

Cell decomposition thm (Knight - Pillay - Steinhorn, 1986)

$n \in \mathbb{N}$.

(I)._n Suppose $x_1, \dots, x_n \subseteq M^n$ def. Then there is a cell decomposition of M^n cptble with each x_i .

(II)._n If $f: X \rightarrow M$ is def. then there is cell-decomposition \mathcal{D} of M^n cptble with X s.t. $f|_C$ is cts for each $C \in \mathcal{D}$.

Propn 1 Suppose $\mathcal{D}_1, \mathcal{D}_2$ are cell decompositions of \mathbb{R}^{n+1} . Then there is a cell decomposition \mathcal{D} of \mathbb{R}^{n+1} s.t. \mathcal{D} is compatible with all the cells in $\mathcal{D}_1 \cup \mathcal{D}_2$ (say \mathcal{D} retains \mathcal{D}_1 and \mathcal{D}_2).

Proof of (I)_{n+1}

By propn 1, we can assume $k=1$.

By propn 2 and (I)_n we get a cell decomposition \mathcal{D} of M^n s.t. for each $C \in \mathcal{D}$ there is an ℓ and a $\tau \in S^{\pm 1} \cap \mathbb{Z}^{2\ell+1}$ s.t.

$$\bar{\tau}(x_n) = \tau \quad \text{for all } x \in C$$

Fix such a C, τ, ℓ . We get definable functions

$$f_1, \dots, f_\ell: C \rightarrow \mathbb{R} \quad \text{s.t.} \quad f_1 < \dots < f_\ell$$

and such that $(f_i, f_{i+1})_C \subseteq X \quad f_0 = -\infty$

$$\text{or} \quad (f_i, f_{i+1})_C \cap X = \emptyset \quad f_{\ell+1} = +\infty$$

(for $i=0, \dots, \ell$)

and $\text{graph } f_i \subseteq X$

or $\text{graph } f_i \cap X = \emptyset$

for $i=1, \dots, \ell$.

By (II)_n we can partition C into finitely many cells so that f is continuous on each cell.

Then apply (I_n) to get a cell decomposition of \mathbb{R}^n compatible with all the resulting cells.

□

Rmk Recall that the definition of σ -minimality requires sets definable with param. to be finite unions of points & intervals.

If we only assume this for sets deft. w.o. param. we don't get σ -minimality.

Counterexample due to Dolich, Miller, Steinhorn (200?)

$(\mathbb{R}, <, V)$ where $V = \{(x, y) \in \mathbb{R}^2 : x - y \in \mathbb{Q}\}$ is the Vitali relation. The only \emptyset -def sets of \mathbb{R} are \emptyset and \mathbb{R} . But over FOS , \mathbb{Q} is definable.

Connectedness

Note that an open interval in M might not be connected, e.g. $\mathbb{Q} = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$

defn A def set $X \subseteq M^n$ is definably connected if X is not the union of two disjoint nonempty definable open subsets of X .

E.g. cells are definably connected.

propn Suppose that $X \subseteq M^{n+m}$ is def. Then there is an $N \in \mathbb{N}$ such that if $x \in M^n$ then X_x has at most N definably connected compacts.

In particular, if $n = m$ then n is σ -minimal.

Determinable choice & coarse selection.

From now on, we assume that M is an o-minimal expansion of an ordered field $(M, <, +, 0, 1)$ (necessarily real closed).

Propn (i) If $X \subseteq M^{m+n}$ is definable and $\pi: M^{m+n} \rightarrow M^m$ is projection to first m coordinates,

then there is a def. map $f: \pi X \rightarrow M^n$ such that $\text{graph } f \subseteq X$.

(ii) If E is a def. eq. reln. on a def. set $X \subseteq M^n$, then E has a def. set representation.

Proof. First we show that if $X \subseteq M^n$ is def. and nonempty then we can definably pick an element $e(X)$ of X .

$n=1$; if X has a least elt., let $e(X)$ be that elt.

now let (a, b) be the leftmost interval of X . Then

$$e(X) = \begin{cases} 0 & \text{if } a = -\infty, b = +\infty \\ b-1 & \text{if } a = -\infty, b \in M \\ a+1 & \text{if } a \in M, b = +\infty \\ \frac{a+b}{2} & \text{if } a, b \in M \end{cases}$$

Finish inductively.

(i) $f(x) = e(x_n)$ for $x \in \pi X$

(ii) $\{e(A) : A \text{ is an eq. class of } E\}$ is a def. set representation.

Propn (Coarse selection) Suppose $X \subseteq M^n$ is definable and $a \in \text{fr } X = (cl X) \setminus X$. Then there is a continuous definable injective $\vartheta: (0, \varepsilon) \rightarrow X$, some $\varepsilon > 0$ s.t. $\lim_{t \rightarrow 0^+} \vartheta(t) = a$ □

Proof Let $|x| = \max \{ |x_1|, \dots, |x_n| \}$



Since $a \in \text{fr } X$, $\{ |a - x| : x \in X \}$ contains arbitrarily small positive elements of M and is definable, so it contains some interval $(0, \varepsilon)$.

If $t \in (0, \varepsilon)$, then

$$\{ x \in X : |a - x| = t \}$$

is nonempty. By def. choice we get a def.

$$\gamma: (0, \varepsilon) \rightarrow X$$

such that $|a - \gamma(t)| = t$ for $t \in (0, \varepsilon)$.

Clearly γ is injective and $\lim_{t \rightarrow 0^+} \gamma(t) = a$.

By monotonicity, we can reduce ε to assume that

γ is continuous. □

Dimension We continue to assume $M = (M, \leq, t_i, 0, 1, \dots)$ is an ω -minimal expansion of a field.

Defn Suppose $X \subseteq M^n$ is definable, $X \neq \emptyset$.

Put

$$\dim X = \max \{ i_1 + \dots + i_n : X \text{ contains an } (i_1, \dots, i_n)\text{-cell} \}.$$

and dimension of \emptyset is $-\infty$.

Lemma If $X \subseteq M^n$ has nonempty interior, and $f: X \rightarrow M^n$ is definable and injective then $f(X)$ contains an open cell.

Proof By induction on n .

If $n=1$, then since X is infinite, $f(X)$ is infinite, and definable, so $f(X)$ contains an open interval.

Suppose $n > 1$. By cell decomposition, we can assume that X is an open cell and that f is continuous.

A poly cell decomposition to $f(X)$, so

$$f(x) = c_1 \cup \dots \cup c_n$$

for cells $c_1, \dots, c_n \subseteq M^n$. Suppose no c_i is open. Since $X = f^{-1}(c_1) \cup \dots \cup f^{-1}(c_n)$

some $f^{-1}(c_i)$ contains an open cell $C \subseteq M^l$, say $C \subseteq f^{-1}(c_1)$. Then

$$f|_C : C \rightarrow c_1$$

is continuous and injective. Since we expand a field, open cells $A \subseteq M^m$ are det. homeomorphic to M^m . So C is det. homeo to M^l

$$\text{A } c_i \dashv \dashv M^l \text{ some } l < n.$$

So we get a det. continuous injective

$$g : M^l \rightarrow M^l$$

$$\begin{aligned} \text{Defn } h : M^l &\rightarrow M^l \\ y &\mapsto g(o, y) \end{aligned}$$

By induction hyp. $h(M^l)$ has interior.

Let $b \in \text{int. } h(M^l)$ and $h(a) = b$ $a \in M^l$.

Since g iscts, so if $x \in M^{l-1} \sim \{a\}$ is suff. small then $g(x, a) \in h(M^l)$.

Then there is $a' \in M^l$ s.t.

$$\begin{aligned} h(a') &= g(o, a') \\ &\vdash \\ g(o, a') & \end{aligned}$$

contradicting injectivity of g .

So $f(X)$ contains an open cell. □

Propn (i) If $X \subseteq Y \subseteq M^n$ are detachable

then $\dim X \leq \dim Y \leq n$

(ii) If $X \subseteq M^n, Y \subseteq M^n$ det. and
 $f : X \rightarrow Y$ is a det. bijection

then $\dim X = \dim Y$.

(iii) If $X, Y \subseteq M^n$ det. then
 $\dim X \cup Y = \max \{\dim X, \dim Y\}$.

Proof (i) is clear.

Proof of (ii) : next time.

(iii) Let $d = \dim X \cup Y$, and $C \subseteq X \cup Y$
an (c_1, \dots, c_n) -cell s.t. $c_1 + \dots + c_n = d$.

Let $\varphi: C \rightarrow M^d$ be a det. bijection.

So

$$M^d = \varphi(C \cap X) \cup \varphi(C \cap Y)$$

and hence one of $\varphi(C \cap X)$, $\varphi(C \cap Y)$

contains an open cell D . So $D \subseteq \varphi(C \cap X)$

Then $\varphi^{-1}(D) \subseteq X$ and

$$\dim \varphi^{-1}(D) = \dim D = d$$

$$\Rightarrow \dim X \geq d \geq \dim X.$$

$$\Rightarrow \dim X = d.$$

□