mathbb{N}$, $r \in \mathbb{R}$

Intuition: Shifting $\gg$ powers $>$ derivatives

Intuition for ①

Usually, the sign of a transseries is defined to be the sign of its leading coefficient.

This will not work for us.

ex. $\underbrace{E(x)E''(x) > E'(x)^2}$ but $\overbrace{E(x)E''(x) - 2E'(x)^2 \underline{< 0}}$

$$= -\underline{E_0(x)^2 E_1(x)^2} + E_0(x)^2 E_1(x) E_2(x)$$

Fix (Boshernitzan 1986): Defines a sequence of functions by $E_0(x) = E(x)$, $E_{d+1}(x) = \dfrac{E_d'(x)}{E_d(x)}$

Facts: • $E^{(d)}(x)$ can be expressed as a poly in $E_0(x), \ldots, E_d(x)$

$\Longrightarrow$ • $E_d(x) > E_{d+1}(x)^n$ all $n \in \mathbb{N}$

① Ordering (relatively simple) finite sums

k: ordered field, coefficients + exponents come from k

X: set of formal variables, $\underline{X \subset L} \rightsquigarrow$ ordered field

Defs • Let G be the multiplicative abelian group generated by $E^{(d)}(x)^a$ for $d \in \mathbb{N}$, $x \in X$, $a \in k$. The E-sums are $\underline{k[G]}$

- let $\underline{H}$ be the (lexicographically ordered) mult. ab. group gen. by $E_d(x)^a$ for $d \in \mathbb{N}$, $x \in X$, $a \in \mathbb{k}$. The E_X-Sums are $\underline{\mathbb{k}((H))}$

Assumptions

- For all $m \in \mathbb{N}$, all $x, y \in X$ with $x > y$, all $a \in k$
  - $E(x-m) > k$ $\quad (E(x-m) = \underbrace{\log \cdots \log}_{m \text{ times}} E(x) > k)$
  - $E(x-m) > E(y-m)^a$
- There is a map $r : X \times X \to \mathbb{Q} \cap (0,1)$ such that for all $x, y \in X$ with $x > y$, we have $x - y < r(x,y)$

  i.e. $\forall x, y \in X \quad \underline{E(x)} < E(y+1) = e^{\underline{E(y)}}$

Define a map $\sigma : k[G] \to k((H))$ so that

$\sigma(s) :=$ " s rewritten in terms of Boshernitzan's seq "

$$\sigma(\underline{E^{(d)}(x)^a}) = \text{"} \, \sigma(E^{(d)}(x))^a \, \text{"}$$

$$= \text{"} \left(E_0(x) E_1(x)^d + \cdots \right)^a \text{"}$$

$$= \text{"} E_0(x)^a E_1(x)^{da} \underline{\left(1 + \frac{\cdots}{E_0(x) E_1(x)^d}\right)^a} \text{"}$$

$$:= E_0(x)^a E_1(x)^{da} \sum_{n=0}^{\infty} \binom{a}{n} \left(\frac{\cdots}{E_0(x) E_1(x)^d} \right)^n$$

Lemma If X and k satify the above assumptions, then σ is injective. Hence, the order on $k((H))$ induces an order on $k[G]$.

Upshot: Knowing how to "correctly" order finite sums of the form in $k[G]$ (to match the true order on germs of $\mathcal{L}$-terms) will be enough to "correctly" order infinite series later on.

② Adapt the log-exp transseries construction to E monomials

van den Dries, Macintyre, and Marker's construction:

- start with $k((x^k))$, k is ordered exponential field
- Inductively add new, larger monomials for increasing levels of exponentiation.

  stage n: $K_n \ni s = \underbrace{s^{\infty}}_{\operatorname{Supp} s^{\infty} > 1} + c + s^{\varepsilon}$ (rel. to K_{n-1})

  $K_{n+1} = K_n((e(A)))$ where $A = \{ s \in K_n : \operatorname{Supp} s > 1 \}$

  End up with $k((x^{-1}))^e$, an exponential field

- Define embedding $\varphi : k((x^{-1}))^e \hookrightarrow k((x^{-1}))^e$ so that $\forall s \in k((x^{-1}))^e$, $\varphi(s)$ has a logarithm

Part 1

Part 2

$\langle E^{(d)}(x)^{a} \rangle$
$\parallel$

What goes wrong if we start with G instead of x^k?

- A logarithm of $E'(x)$ will not naturally arise in part 2
  - $\log E'(x) = E(x-1) + \log E'(x-1)$
    $= E(x-1) + E(x-2) + \cdots + E(x-m) + \log E'(x-m)$
- Some monomials in G are "small" relative to others

ex. $\dfrac{E'(x)^{1/2}}{E(x)^{1/2}}, E(x) \in G$ but $\exp\left(\dfrac{E'(x)^{1/2}}{E(x)^{1/2}}\right) = \exp\left(E'(x-1)^{1/2}\right) < E(x)$

- G has reverse well ordered subsets of asymptotic monomials

ex. $\displaystyle\sum_{n=1}^{\infty} c_n E(x)^{1/n} \underline{E'(x)}^{2-2/n} E''(x)^{1/n}$

Recall: $\underline{E(x)E''(x)} > E'(x)^2$ but $E(x)E''(x) - 2E'(x)^2 < 0$

$$\lim_{x\to\infty} \frac{E(x)E''(x)}{E'(x)^2} = 1$$

Solutions?

- Include $\log E'(x)^a$, $x \in X$, $a \in k$, as monomials at stage 0.
  (Also strengthen lemma of (1))
- Include $\exp(\text{"small" infinite el. of } G)$ as monomials at stage 0.
  At later stages, only add new monomials for $\exp(\text{"large"})$

adaptation of part 1

- Just don't allow infinite sums of "close" monomials.

let $G_\star$ = our full group of starting monomials (G, logs, exp(small))

For $g, h \in G_\star$, formalize $g \sim h$ "close"

let $k((G_\star))_\sim$:= ring whose el. are sums of the form $s = \sum_{g \in G_\star} c_g g$ where

stage 0

- $\mathrm{Supp}(s) = \{g : c_g \neq 0\}$ is reverse well ordered
- For each $w \in G_\star/\sim$, $\{g \in w : c_g \neq 0\}$ is finite

$k((G_\star))_\sim$ can be ordered via $\boxed{1}$,
if X, k satisfy the assumptions

Part 2: Close under log (and exp, $(\cdot)^{-1}$)

ex 1. $\log E(x) = E(x-1) \notin$ part 1 construction

ex 2. $\exp\left(\frac{E(x)}{E'(x)}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{E(x)}{E'(x)}\right)^n}{n!} \notin$ part 1 construction

$\left(\frac{E(x)}{E'(x)}\right)^n \sim \frac{E(x)}{E'(x)} = \frac{1}{E'(x-1)}$

ex 3. $\frac{1}{E'(x) + E(x)} = \frac{1}{E'(x)}\left(\frac{1}{1 + \frac{E(x)}{E'(x)}}\right) = \frac{1}{E'(x)} \sum_{n=0}^{\infty}\left(-\frac{E(x)}{E'(x)}\right)^n$

Let $X-m = \{x-m : x \in X\}$, $m \in \mathbb{N}$

let K_{X-m} = ring built from $k, X-m$ via part 1

$D_X(k) = \varinjlim \underbrace{\left(K_X \hookrightarrow K_{X-1} \hookrightarrow \cdots\right)}$

$\hookrightarrow$ an ordered exponential-logarithmic field

(3) Use (2) repeatedly to build a field closed under $E, E', \ldots, L$

Let $k \models Th(\mathbb{R}_{an,exp}, \in)$

Let $F_0 = k((\tau^{-1}))^{le}$, $\tau > k$

For $f, g \in F_0$, define $f \approx g$ if $f-g$ is finite

Idea:

- The finite subsets of $(F_0/\approx)_{+\infty}$ form a directed set.
- For each finite subset, we will use (2) to build a field.
  - α_1 smallest
    Let $X_{\alpha_1} \subset \alpha_1$ be maximal satisfying assumptions

    Run (2) to build $D_{X_{\alpha_1}}(k')$

- α_2 2^{nd} smallest

  Let $X_{\alpha_2} \subset \alpha_2$ be maximal satisfying assumptions

  Run (2) to build $D_{X_{\alpha_2}}(D_{X_{\alpha_1}}(k'))$

  ⋮

- F_1 will be the direct limit of this directed system.

- Inductively build F_n, $n \in \mathbb{N}$

  $N_k = \varinjlim (F_0 \hookrightarrow F_1 \hookrightarrow \cdots\cdots)$

  Field of germs of $\mathcal{L}_E$-terms