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Promo talk for “Tame control theory” (second module of grad course “Tame phenomena over the real field”.)

Unless I say otherwise:

A **vector field** is a map $F\colon X \to \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $X \subseteq \mathbb{R}^n$.

A **solution of $\boldsymbol{F}$** is a differentiable $\gamma\colon I \to X$ on some infinite connected $I \subseteq \mathbb{R}$ such that $\gamma' = F \circ \gamma$.

A **trajectory of $\boldsymbol{F}$** is the image $\{\, \gamma(t) : t \in I \,\}$ of a solution γ of F.

Motivating, but ill-posed, questions:

What should it mean for a collection of vector fields to be “tame”?

(Or “mutually well behaved”, or “natural”, or ...)

Similarly for collections of trajectories of vector fields.

Similarly for collections of pairs (Γ, F) of trajectories Γ of vector fields F.

If familiar with first-order logic, some by-now routine conventions:

“definable” $:=$ “definable with parameters” (mostly for convenience)

$\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot\,)$

$\mathfrak{R} :=$ an expansion (in the sense of definability) of $\overline{\mathbb{R}}$

If not familiar with first-order logic:

Regard $\mathfrak{R}$ as a sequence $\mathfrak{R} := (\mathfrak{R}_m)_{m=1}^{\infty}$ such that for all m:

- $\mathfrak{R}_m$ is a boolean algebra of subsets of $\mathbb{R}^m$
- $A \in \mathfrak{R}_m \Rightarrow A \times \mathbb{R} \in \mathfrak{R}_{m+1}$
- $A \in \mathfrak{R}_{m+1} \Rightarrow$ projection of A on first m variables is in $\mathfrak{R}_m$
- $f \in \mathbb{R}[x_1, \ldots, x_m] \Rightarrow f^{-1}(0) \in \mathfrak{R}_m$.

definable in $\mathfrak{R}$ $\cong$ member of some $\mathfrak{R}_m$

(Definability is always with respect to some given structure.)

$\mathcal{T}$:= some collection of trajectories of definable (in $\mathfrak{R}$) vector fields

The Game: What can we say, *relative to* $\mathfrak{R}$, about $(\mathfrak{R}, (\Gamma)_{\Gamma\in\mathcal{T}})$?

If every $\Gamma \in \mathcal{T}$ is definable (in $\mathfrak{R}$), then trivially, we win. Not interesting,

but of course, we might not *know* which Γ are definable.

Fact. An expansion of $\overline{\mathbb{R}}$ defines all trajectories iff it defines $\mathbb{Z}$.

If $\mathfrak{R}$ defines $\mathbb{Z}$, then trivially, we win.

If $\mathfrak{R}$ avoids $\mathbb{Z}$, but $(\mathfrak{R}, (\Gamma)_{\Gamma \in \mathcal{T}})$ does not, then Pyrrhic victory at best.

We usually assume that $\mathfrak{R}$ avoids $\mathbb{Z}$, and try to rule out $\mathcal{T}$

such that $(\mathfrak{R}, (\Gamma)_{\Gamma \in \mathcal{T}})$ defines $\mathbb{Z}$.

What does nondefinability of $\mathbb{Z}$ imply about the vector fields of $\mathfrak{R}$?

What else should we assume about $\mathfrak{R}$ (and *why*)?

What else should we assume about the trajectories we consider?

What else should we assume about the vector fields we consider?

Let ω range over nonzero reals.

$$\mathbb{F}_\omega := (x - \omega y,\ \omega x + y) : \mathbb{R}^2 \to \mathbb{R}^2 \quad (\ \cong\ z \mapsto (1 + i\omega)z : \mathbb{C} \to \mathbb{C}\,)$$

$$\mathbb{M}_\omega := \{\, \exp(2\pi k/\omega) : k \in \mathbb{Z}\,\}$$

Main Classification. Exactly one of the following holds (for $\mathfrak{R}$).

- No $\mathbb{M}_\omega$ is definable.
- Some $\mathbb{M}_\omega$ is definable, but not any nontrivial trajectories of $\mathbb{F}_{q\omega}$, $0 \neq q \in \mathbb{Q}$.
- For some ω, all trajectories of $\mathbb{F}_{q\omega}$ are definable, $0 \neq q \in \mathbb{Q}$,

  but no unbounded trajectories of any $\mathbb{F}_\tau$ with $\tau \notin \mathbb{Q}\omega$.
- $\mathfrak{R}$ defines all trajectories.

(Essentially, a corollary of a result of Hieronymi.)

Examples

$\overline{\mathbb{R}}$ defines no $\mathbb{M}_\omega$.

Expansion of $\overline{\mathbb{R}}$ by $\mathbb{M}_\omega$ defines no nontrivial trajectories of any $\mathbb{F}_\tau$.

Expansion of $\overline{\mathbb{R}}$ by all trajectories of $\mathbb{F}_{q\omega}$, $0 \neq q \in \mathbb{Q}$, defines no unbounded trajectories of any $\mathbb{F}_\tau$ with $\tau \notin \mathbb{Q}\omega$.

Exercise. If Γ is a nontrivial trajectory of $\mathbb{F}_\omega$, then $(\overline{\mathbb{R}}, \mathbb{M}_\omega, \Gamma)$ defines all trajectories of $\mathbb{F}_{q\omega}$, $0 \neq q \in \mathbb{Q}$.

$\mathfrak{R}$ is **o-minimal** if every definable set has only finitely many connected components.

Evidently, if $\mathfrak{R}$ is o-minimal, then no $\mathbb{M}_\omega$ is definable.

Fact. If $\mathfrak{R}$ is o-minimal and defines no x^r with $r \notin \mathbb{Q}$, then $(\mathfrak{R}, \mathbb{M}_\omega)$ defines no unbounded trajectories of any $\mathbb{F}_\tau$ with $\tau \notin \mathbb{Q}\omega$.

(Much more is true, but now is not the time for details.)

Fact. If $\mathfrak{R}$ is o-minimal, then so is the expansion of $\mathfrak{R}$ by all compact trajectories of linear planar vector fields.

("linear" means "homogeneous linear" unless stated otherwise)

Open!

If $\mathfrak{R}$ is o-minimal and defines no x^r with $r \notin \mathbb{Q}$, is the same true of the expansion of $\mathfrak{R}$ by all compact trajectories of linear planar vector fields?

Main Contention. We should play the game only over $\mathfrak{R}$ that:

- is o-minimal,
- defines no irrational power functions,
- defines all compact trajectories of linear planar vector fields

(at the very least—we could require more).

Prototypical example. $\mathfrak{R} = \mathbb{R}_{\mathrm{an}} :=$ the globally subanalytic sets.

“Mathematical control theory is the area of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems” —E. Sontag

Contention. We should first consider only trajectories Γ of definable vector fields F that are also "locally a trajectory" of F:

$$(\forall x \in \Gamma)(\forall \epsilon > 0)(\exists \delta > 0),\ (\delta < \epsilon\ \&\ \Gamma \cap B(x, \delta) \text{ is a trajectory of } F)$$

Have to call this *something*, so maybe "regular".

(If needed, we can also consider finite unions of such.)

Anti-examples

$\{ (\cos t, \sin t, \cos \sqrt{2}t, \sin \sqrt{2}t) : t \in \mathbb{R} \}$ is a trajectory of $(\mathbb{F}_1, \mathbb{F}_{\sqrt{2}})$.

Dense and codense in $S^1 \times S^1$.

Nonperiodic trajectories of Rössler or Lorenz attractors.

Given a metric space (X, d), $E \subseteq X$ and $r > 0$, let $\mathrm{N}_r(E) \in \mathbb{N} \cup \{+\infty\}$

be the number of balls of radius r needed to cover E.

The **Assouad dimension** of E, $\quad \mathrm{Dim}\, E :=$

$$\inf_{s \in \mathbb{R}} (\exists C > 0)(\forall x \in E)(\forall\, 0 < r < R) \left[\mathrm{N}_r\big(E \cap B(x,R)\big) \leq C \left(\frac{R}{r}\right)^s \right]$$

Motto: "$\mathrm{Dim}\, E$ measures the size of E in all scales".

Example (!!!) $\operatorname{Dim} \mathbb{Z} = 1 = \operatorname{Dim}(\{0\} \cup \{ 1/k : k \in \mathbb{Z} \})$

Empirical observation: All dimensions (on $\mathbb{R}^n$, in the usual metric) commonly encountered in GMT, fractal geometry and analysis on metric spaces are bounded below by topological dimension and above by Dim.

Special case of a result of Hieronymi and M.

If E is a finite union of regular trajectories of *any* vector fields and

$(\mathbb{R}, +, \cdot, E)$ does not define $\mathbb{Z}$, then $\operatorname{Dim} E = 1$.

But this uses only that E:

- does not define $\mathbb{Z}$ over $\overline{\mathbb{R}}$
- has topological dimension 1
- is a boolean combination of closed sets

What, if anything, can be concluded about $\operatorname{Dim}\ 1$ trajectories of

definable vector fields if The Contention holds for $\mathfrak{R}$?