Recall $X(e,H) = \{ q \in X : [\mathbb{Q}(q):\mathbb{Q}] \le e, H(q) \le H \}$.

Diophantine proposition

Suppose that k, n, e, d are positive integers, with $k < n$ and $d \ge (e+1).n$. Then there exist a positive integer r, and $c, \varepsilon > 0$ with the following property.

Suppose $\phi : I^k \to I^n$ is C^r, with $|\phi^{(\alpha)}(x)| \le 1$ for $|\alpha| \le r$, and let $X = I m \phi$. Then for $H \ge 1$ the set $X(e,H)$ is contained in at most cH^{ε} alg. hypersurfaces of degree $\le d$.

Moreover $\varepsilon \to 0$ as $d \to \infty$.

For $d \ge 0$, $n \ge 1$, put $D_n(d) = \binom{n+d}{n}$.

Suppose $q \in \mathbb{Q}$ has $H(q) \le H$, $q \ne 0$ then $|q| \ge \frac{1}{H}$.

Lemma Suppose $q \in \mathbb{R}^n$, with $[\mathbb{Q}(q):\mathbb{Q}] \le e$, $H(q) \le H$, and $f \in \mathbb{Z}[x_1, \dots, x_n]$, has degree $\le d$, $f(q) \ne 0$. Then

$$|f(q)| \ge \frac{1}{(D_n(d)\,|f|\,H^{d.n})^e}$$

Lemma Let $M, N \in \mathbb{N}$, $M \le N$, and A is an $M \times N$ matrix, with rows $a_1, \dots, a_M$ s.t $|a_i|_2 \ge 1$. Put $\Delta = |a_1|_2 \cdots |a_M|_2$.

If $Q \ge 2\sqrt{N}\,\Delta^{1/N}$ then there exists $f \in \mathbb{Z}^N \setminus \{0\}$ with $|f| \le Q$ and $|Af| \le (2\sqrt{N})^{N/M} Q^{1-N/M} \Delta^{1/M}$.

Propn Suppose k, d, e, n, b positive integers, with $k < n$.

$D_n(d) \geq (e+1) D_k(b)$. Then there is a $c > 0$ with the following property.

Let $\phi : I^k \to I^n$ C^{b+1}, with $|\phi^{(\alpha)}(x)| \leq 1$

$\forall |\alpha| \leq b+1$ and $x \in I^k$, $X = \operatorname{Im} \phi$.

Then for any $H \geq 1$, there is $N \leq c H^{(k+1) n e \frac{d}{b}}$

and polynomials $f_1, \dots, f_N \in \mathbb{Z}[X_1, \dots, X_n] \setminus \{0\}$ degree $\leq d$

s.t. if $q \in X(e, H)$ then

$$f_i(q) = 0 \text{ for some } i.$$

Proof of diophantine propn. Suppose k, n, e, d, $k < n$, $d \geq (e+1)n$.

$D_k(b)$ is strictly increasing in b, so there is a unique b

s.t.
$$(e+1) D_k(b) \leq D_{k+1}(d) < (e+1) D_k(b+1).$$

Computation leads to

$$e + 1 > \frac{d+1}{k+1} \left(d/b \right)^k$$

and so
$$\underline{\frac{d}{b}} < \left(\frac{(e+1)(k+1)}{d+1} \right)^{1/k}$$
$$\to 0 \quad \text{as } d \to \infty$$

We then apply propn above, put $r = b+1$,

$$\varepsilon = (k+1) n e \frac{d}{b} \longrightarrow 0 \quad \text{as } d \to \infty.$$

$\square$

Sketch of proof of propn

Suppose $H \geq 1$, c, c' etc below are independent of H.

Put
$$r = \frac{c'}{H^{k+1/k \cdot n e d/b}}$$

where c' is small. I^k is contained in the union of $N \leq (1 + 1/r)^k \leq \underline{2^k c'^{-k}} H^{(k+1) n e d/b}$

closed boxes of side length r

Let V be such a box. We find $f = f_V$ s.t. $f(q) = 0$ for $q \in X(e, H)$, $q = \phi(z)$ some $z \in V \cap \mathbb{Z}^k$. We then vary V to get $f_1, \dots, f_N$.

Write $f(x_1, \dots, x_n) = \sum_{|i| \le d} f_i \, x_1^{i_1} \cdots x_n^{i_n}$ $\quad f_i \in \mathbb{Z}$ are to be determined. Fix $a \in V \cap \mathbb{Z}^k$.

For $\alpha \in \mathbb{N}^k$, $|\alpha| \le b$, let A_α be the vector

$$\frac{r^{b-|\alpha|}}{\left\| \left(\frac{D^\alpha \phi^i(a)}{\alpha!} \right)_{|i| \le d} \right\|_2} \cdot \frac{D^\alpha \phi^i(a)}{\alpha!}$$

where $\phi^i = \phi_1^{i_1} \cdots \phi_n^{i_n}$.

We get $M = D_k(b)$ rows A_α, $N = D_n(d)$ columns, we apply Lemma to this matrix.

A calculation shows that $\Delta = r^{-b/_{k+1} D_k(b)}$

and we put $Q = r^{-\frac{b+k+1}{(e+1)(k+1)}}$

Can check that $Q \geq 2\sqrt{N} \, \Delta^{1/N}$ provided c' is suff. small.

By the Lemma there are f_i not all zero with $|f| \le Q$ and

$$|Af| \le \frac{c \, \Delta^{1/D_k(b)}}{Q^e}$$

$$\| \quad \left| \frac{r^{b-|\alpha|}}{\left\| \left(\frac{D^\alpha \phi^i(a)}{\alpha!} \right)_{|i| \le d} \right\|_2} \cdot \sum_{|i| \le d} f_i \frac{D^\alpha \phi^i(a)}{\alpha!} \right|$$

Put $g(z) = f(\phi_1(z), \ldots, \phi_n(z))$.

Expand g as follows

$$g(z) = \sum_{|\alpha| \leq b} \left(\sum_{|i| \leq d} f_i \frac{D^\alpha \phi^i(a)}{\alpha!} \right) (z-a)^\alpha$$

$$+ \sum_{|\alpha| = b+1} \left(\sum_{|i| \leq d} f_i \frac{D^\alpha \phi^i(\xi_z)}{\alpha!} \right) (z-a)^\alpha$$

where ξ_z lies on the line segment between a and z.

We have:

$$\left| \sum_{|\alpha| \leq b} \left(\sum_{|i| \leq d} f_i \frac{D^\alpha \phi^i(a)}{\alpha!} \right) (z-a)^\alpha \right|$$

$$\leq c \frac{\Delta^{1/D_n(b)}}{Q^e} \cdot r^b \cdot \sum_{|\alpha| \leq b} \left(\sum_{|i| \leq d} \frac{D^\alpha \phi^i(a)}{\alpha!} \right)^{1/2}$$

$$\leq c\, r^\sigma \qquad \sigma = \frac{e(b+l+1)}{(e+1)(l+1)} + \frac{bk}{k+1}.$$

For the remainder, we get

$$|\cdot| \leq c\, |f|\, r^{b+1} \leq c\, Q\, r^{b+1}$$

$$\leq c\, r^\sigma$$

So we get $|g(z)| \leq c\, r^\sigma$

Suppose $q = \phi(z) \in X(e, H)$ $z \in V$.

If $f(q) \neq 0$ then by Liouville Lemma

$$|f(q)| \geq \frac{1}{(D_n(d)\, Q\, H^{d_n})^e} \qquad \text{(since } |f| \leq Q)$$

So that $$c r^\sigma \geq \frac{1}{(D_n(d)\, Q\, H^{d_n})^e}$$

Hence
$$\underbrace{r^{\sigma} Q^{e} H^{dne}}_{= c'^{\,bk/k+1}} \geq \underline{\frac{1}{c\, D_n(d)}}$$

Choosing c' sufficiently small, we get a contradiction, so that $f(q)=0$.

$\square$

<u>Pila - Wilkie</u>

Now $\tilde{\mathbb{R}} = (\bar{\mathbb{R}}, \ldots)$ o-minimal.

<u>Thm</u> Suppose $X \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is definable, let $e \geq 1$, $\varepsilon > 0$. Then there is $c > 0$ s.t. for all $a \in \mathbb{R}^m$ and all $H \geq 1$,
$$\# X_a^{trans}(e, H) \leq c H^{\varepsilon}$$

We give the proof following Bhardwaj & van den Dries.

<u>Lemma</u> Suppose $S \subseteq \mathbb{R}^n$ is semialgebraic, $f : S \to \mathbb{R}^m$ semialgebraic and injective. If $X \subseteq S$ is s.t. $f|_X : X \to Y = f(X)$ is a homeomorphism, then $f(X^{alg}) = Y^{alg}$ & $f(X^{trans}) = Y^{trans}$.

Pf/. Clear that $f(X^{alg}) \subseteq Y^{alg}$.

Suppose $C \subseteq Y$ a connected infinite semialgebraic set. Then $f^{-1}(C)$ is semialgebraic, and contained in X and connected and infinite. So $f^{-1}(C) \subseteq X^{alg}$, so $f^{-1}(Y^{alg}) \subseteq X^{alg}$ & $f(X^{alg}) = Y^{alg}$ $\square$

<u>Proof of Pila - Wilkie</u>

By induction on n. Using the maps $x \mapsto \pm x^{\pm 1}$ which preserve points of degree $\leq e$, height $\leq H$ we can assume $X_a \subseteq [0,1]^n$, and so using induction, that $X_a \subseteq I^n$ (for each a).