Transseries, Model Theory, and Hardy Fields

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Introductory remarks

Everything I say is based on joint work with Matthias Aschenbrenner and Joris van der Hoeven.

The focus of our book *Asymptotic Differential Algebra and Model Theory of Transseries* (Annals of Mathematics Studies 195, 2017, Princeton University Press) was on understanding the differential field $\mathbb{T}$ of transseries. In the process we developed a wide ranging theory of valued differential fields.

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To apply this to Hardy fields we had to refine the algebraic machinery from that book. But Hardy fields are algebraic as well as analytic in nature, so we used also analysis (real, complex, functional, ODE). This work on Hardy fields took place in the last five years, and I expect it to have benefits beyond Hardy fields.

More introductory remarks

In the first of these two workshop talks I give an overview of results in our book: I discuss the key facts about $\mathbb{T}$ and its model theory, and Hardy fields will just be mentioned in passing.

In the second talk I will introduce Hardy fields in more detail, and state our two conjectures from 2017 about them. About a year ago the first conjecture was established, and last August the second, but we are still working on a good exposition. I will outline the ideas behind their proofs in this second talk.

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In both talks I will mention open problems, and in the second talk I might also speculate on potential connections to o-minimality. In some sense the topic of Hardy fields is a kind of one-variable version of o-minimality, since the (germs of) the one-variable functions in a Hardy field cannot oscillate.

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These talks are meant as an introduction to my course with the same title. There we treat things in in more detail, with careful definitions and proofs, although I will also rely on the literature now and then. With some luck you might solve some open problems during this semester.

What are transseries?

Also called **logarithmic-exponential series**, they are formal series in a variable x. Example:

$$e^{e^x + e^{x/2} + e^{x/4} + \cdots} - 3\,e^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 1 + x^{-1} + x^{-2} + \cdots + e^{-x} + e^{-x^2}.$$

Think of x as positive infinite: $x > \mathbb{R}$. The monomials here, called **transmonomials**, are arranged from left to right in decreasing order, with real coefficients.

The field $\mathbb{T}$ of transseries is basically obtained by starting with ordinary Laurent series in x^{-1} over $\mathbb{R}$ and closing off under exponentiation, taking logarithms, and (reasonable) infinite sums; the precise construction requires care.

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The transmonomials form a subgroup of $\mathbb{T}^{>}$, and the set of transmonomials occurring in a given transseries with nonzero coefficient is countable and *reverse well-ordered* in the ordering of $\mathbb{T}$. (In the example above the corresponding ordinal is $\omega + 2$, the $+2$ due to the two transmonomials e^{-x} and e^{-x^2} at the end. Sums and products are defined as you would expect with series, the "reverse well-ordered" restriction making products possible.

Exponentiating, and taking logarithms of transseries

To *exponentiate* a transseries f, let f_∞, c, and ε be its infinite part, its constant term, and its infinitesimal part, so $f = f_\infty + c + \varepsilon$. Then

$$\exp(f) \;=\; \mathrm{e}^f \;:=\; \mathrm{e}^{f_\infty} \cdot \mathrm{e}^c \cdot \mathrm{e}^\varepsilon\,.$$

Here e^{f_∞} is a transmonomial (all transmonomials have this form), e^c is just the real number the notation suggests, and $\mathrm{e}^\varepsilon := \sum_{n=0}^\infty \varepsilon^n/n!$, expanded as a series in the usual way. To take the logarithm of $f > 0$ we factor out its leading term, giving $f = c\,\mathrm{e}^g(1+\varepsilon)$ with c a positive real number, e^g the leading transmonomial of f, and infinitesimal ε; then

$$\log f \;:=\; g + \log c + \sum_{n=1}^\infty (-1)^{n-1}\varepsilon^n/n,$$

indicating the infinite part g of $\log f$, its constant term $\log c$, and its infinitesimal part $\sum_{n=1}^\infty (-1)^{n-1}\varepsilon^n/n$. Exponentiation is an isomorphism of the ordered additive group of $\mathbb{T}$ onto the multiplicative group $\mathbb{T}^>$, with inverse given by log.

Restrictions on forming transseries

Let e_n be the nth iterate of x under exp and ℓ_n its nth iterate under log, for example $e_3 = \mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}$ and $\ell_2 = \log(\log x)$. The e_n and ℓ_n are exactly the positive infinite transseries all whose iterated logarithms are transmonomials. The key restriction in forming infinite sums is that for each transseries there is a finite bound on the "nesting" of exp and log in its transmonomials: series like

$$e_0^{-1} + e_1^{-1} + e_2^{-1} + e_3^{-1} + \cdots, \qquad \ell_0^{-1} + (\ell_0\ell_1)^{-1} + (\ell_0\ell_1\ell_2)^{-1} + \cdots$$

are not in $\mathbb{T}$; they do belong to a certain natural "spherical completion" of $\mathbb{T}$, which however is no longer closed under exponentiation.

By the way, the strictly increasing sequence $e_0, e_1, e_2, \ldots$ is cofinal in $\mathbb{T}$, and the strictly decreasing sequence $\ell_0, \ell_1, \ell_2, \ldots$ is coinitial in the set $\mathbb{T}^{>\mathbb{R}}$ of positive infinite elements of $\mathbb{T}$.

$\mathbb{T}$ as an ordered differential field

- $\mathbb{T}$ is a real closed ordered field extension of $\mathbb{R}$.
- Every $f \in \mathbb{T}$ can be *differentiated* term by term:

$$\left(\sum_{n=0}^{\infty} n! x^{-1-n} e^{x}\right)' = \frac{e^{x}}{x}.$$

- The resulting *derivation* $f \mapsto f' \colon \mathbb{T} \to \mathbb{T}$ on the field $\mathbb{T}$ has *constant field* $\{f \in \mathbb{T} : \ f' = 0\} = \mathbb{R}$.
- Every $f \in \mathbb{T}$ has an *antiderivative* in $\mathbb{T}$.

The dominance relation $\preccurlyeq$ on $\mathbb{T}$

For $f, g \in \mathbb{T}$,

$$
\begin{aligned}
f \preccurlyeq g \quad &:\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some positive constant } c \\
f \asymp g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \text{ and } g \preccurlyeq f \\
f \prec g \quad &:\Longleftrightarrow \quad f \preccurlyeq g \text{ and } f \not\asymp g
\end{aligned}
$$

For example $\quad 0 \prec \mathrm{e}^{-x} \prec x^{-10} \prec 1 \prec \log x \prec x^{1/10} \prec \mathrm{e}^{x} \prec \mathrm{e}^{\mathrm{e}^{x}}$

$f > \mathbb{R} \Rightarrow f' > 0$, and we can differentiate and integrate dominance:

$$f \preccurlyeq g \iff f' \preccurlyeq g' \qquad \text{for nonzero } f, g \not\asymp 1.$$

$\mathbb{T}$ as an ordered valued differential field

We shall consider $\mathbb{T}$ as a *valued ordered differential field*, and model-theoretically as an $\mathcal{L}$-structure where the language $\mathcal{L}$ has primitives

$$0, \quad 1, \quad +, \quad -, \quad \cdot\,, \quad \partial \text{ (derivation)}, \quad \leqslant \text{ (ordering)}, \quad \preccurlyeq \text{ (dominance)}.$$

More generally, let K be any ordered differential field with constant field $C = \{f \in K : \ f' = 0\}$. This yields a dominance relation $\preccurlyeq$ on K by

$$f \preccurlyeq g \ :\Longleftrightarrow \ |f| \leqslant c|g| \text{ for some positive } c \in C$$

and we view K accordingly as an $\mathcal{L}$-structure. We also introduce the valuation ring $\mathcal{O}$ of K,

$$\mathcal{O} \ := \ \{f \in K : f \preccurlyeq 1\} \ = \ \text{convex hull of } C \text{ in } K$$

with its maximal ideal $\scriptstyle{\mathcal{O}}$ $:= \ \{f \in K : \ f \prec 1\}$ of infinitesimals.

H-fields

An H-**field** is an ordered differential field K such that:

1. $f > C \Rightarrow f' > 0$;
2. $\mathcal{O} = C + \mathcal{o}$.

Examples: any differential subfield of $\mathbb{T}$ that contains $\mathbb{R}$.

In particular, $\mathbb{T}$ is an H-field, but $\mathbb{T}$ has further basic elementary properties that go beyond this: its derivation is *small*, and it is *Liouville closed*.

Here an H-field K is said to have **small derivation** if it satisfies $f \prec 1 \Rightarrow f' \prec 1$, and is said to be **Liouville closed** if it is real closed and for all $f \in K$ there are $g, h \in K^\times$ such that $g' = h^\dagger = f$ (where $h^\dagger := h'/h$).

DIVP

We say that an H-field K has **DIVP** (the Differential Intermediate Value Property) if for every differential polynomial $P(Y) \in K[Y, Y', Y'', \ldots]$ and all $f < g$ in K with $P(f) < 0 < P(g)$ there is a $y \in K$ such that $f < y < h$ and $P(y) = 0$.

Theorem

The theory of $\mathbb{T}$ is completely axiomatized by:

- *being an H-field with small derivation;*
- *being Liouville closed;*
- *having DIVP.*

Actually, DIVP is a bit of an afterthought. We mention it here mainly for expository reasons.

In our book we proved the theorem above with DIVP replaced by two more fundamental conditions on an H-field, namely of being **ω-free and newtonian**.

We only noticed afterwards the equivalence of these two conditions with DIVP, for Liouville closed H-fields. These two conditions are somewhat technical, so I won't give a precise definition right now.

Being **ω-free** says something about solutions of homogeneous 2nd order linear differential equations. This property is amazingly robust and is in particular preserved when passing to any differentially algebraic H-field extension.

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Newtonianity can be thought of as a powerful version of "differential henselianity".

Model completeness

Terminology: an H-**closed field** is a Liouville closed H-field having DIVP. So the above theorem says: *The theory of H-closed fields with small derivation is complete, and $\mathbb{T}$ is a model of it*. But what we really proved throws more light on definability in $\mathbb{T}$ and the intrinsic significance of the concept of H-field:

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(This model complete theory is not complete: it has models whose derivation is not small, for example $\mathbb{T}$ with its usual derivation $\frac{d}{dx}$ replaced by $-\frac{d}{dt} = x^2\frac{d}{dx}, \quad t := x^{-1}$.)

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Byproduct of the proof:

Theorem

If K *is* H*-closed, then* K *has no proper* d*-algebraic* H*-field extension with the same constants.*

Elimination Theory

This model completeness is the main step towards an elimination theory for H-closed fields, which is really the main result of our book. This involves the following (definable) subsets of an H-closed field K, where $b^\dagger := b'/b$ for $b \in K^\times$:

$$\begin{aligned} \Lambda(K) &:= \{-a^{\dagger\dagger} : \ a \in K,\ a \succ 1\}, \\ \Omega(K) &:= \{a \in K : \ 4y'' + ay = 0 \text{ for some } y \in K^\times\}. \end{aligned}$$

These sets are downward closed in the ordering of K; for example, for $K = \mathbb{T}$,

$$\begin{aligned} \Lambda(\mathbb{T}) &= \{f \in \mathbb{T} : \ f \ < \ \ell_0^{-1} + (\ell_0\ell_1)^{-1} + \cdots + (\ell_0\ell_1\cdots\ell_n)^{-1} \text{ for some } n\}, \\ \Omega(\mathbb{T}) &= \{f \in \mathbb{T} : \ f \ < \ \ell_0^{-2} + (\ell_0\ell_1)^{-2} + \cdots + (\ell_0\ell_1\cdots\ell_n)^{-2} \text{ for some } n\}. \end{aligned}$$

Quantifier Elimination

We now augment the language of ordered valued differential fields with symbols Λ, Ω that name in each H-closed field the binary relations on K given by

$$a\Lambda b \ :\Leftrightarrow \ a \in \Lambda(K)b, \qquad a\Omega b \ :\Leftrightarrow \ a \in \Omega(K)b.$$

Elimination Theorem: *Any formula $\phi(y_1, \ldots, y_m)$ in this extended language is equivalent to a quantifier-free formula $\phi^{qf}(y_1, \ldots, y_m)$, uniformly for all H-closed fields.*

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Consequences for definability in $\mathbb{T}$ as an ordered differential field:

- *If the set $S \subseteq \mathbb{T}$ is definable in $\mathbb{T}$, then for some $f \in \mathbb{T}$, either all $y > f$ in $\mathbb{T}$ are in S, or all $y > f$ in $\mathbb{T}$ are outside S: $\mathbb{T}$ is o-minimal at infinity.*
- *If the set $S \subseteq \mathbb{T}^n$ is definable in $\mathbb{T}$, then $S \cap \mathbb{R}^n$ is semialgebraic.*
- *$\mathbb{T}$ has NIP.*

Dimension Theory, Uniform Finiteness

We now have powerful tools to go further. For example, we have a rough dimension theory, assigning to any definable set $S \subseteq \mathbb{T}^n$ a dimension $\dim S \in \{-\infty, 0, 1, \dots, n\}$, such that:

$$\begin{aligned} \dim S = n &\iff S \text{ has nonempty interior in } \mathbb{T}^n, \\ \dim S < n &\iff S \subseteq \{y \in \mathbb{T}^n : \ P(y) = 0\} \text{ for some } P \in \mathbb{T}\{Y_1, \dots, Y_n\}^{\neq}, \\ \dim S = 0 &\iff S \neq \emptyset \text{ and } S \text{ is discrete.} \end{aligned}$$

NB: $\mathbb{R}$ is discrete as a subset of $\mathbb{T}$, so $\dim \mathbb{R} = 0$.

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NB: $\mathbb{R}$ is discrete as a subset of $\mathbb{T}$, so $\dim \mathbb{R} = 0$.

More subtle is **Uniform Finiteness**: For any definable $S \subseteq \mathbb{T}^{m+n}$ there is a bound $B \in \mathbb{N}$ such that all *finite* sections $S(y)$ with $y \in \mathbb{T}^m$ have size $\leqslant B$.

Open problems

All the above is really describing the model theory of $\mathbb{T}$ as a *differential field*, the ordering and valuation being definable in it. But exponentiation on $\mathbb{T}$ is not definable in $\mathbb{T}$, although its restriction to the set of infinitesimals is.

So what about $\mathbb{T}$ as a differential *exponential* field? Is it "tame"' (in the sense of having a viable model theory)? Elliot Kaplan has results going into this direction, for example, expanding $\mathbb{T}$ with exponentiation and sine restricted to $[-1, 1]$ is tame in this sense.

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Is the H-subfield $\mathbb{T}_{\log}$ of $\mathbb{T}$ consisting just of the logarithmic transseries tame? A transseries is said to be *logarithmic* if every transmonomial in it is of the form $\ell_0^{r_0}\cdots\ell_n^{r_n}$ with $r_0,\dots,r_n \in \mathbb{R}$. Allen Gehret has partial results in this direction.

In contrast, the H-subfield of $\mathbb{T}$ consisting just of the *exponential* transseries, which do not involve log, is known to be very wild: it interprets second order number theory.

THANKS FOR YOUR ATTENTION!