A TALK ON EXPANSIONS OF $(\mathbb{Z}, +, <)$ BY BEATTY SEQUENCES

Let $r > 1$ be an irrational.

Beatty sequence generated by r is $\mathcal{B}_r = (\lfloor nr \rfloor)_{n>0}$

(Characteristic) Sturm word of slope r is $S_r = \left(\lfloor \frac{n+1}{r} \rfloor - \lfloor \frac{n}{r} \rfloor \right)_{n>0}$

Let $P_r^+ := \{ \lfloor nr \rfloor : n > 0 \}$ be the set of terms of $\mathcal{B}_r$,

$\quad$ & $P_r := \{ \lfloor nr \rfloor : n \in \mathbb{Z} \setminus \{0\} \} = P_r^+ \cup (-P_r^+ - 1)$.

<u>Aim</u>: Study $(\mathbb{Z}, +, -, 0, 1, <, P_r)$.

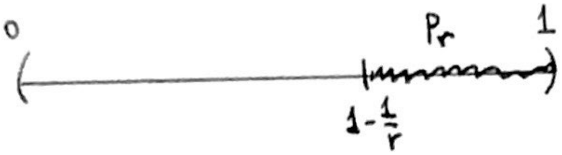
Some Observations:

- For $n \in \mathbb{Z}$: $n \in P_r \iff S_n = 1$.

  (<u>"Proof"</u>: let $n = \lfloor kr \rfloor$ for $k \in \mathbb{Z}$. Then $kr - 1 < n < kr$ and hence $k - \frac{1}{r} < \frac{n}{r} < k$. & $k < \frac{n}{r} + \frac{1}{r} < k + \frac{1}{r}$. So $\lfloor \frac{n}{r} \rfloor = k-1$ and $\lfloor \frac{n+1}{r} \rfloor = k$.)

- $S_n = 1 \iff \lfloor \frac{n+1}{r} \rfloor - \lfloor \frac{n}{r} \rfloor = 1 \iff \frac{n+1}{r} - \underbrace{\left\{ \frac{n+1}{r} \right\}}_{\left\{\frac{n}{r}\right\} + \frac{1}{r} - {}^{0}_{1}} - \frac{n}{r} + \left\{ \frac{n}{r} \right\} = 1$

  $\iff \left\{ \frac{n}{r} \right\} + \frac{1}{r} > 1 \iff \left\{ \frac{n}{r} \right\} > 1 - \frac{1}{r}$.



- Carry this to the unit circle $\mathbb{S}$ (in $\mathbb{C}$):

$$e : \mathbb{R} \to \mathbb{S}, \quad x \mapsto \exp(2\pi i x) \qquad , \qquad h : \mathbb{Z} \to \mathbb{S}, \quad n \mapsto e(\tfrac{n}{r})$$

Topology of $\mathbb{S}$ induced (from $\mathbb{C}$) is generated by "orientation intervals": For $\alpha = e(a)$, $\beta = e(b)$ with $0 < b - a < 1$: $(\alpha, \beta) = \{ e(c) : c \in (a,b) \}$

Putting these together: $\Gamma_r := h(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ as an abelian group, and $n \in P_r \iff h(n) \in (h(-1), 1)$

$$\left(h(-1) = e\left(\tfrac{-1}{r}\right) = e\left(\tfrac{r-1}{r}\right) = e\left(1 - \tfrac{1}{r}\right) \right)$$

<u>Then</u>: $(\mathbb{Z}, +, -, 0, 1, P_r) \simeq (\Gamma_r ; \cdot, {}^{-1}, 1, \eta, P_r)$

$(\eta = h(-1), \quad P_r = (h(-1), 1) \cap \Gamma_r)$

$(\Gamma_r, \cdot, {}^{-1}, 1, \eta, P_r)$ is a reduct of $(\Gamma_r, \cdot, {}^{-1}, 1, \eta, \mathcal{O})$. However, it's easy to see that they are interdefinable; even in a quantifier-free way!

Let $T = Th(\Gamma_r, \cdot, {}^{-1}, 1, \eta) = Th(\mathbb{Z}, +, -, 0, 1)$, let $L = \{\cdot, {}^{-1}, 1, c, P\}$

Let G, H be L-structures, both models of T; suppose also that they are $\aleph_1$-saturated. Also let $f : A \xrightarrow{\sim} B$ be an isomorphism between L-substructures of G & H, and take $\alpha \in G \setminus A$. How can we extend f to α? (Assume A and B are pure in G and H.)

We may extend f to $\langle A \cup \{\alpha\} \rangle_G = \{ \sqrt[m]{a\alpha^n} : n \in \mathbb{Z}, m > 0, a \in A \}$, respecting the group structure. But how about $\sqrt[m]{a\alpha^n}$ being in P_r? It's easy to see that it's enough to control $a\alpha^n$. So let $a_1, \dots, a_\ell \in A$ $n_1, n_2, \dots, n_\ell \in \mathbb{Z}$ and $I \subseteq \{1, \dots, \ell\}$ with $a_i \alpha^{n_i} \in P_r \iff i \in I$.

Can we find β with $f(a_i)\beta^{n_i} \in P_r \iff i \in I$ (& the group theoretic properties as well.) In general, the answer is, of course, no. So we need to add axioms concerning these. Let's observe what happens in Γ_r.

<u>Lemma</u>: Let $\alpha, \beta, \gamma \in S$ with $\alpha < \beta$ and $m > 0$. Then

$$\gamma^m \in (\alpha, \beta) \iff \gamma \in \bigcup_{s=0}^{m-1} \left(\zeta_m^s \alpha^{1/m}, \zeta_m^s \beta^{1/m} \right) \qquad \left(\begin{array}{l} \alpha = e(a),\ a \in (0,1] \\ \alpha^{1/m} := e\left(\frac{a}{m}\right) \end{array} \right)$$

Using this, one can prove the following:

$$\Gamma_r \models \forall \vec{x} \forall \vec{y} \left(\exists z \bigwedge_{i=1}^{\ell} z^{m_i} \in (x_i, y_i) \longleftrightarrow \underbrace{\bigvee_{i=1}^{\ell} \bigwedge_{j \neq i} \left(\bigvee_{k=1}^{m'_{ij}-1} (y_j x_j^{-1})^{k+1} <_\sigma (y_j x_j^{-1})^{k} \vee x_i^{m'_{ji}} \in \left(x_j^{m'_{ij}}, y_j^{m'_{ij}} \right) \right)}_{\varphi_{\vec{m}}(\vec{x}, \vec{y})} \right)$$

$\left(m_{ij} = \gcd(m_i, m_j),\ m'_{ij} = \dfrac{m_i}{m_{ij}},\ \alpha <_\sigma \beta \text{ means } \sigma(0, \alpha, \beta) \right)$

Note that $\varphi_{\vec{m}}(\vec{x}, \vec{y})$ is a quantifier-free formula.

We add, for each $\vec{m} \in \mathbb{Z}^{\ell}$ and $I \subseteq \{1, \ldots, \ell\}$, the following axiom:

$$\forall \vec{x} \forall \vec{y} \left(\exists z \bigwedge_{i \in I} z^{m_i} \in (x_i, y_i) \wedge \bigwedge_{j \notin I} z^{m_j} \notin (x_j, y_j) \longleftrightarrow \varphi_{\vec{m}, I}(\vec{x}, \vec{y}) \right)$$

where $\varphi_{\vec{m}, I}$ is the appropriate boolean combination of $\varphi_{\vec{m}}$'s.

Now if the earlier $\mathcal{G}, \mathcal{H}$ are models of this richer theory, then f can be extended.

Here is a biproduct of the argument above: If $\varphi_{\vec{m}, I}(\vec{\alpha}, \vec{\beta})$ holds, then the set of γ with $\gamma^{m_i} \in (\alpha_i, \beta_i)$ for $i \in I$ and $\gamma^{m_j} \notin (\alpha_j, \beta_j)$ for $j \notin I$ is a finite union "convex sets". So among them we may choose γ to be from a given dense (wrt orientation) subset, like $\Gamma_r^+ = h(\mathbb{N})$! So f can be extended to respect ordering as well!

Of course, there might not be any such f. So this theory is not complete. For that purpose, it's enough to add "the type of η": For any $n \in \mathbb{Z}$, if $n \in P_r$, then we add the axiom $\eta^n \in P_r$ and if $n \notin P_r$, then we add the axiom $\eta^n \notin P_r$.

We get QE after adding relation symbols for the following sets:

$D_m^+ := \{\alpha \in \Gamma_r : \alpha = \beta^m \text{ for some } \beta \in P_r\}$, $D_m^- := \{\alpha \in \Gamma_r : \alpha = \beta^m \text{ for some } \beta \notin P_r\}$.

Another consequence is that infinite definable subsets are $\underline{\underline{\text{not}}}$ sparse:

Let $X \subseteq \mathbb{Z}$ be definable in $(\mathbb{Z}, +, <, P_r)$. If X is infinite, then there is $N = N(X) \in \mathbb{N}_{>0}$ such that for every $x \in X$, we have

$$\{x-N, \ldots, x-1, x, x+1, \ldots, x+N\} \cap X \neq \emptyset.$$