Parametrizations and complexity of preparation

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Wilkie’s Conjecture & parametrization

The Pila-Wilkie theorem

Statement

Theorem

Let $X \subset \mathbb{R}^n$ be definable in an o-minimal structure on $\mathbb{R}$. For each $\epsilon > 0$ there exists a constant c such that for any $H \in \mathbb{N}$:

$$\left| X^{\text{trans}}(\mathbb{Q}, H) \right| \leq cH^{\epsilon}.$$

Conjecture (Wilkie)

Let $X \subset \mathbb{R}^n$ be definable in $\mathbb{R}_{\exp}$. Then there exist constants c_1, c_2 such that for any $H \in \mathbb{N}, H > e$:

$$\left| X^{\text{trans}}(\mathbb{Q}, H) \right| \leq c_1 \log(H)^{c_2}.$$

The Pila-Wilkie theorem
Idea of the proof

- By induction on $\dim(X)$. The case $\dim(X) = 0$ follows by o-minimality.
- Determinant method (Bombieri, Pila): find few (cH^ϵ) algebraic hypersurfaces $\mathscr{H}_i$ of degree d sufficiently large such that

$$X(\mathbb{Q}, H) \subset \bigcup_i \mathscr{H}_i.$$

- We only have to consider the intersections $X \cap \mathscr{H}_i$ where the dimension drops. The proof follows by induction on $\dim(X)$.

The Pila-Wilkie theorem
Determinant method

Bombieri-Pila

Let $m, n, d \in \mathbb{N}$ with $m < n$. There exist constants $r = r(m,n,d)$, $\epsilon = \epsilon(m,n,d)$ and $c = c(m,n,d)$ with the following property.

If $X = \mathrm{im}(\phi)$ for some C^r-map $\phi : (0,1)^m \to \mathbb{R}^n$ with $|\phi|_r \leq 1$, then

$$X(\mathbb{Q}, H) \subset \bigcup_{i=1}^{cH^\epsilon} \mathscr{H}_i,$$

Where $\mathscr{H}_i$ is an algebraic hypersurface of degree at most d. Moreover, $\epsilon \to 0$ for $d \to \infty$.

The Pila-Wilkie theorem
Determinant method

(2020, Cluckers, Pila, Wilkie)
Let $m, n, H \in \mathbb{N}$ with $m < n$ and H sufficiently large. There exist constants $r = \text{poly}_{m,n}(\log H)$ and $C = C(m, n)$ with the following property.

If $X = \text{im}(\phi)$ for some C^r-map $\phi : (0,1)^m \to \mathbb{R}^n$ with $|\phi|_r \leq 1$, then $X(\mathbb{Q}, H)$ is contained in the union of C algebraic hypersurfaces of degree at most $[\log(H)^{m/(n-m)}]$.

The Pila-Wilkie theorem

Main technical problems

- Applying the determinant method: how to obtain, in general, $X = \mathrm{im}(\phi)$?
  - Solution: C^r-parametrization theorem for o-minimal structures.
- The induction step: parametrizing $X \cap \mathscr{H}_i$ yields a constant $c = c(X \cap \mathscr{H}_i, \epsilon)$, which depends on H.
  - Solution: prove the C^r-parametrization theorem uniformly: consider $X \cap \mathscr{H}_i$ as a family of sets $\{X \cap \mathscr{H}_t \mid t \in T\}$ depending on some parameter space T, and show that the constant $c(X \cap \mathscr{H}_t, \epsilon)$ only depends on the "family".

Wilkie's Conjecture

Main technical problems

1. Apply the C^r-parametrization theorem to obtain that $X = \bigcup_{i=1}^{N} \mathrm{im}(\phi_i)$, in order to apply the determinant method.

- $r = \mathrm{poly}_{m,n}(\log H)$
- N depends on r
- In the general Pila-Wilkie theorem: N is not explicit.

To construct: C^r-parametrizations, with $N = \mathrm{poly}(r)$.

Wilkie's Conjecture
Main technical problems

2. Apply the determinant method: for each $\mathrm{im}(\phi_i)$, we obtain a constant amount (depending only on m, n) of hypersurfaces of degree at most $[\log(H)^{m/(n-m)}]$.

   - We have obtained $\mathrm{poly}_{m,n}(\log H)$ intersections of X with hypersurfaces of degree at most $\mathrm{poly}_{m,n}(\log H)$.

3. Apply C^r-parametrization to each intersection.

   - The uniformity of the number of charts of the parametrization so far only holds for the coefficients of the defining polynomials, but depends on the degree!

To construct: a uniform C^r-parametrization that parametrizes these intersections, with a number of charts that is polynomial in d.

Wilkie's Conjecture

Parametrizations

FIELDS

Conjecture (2010, Pila)
Suppose that X is definable in $\mathbb{R}_{\exp}$ and let $\mathscr{H}$ be an algebraic hypersurface of degree at most d. Then $X \cap \mathscr{H}$ has a C^r-parametrization consisting of at most $\text{poly}_{m,n}(d, r)$ charts.

- Original statement: $X \cap \mathscr{H}$ is $(c_1 d^{c_2}, c_3 d^{c_4}, c_5)$-mild.
  - Instead of $\|\phi\|_r \leq 1$, impose the bounds $\|\phi^{(k)}\|_\infty \leq A^k (k!)^{1+C}$ $(\forall k \in \mathbb{N}^m)$
  - Eliminates dependence on r.
  - Mild parametrization

Parametrizations

Recent results

- Binyamini-Novikov (2019): C^r-parametrization theorem for subanalytic sets: $c(X)r^{\dim(X)}$ charts. Moreover, if X is semi-algebraic, then $c(X) = \text{poly}_{m,n}(d)$.
  - Also mild parametrization version
- Cluckers-Pila-Wilkie (2020): C^r-parametrization theorem for power-subanalytic sets: $c_1(X)r^{c_2(X)}$ charts.
  - V.H. (2021): the construction can be improved to also obtain $c_2(X) = \dim(X)$, and a mild parametrization version.
  - No extra information in semi-algebraic case

Parametrization & preparation

C^r-parametrization
Proof strategy Cluckers-Pila-Wilkie

- Decompose X into finitely many "simple" pieces. Each "simple" piece is the image of a map $\phi : (0,1)^{\dim(X)} \to X$ (cell decomposition).
- For each ϕ: bound the C^1-norm and reduce to a nice form.
- Use the power substitution $x \mapsto x^r$: the C^r-norm of ϕ is now bounded. Subdividing $(0,1)^{\dim(X)}$ into $r^{\dim(X)}$ smaller cubes, this bound no longer depend on r.

Substitution: example

Let $\phi : C \to \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1, x_2^2/x_1)$, where $C = \{(x_1, x_2) \in (0,1)^2 \mid x_2 < x_1\}$.
After the substitution $(x_1, x_2) \mapsto (x_1^r, x_2^r)$, we obtain the function $(x_1^r, x_2^{2r}/x_1^r)$ on the same domain, which indeed has bounded C^r-norm.

C^r-parametrization
Proof strategy Cluckers-Pila-Wilkie

- To obtain the nice form, they use a preparation theorem.

(Lion-Rolin, 1997)
Let $f(x, y) : \mathbb{R}^{n+1} \to \mathbb{R}$ be a globally subanalytic function. Then there is a decomposition of $\mathbb{R}^{n+1}$ into finitely many cylinders C_i such that for each i $f|_{C_i}(x, y) = f_i(x)$ or $f|_{C_i}$ is prepared, i.e.

$$f(x, y)|_{C_i} = a(x)|y - \theta(x)|^r u(x, |y - \theta(x)|),$$

for some globally subanalytic functions $a(x)$ and $\theta(x)$, $r \in \mathbb{Q}$, $y \neq \theta(x)$ and where u is a special unit on $\{(x, |y - \theta(x)|\ |\ (x, y) \in C_i\}$.

Preparation
D. Miller's version

- Globally subanalytic $\iff$ definable in $\mathbb{R}_{\text{an}}$
- Investigation of classes $\mathscr{W} \subseteq$ an, closed under preparation.
  - E.g. Algebraic functions $\subsetneq$ differentially algebraic functions $\subsetneq$ analytic functions
- O-minimal structures $\mathbb{R}_{\mathscr{W}}$ and $\mathbb{R}_{\mathscr{W}}^{K}$
- Main ingredient: desingularization algorithm from RSW03.

Preparation
Complexity Problems

- Complexity of geometric operations with cylinders & cell decomposition
  - Pfaffian sets (sharply o-minimal structures)
  - Examples of $\mathscr{W} \subseteq$ pfaff?
- How many cylinders does the desingularization algorithm produce?
  - Transform a power series $P(x, y)$ into $x^\mu U(x, y)$
  - Holds in some chart on a neighborhood of the origin

Preparation
Example 1 (D. Miller)

1. $f(x, y) = y^2 - x$
2. $(f \circ p^2_{1,1})(x, y) = y^2 - x^2$
   a) $(f \circ p^2_{1,1} \circ b^{1,2}_1)(x, y) = ((y+1)x)^2 - x^2 = x^2 y(y+2)$

- We find:

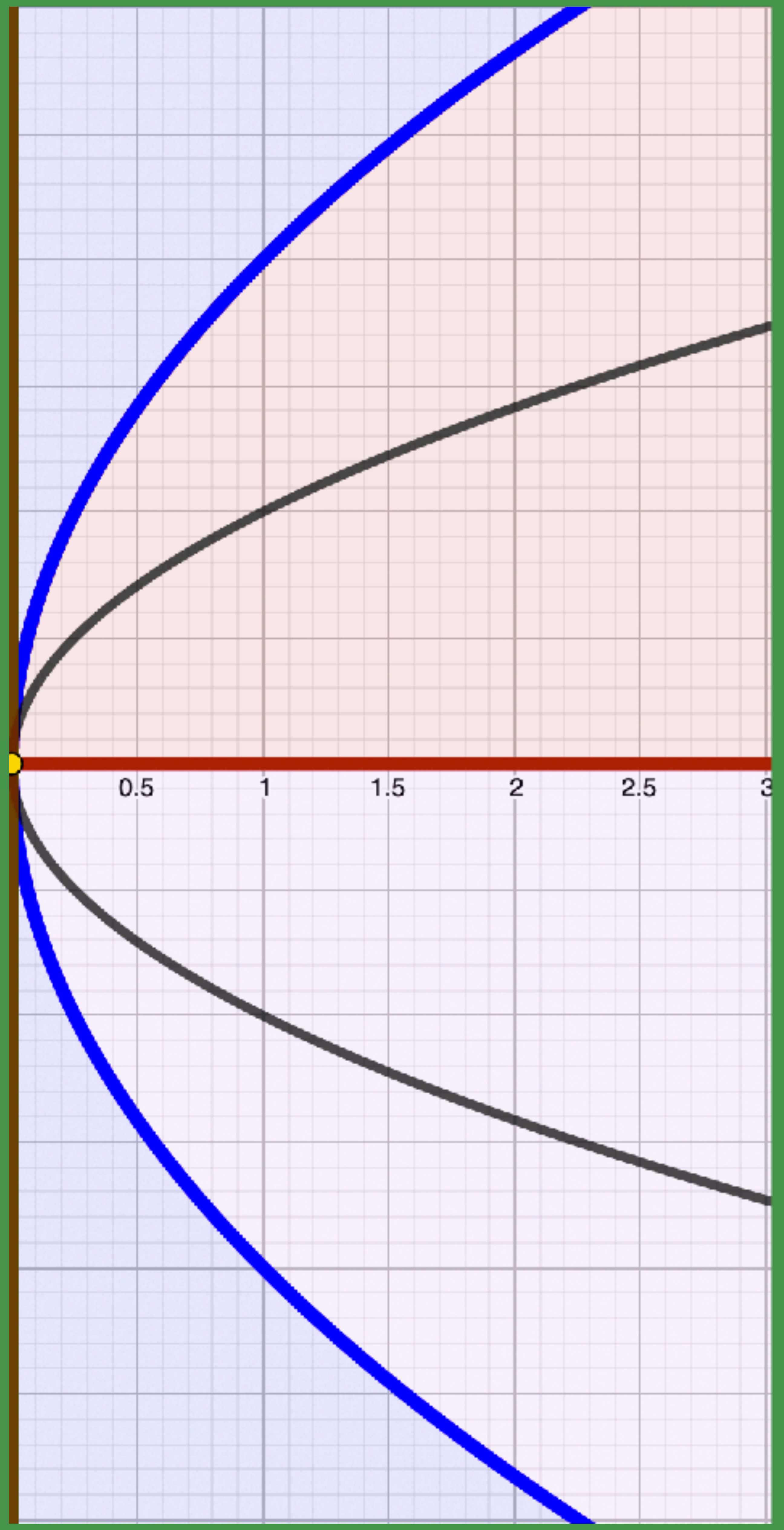
$$f(x, y) = x\left(y/\sqrt{x} - 1\right)\left((y/\sqrt{x} - 1) + 2\right) = \sqrt{x}(y - \sqrt{x})\left(\frac{y - \sqrt{x}}{\sqrt{x}} + 2\right)$$

- $(y+2)$ is a unit on, for example, $-1 < y < 1$. The prepared form holds on $C = \{(x, y) \in \mathbb{R}^2 \mid 0 < x, 0 < y < 2\sqrt{x}\}$.

Preparation

Example 1 (D. Miller)

- $f(x,y) = -x$ on $C_0 = \{0 < x, y = 0\}$
- $f(x,y) = \sqrt{x}(y - \sqrt{x})\left(\frac{y - \sqrt{x}}{\sqrt{x}} + 2\right)$ on
  $C_1 = \{0 < x, 0 < y < 2\sqrt{x}\}$
- $f(x,y) = y^2\left(1 - \frac{x}{y^2}\right)$ on
  $C_2 = \{0 < x, 2\sqrt{x} < y\}$ and $C_3 = \{0 < x, y < -2\sqrt{x}\}$
- $f(x,y) = -\sqrt{x}(y + \sqrt{x})\left(-\frac{y + \sqrt{x}}{\sqrt{x}} + 2\right)$ on
  $C_4 = \{0 < x, -2\sqrt{x} < y < 0\}$



Decomposition into cylinders to prepare $f(x,y) = y^2 - x$

Preparation

Back to parametrization

- We want to show that $X \cap \mathscr{H}$ is $(c_1 d^{c_2}, c_3 d^{c_4}, c_5)$-mild.
- Positive answers to the complexity problems establish “$c_1 d^{c_2}$”.
- Still have to check “$c_3 d^{c_4}$”.