The Pila-Wilkie Thm

Setting: $\tilde{\mathbb{R}} = (\overline{\mathbb{R}}, \dots)$ an o-minimal expansion of the ordered field of reals.

Suppose $q = a/b$, in lowest terms with $b \neq 0$, then the height of q is $H(q) = \max\{|a|, |b|\}$.

If $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$, put $H(q) = \max\{H(q_1), \dots, H(q_n)\}$.

For $X \subseteq \mathbb{R}^n$, and $H \geq 1$, let

$$X(\mathbb{Q}, H) = \{ q \in X \cap \mathbb{Q}^n : H(q) \leq H \}.$$

Aim of Pila-Wilkie is to give good bounds on

$\# X(\mathbb{Q}, H)$ as $H \to \infty$.

<u>thm</u> (Pila) Suppose $f: [0,1] \to \mathbb{R}$ is analytic and transcendental. Let $X = \text{graph} f$. Then for all $\varepsilon > 0$ there is a $c > 0$ s.t. for $H \geq 1$,

$$\# X(\mathbb{Q}, H) \leq c H^{\varepsilon}$$

In general, a def set might contain a rational line in which case it will have lots of rational points. Following Pila, we define:

<u>algebraic</u> <u>part</u> of X : X^{alg} = union of all connected infinite $\overline{\mathbb{R}}$-def subsets of X

<u>transcendental</u> <u>part</u> of X : $X^{trans} = X \setminus X^{alg}$

X^{alg} could be quite complicated.

<u>thm</u> (Pila-Wilkie) Suppose $X \subseteq \mathbb{R}^n$ is definable. Then for all $\varepsilon > 0$ there is a $c > 0$ st. for all $H \geq 1$,

$$\# X^{trans}(\mathbb{Q}, H) \leq c H^{\varepsilon}.$$

Two main ingredients for the proof:

<u>thm</u> Suppose $X \subseteq (0,1)^n$ is def., $r \in \mathbb{N}$. Then there exist $\phi_1, \dots, \phi_k : (0,1)^{\dim X} \to X$ C^r definable maps s.t. $|\phi_i^{(\alpha)}(x)| \leq 1$ for all $x \in (0,1)^{\dim X}$, $|\alpha| \leq r$

and $\bigcup \operatorname{Im} \phi_i = X$.

(Earlier versions of this due Yomdin, Gromov)

Thm $k, n \in \mathbb{N}$, $k < n$. For each $d \in \mathbb{N}$ there exist $r \in \mathbb{N}$, $c, \varepsilon > 0$ with the following property. If $\phi : (0,1)^k \to (0,1)^n$ is a C^r map with $|\phi^{(\alpha)}(x)| \leq 1$ for $x \in (0,1)^k$, $|\alpha| \leq r$, and $X = \operatorname{Im} \phi$ then for $H \geq 1$, the set $X(\mathbb{Q}, H)$ is contained in at most cH^{ε} algebraic hypersurfaces of degree at most d, and $\varepsilon \to 0$ as $d \to \infty$.

(Has origins in work of Bombieri & Pila.)

Some references:

- Pila-Wilkie (2006)
- Bhardwaj - van den Dries (2021)
  Neer giving seminar on Tuesday at 10:10am.
- Binyamini & Novikov (2021)

Remarks.

(i) The cH^{ε} is best possible. Examples by Surroca, Pila showing cH^{ε} is best possible in $\mathbb{R}_{an}$.

(ii) Hope to do better, say for $\mathbb{R}_{exp} = (\overline{\mathbb{R}}, \exp)$.
Wilkie conjectures a bound of the form $c(\log H)^k$ for definable sets in $\mathbb{R}_{exp}$.
Binyamini - Novikov (2017) prove such bounds for $(\overline{\mathbb{R}}, \exp|_{[0,1]}, \sin|_{[0,1]})$.
For certain other classes of sets bounds of this form are known.

(iii) The constant c is ineffective.

(iv) There are various stronger results, e.g. bounds on

$$\# X^{trans}(d, H) = \# \left\{ \alpha \in X : \begin{array}{l} [\mathbb{Q}(\alpha):\mathbb{Q}] \leq d \\ H(\alpha) \leq H \end{array} \right\}$$

$$\leq cH^{\varepsilon} \quad \text{where } c = c(d).$$

Here we hope for (in nice structures)

$$c \, d^k (\log H)^c$$

Such bounds also have applications.

(v) Improvements to parametrization result.

Now work in $\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$ o-minimal expansion of a field. Let $I = (0,1)$ (in M).

<u>defn</u> A <u>basic</u> <u>cell</u> $C \subseteq I^n$ is a product of n copies of I or $\{0\}$. A continuous map $\phi : C \to I^n$ is <u>cellular</u> if

(i) $\phi = (\phi_1, \dots, \phi_n)$, and for each i the function ϕ_i depends only on $(x_1, \dots, x_i)$.

So $\phi(x) = (\phi_1(x_1), \phi_2(x_1, x_2), \dots, \phi_n(x_1, \dots, x_n))$.

(ii) for each i, and each $(x_1, \dots, x_{i-1}) \in$ proj. of C to first $i-1$ coordinates, the function

$$x_i \mapsto \phi_i(x_1, \dots, x_{i-1}, x_i)$$

is strictly increasing.

<u>Rmk</u> Suppose $\phi : C \to I^n$, $\psi : C' \to C$ are cellular. Then so is $\phi \circ \psi : C' \to I^n$.

wlog $\phi = (\phi_1, \dots, \phi_n)$, $\psi = (\psi_1, \dots, \psi_n)$. $x \in C'$.

$$\begin{aligned} (\phi \circ \psi)_1(x) &= \phi_1(\psi_1(x), \dots, \psi_n(x)) \\ &= \phi_1(\psi_1(x)) && \text{since } \phi \text{ is cellular} \\ &= \phi_1(\psi_1(x_1)) && \text{— } \psi \text{ —} \end{aligned}$$

$$\begin{aligned} (\phi \circ \psi)_2(x) &= \phi_2(\psi_1(x), \dots \psi_n(x)) \\ &= \phi_2(\psi_1(x), \psi_2(x)) \\ &= \phi_2(\psi_1(x_1), \psi_2(x_1, x_2)) \end{aligned}$$

Eg $(\phi \circ \psi)_2(x_1, x_2) = \phi_2(\psi_1(x_1), \psi_2(x_1, x_2))$ is strictly increasing in x_2.

<u>Notation</u> For $f : U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$ C^r

we put $$\|f\|_r = \frac{\max_{|\alpha| \leq r} \sup_{\substack{x \in U \\ i=1,\dots,m}} |D^{(\alpha)} f_i(x)|}{\alpha!}$$

<u>Defn</u> Let $r \in \mathbb{N}$, $X \subseteq I^n$ def.

A <u>cellular</u> <u>r-parameterization</u> is a finite set Φ of definable C^r cellular maps s.t. $\|\phi\|_r \leq 1$ for each $\phi \in \Phi$ and such that

$$\bigcup_{\phi \in \Phi} \operatorname{Im} \phi = X.$$

Suppose $Y \subseteq I^m$, $F : X \to Y$ def.

A <u>cellular r-parameterization</u> of F is a cellular r-parameterization Φ of X such that

$$\|F \circ \phi\|_r \leq 1$$

for each $\phi \in \Phi$.

<u>Thm</u> (Parameterization, following Binyamini & Novikov)

$n \in \mathbb{N}$.

$(\mathrm{I})_n$ If $r \in \mathbb{N}$, and $X \subseteq I^n$ is def. then X has a cellular r-parameterization.

$(\mathrm{II})_n$ If $r, m \in \mathbb{N}$, $X \subseteq I^n$, $Y \subseteq I^m$, $F : X \to Y$ def. then F has a cellular r-parameterization.

We can almost compose cellular r-param.

Say Φ is a c.r.p of $X \subseteq I^n$ and that for each $\phi \in \Phi$, $\phi : C \to X$ we have a c.r.p Ψ_ϕ of C.

Let $\psi \in \Psi_\phi$. Computation shows that

$$\|\phi \circ \psi\|_r \leq c_{n,r}$$

Cover $C' = \mathrm{dom}\, f$ with $(c+1)^k$ boxes, where $k = \dim C'$, $c = c_{n,r}$, each box is a translate of $(0, 1/c)^k$. Then composing with linear maps for each box, we get a bound of 1 in place of $c_{n,r}$.

We call this trick at the end 'linear substitution'.