defn A **basic cell** $C \subseteq I^n$ is a product of n copies of I or $\{0\}$. A continuous map $\phi : C \longrightarrow I^n$ is **cellular** if

(i) $\phi = (\phi_1, \ldots, \phi_n)$, and for each i the function ϕ_i depends only on $(x_1, \ldots, x_i)$.
So $\phi(x) = (\phi_1(x_1), \phi_2(x_1, x_2), \ldots, \phi_n(x_1, \ldots, x_n))$.

(ii) for each i, and each $(x_1, \ldots, x_{i-1}) \in$ proj. of C to first $i-1$ coordinates, the function
$$x_i \mapsto \phi_i(x_1, \ldots, x_{i-1}, x_i)$$
is strictly increasing.

Defn Let $r \in \mathbb{N}$, $X \subseteq I^n$ def.
A **cellular** **r-parameterization** is a finite set Φ of definable C^r cellular maps s.t. $\|\phi\|_r \leq 1$ for each $\phi \in \Phi$ and such that
$$\bigcup_{\phi \in \Phi} \operatorname{Im} \phi = X.$$
Suppose $Y \subseteq I^m$, $F : X \to Y$ def.
A **cellular r-parameterization of F** is a cellular r-parameterization Φ of X such that
$$\| F \circ \phi \|_r \leq 1$$
for each $\phi \in \Phi$.

Thm (Parameterization, following Binyamini & Novikov)

$n \in \mathbb{N}$.

$(\mathrm{I})_n$ If $r \in \mathbb{N}$, and $X \subseteq I^n$ is def. Then X has a cellular r-parameterization.

$(\mathrm{II})_n$ If $r, m \in \mathbb{N}$, $X \subseteq I^n$, $Y \subseteq I^m$, $F: X \to Y$ def. Then F has a cellular r-parameterization.

From now on we assume $\mathcal{M}$ $\aleph_0$-saturated.

Lemma Let $n \in \mathbb{N}$, $r \in \mathbb{N}$ and suppose that every $F: X' \to I$ def. on a cell X' of dimension n, has a cellular r-parameterization. Then every definable $G: X \to I^m$ and $m \in \mathbb{N}$, where $\dim X = n$, has a cellular r-parameterization.

Proof Fix $G = (G_1, \dots, G_m)$. Let Φ_1 be a cellular r-param. of G_1. It is enough to find a cellular r-param of $G \circ \phi$ for $\phi \in \Phi_1$. So we can suppose $\|G_1\|_r \leq 1$.

Then get a cellular r-param Φ_2 of G_2. If $\phi \in \Phi_2$ $\|G_1 \circ \phi\|_r \leq c_{n,r}$. Keep going in this way and get $\|G_i\|_r \leq c_{n,r}$. Then use linear substitution to get a bound of 1. $\square$

We now start proof of $(\mathrm{I})_1$.

Lemma Suppose $r \geq 2$, and $f: I \to I$ is definable C^{r-1}, $\|f\|_{r-1} \leq 1$. Then f has cellular r-param.

Proof. By smooth monotonicity, we can split into intervals on which f is C^r, and on which $f^{(r)}$ is strictly monotonic, and non-vanishing.

W.l.o.g we assume $f^{(r)}$ strictly decreasing on I, $f^{(r)} > 0$.

By the mean value thm (in $\mathcal{M}$), for $x \in I$, there is a $\xi_x \in (0, x)$ s.t.

$$\frac{c_r}{x} \geq \frac{f^{(r-1)}(x) - f^{(r-1)}(0)}{x} = f^{(r)}(\xi_x) \geq f^{r}(x)$$

Let $g(x) = f(x^2)$. Then $\|g\|_{r-1}$ is bounded, and when we compute $g^{(r)}$ we get

$$g^{(r)}(x) = \text{sum of some bounded terms} + c_r\, f^{(r)}(x^2)\, x^r$$

By the above $f^{(r)}(x^2)x^r \leq c_r\, x^{r-2}$

which is bounded, since $r \geq 2$.
So $\|g\|_r \leq C_r$, and we finish using linear substitutions.

<u>Lemma</u> Suppose $f: I \to I$ is a def. function, $r \in \mathbb{N}$.
Then there exist def C^r $\phi_1, \dots, \phi_k : I \to I^2$ whose images cover graph f such that $\|\phi_i\|_r \leq 1$, and each coordinate of each ϕ_i is either strictly monotonic or constant.

<u>Proof.</u> By smooth monotonicity, we can assume f is C^1 on I and that f is either constant on I or strictly monotonic, and s.t.

$$f' < -1, \text{ or } -1 \leq f' < 0 \text{ or } 0 < f' \leq 1 \text{ or } f' > 1$$

on I. We can then assume $0 < f' \leq 1$ by using f^{-1}, if necessary, in place of f, and perhaps reversing orientation of I.

Then we apply the previous Lemma $r-1$ times to get a parameterization Φ of I s.t. $\|f\circ\phi\|_r \leq 1$

So we take $\{(\phi, f\circ\phi) : \phi \in \Phi\}$

as our parameterization of the graph. $\square$

We can now prove (II)$_1$. It suffices to do this for a def fn $f: I \to I$. We apply the previous

Lemma to get a parameterization Φ of the graph

If $\phi \in \Phi$ has strictly decreasing last coordinate, we compose with $1-x$, this ensures that

$$\{ \phi_1 \quad : \quad \phi = (\phi_1, \phi_2) \in \Phi \}$$

is cellular r-param. of I, and since

$$\| \phi_2 \|_r \leq 1$$
$$\|$$
$$\| f \circ \phi_1 \|_r$$

so we have a cellular r-param. of f. $\square$

Inductively, we assume $(\mathrm{I})_m, (\mathrm{II})_m$ hold for $m \leq n$.

We must prove $(\mathrm{I})_{n+1}$.

By cell decomposition, we can suppose $X \subseteq I^{n+1}$ is a cell.

If $X = \text{graph } f$, for $f : X' \to I$ cts def, $X' \subseteq I^n$, we apply $(\mathrm{II})_n$ to f, to a cellular r-param of f. Let $\phi \in \Phi$. So $\phi : C' \to I^n$ for some basic cell $C' \subseteq I^n$. Put $C = C' \times \{0\}$ and define

$$\psi : C \to I^{n+1}$$
$$(x, 0) \mapsto (\phi(x), f \circ \phi(x)).$$

Since $\| f \circ \phi \|_r \leq 1$ taking all such ψ gives a cellular r param. of X.

Otherwise $X = (f, g)_{X'}$. Apply $(\mathrm{II})_n$ to the map (f, g), we get a cellular r param. Φ of (f, g). Let $\phi \in \Phi$, so $\phi : C' \to I^n$ $C' \subseteq I^n$ a basic cell. Put $C = C' \times I$, and define

$$\psi : C \to I^{n+1}$$
$$(x', x_{n+1}) \mapsto \left(\phi(x'), \; g \circ \phi(x') x_{n+1} + (1 - x_{n+1}) . f \circ \phi(x') \right)$$

then since $\| g \circ \phi \|_r , \| f \circ \phi \|_r \leq 1$

we have $\| \psi \|_r \leq c$, and we use linear substitution to finish.

$\square (\mathrm{I})_{n+1}$

We now assume $(\mathrm{I})_m$, $m \leq n$, and $(\mathrm{II})_m$ for $m < n$ and show $(\mathrm{II})_n$ holds.

As an exercise, using $\aleph_0$-saturation & def choice, show that $(\mathrm{II})_{n-1}$ implies the following:

Suppose $F : I \times I^{n-1} \to I$ is def. Then there is a partition $I_1, \dots, I_k$ of I into def sets s.t. for each $j = 1, \dots, k$, there exists Φ_j a set of def maps $\phi : I_j \times C \to I^{n-1}$, s.t. if $a \in I_j$ then $\Phi_{j,a} = \{ \phi(a, \cdot) : \phi \in \Phi_j \}$ is a cellular r-param of $F(a, \cdot)$.

Now suppose $F : X \to I$, $X \subseteq I^n$ definable. We want a cellular r-param. of F. By applying $(\mathrm{I})_n$ to X and working with each chart separately, we can assume that X is a basic cell. If this has any $\{0\}$ coordinates, we can use $(\mathrm{II})_m$, $m < n$ to finish.

So we can assume $F : I^n \to I$.

We next show that we can assume that F is C^r, and that $\| F(a, \cdot) \|_r \leq 1$ for each $a \in I$.

For this, we apply the uniform version of $(\mathrm{II})_{n-1}$ to $F : I \times I^{n-1} \to I$. We get finitely many Φ_j as above. Partitioning and rescaling, we can assume each I_j is I.

Fix one $\phi : I \times C \to I^{n-1}$, $C = I^{n-1}$. Put

$$G(x_1, \dots, x_n) = F(x_1, \phi(x_1, \dots, x_n))$$

Then for each $a \in I$, we have $\| G(a, \cdot) \|_r \leq 1$

By smooth cell decomposition, there is a def $Z \subseteq I^n$ with $\dim Z < n$ s.t. $G|_{I^n \setminus Z}$ is C^r.

Using $(\mathrm{I})_m$, $m < n$ to find a cellular r-param Φ_Z of Z. Since $\dim Z < n$, if $\phi \in \Phi_Z$ has domain C, then C has a $\{0\}$ coordinate and we can apply $(\mathrm{II})_{n-1}$ to $G \circ \phi$, for each ϕ.

So we're left with $G|_{I^n \setminus Z}$. By $(I)_n$ we can find a cellular r param Φ of $I^n \setminus Z$.

Fix $\phi \in \Phi$, $\phi : C \to I^n \setminus Z$, $C \approx I^n$.

Then $G \circ \phi$ is C^r, and if $\alpha \in \mathbb{N}^{n-1}$, $|\alpha| \leq r$, $a \in I$, we have

$$|D^\alpha (G \circ \phi)(a, \cdot)|$$

$$= |D^\alpha (G(\phi_1(a), \cdot) \circ (\phi_2, \dots, \phi_n))|$$ since ϕ is cellular.

$$\leq c_{n,r}$$ since $\|G(b, \cdot)\|_r \leq 1$ for each $b \in I$

and $\|\phi\|_r \leq 1$

So by linear substitution we can assume $\|G \circ \phi(a, \cdot)\|_r \leq 1$ for each $a \in I$.

So replacing F with $G \circ \phi$, for ϕ, we may assume $F : I^n \to I$ is C^r and that

$\|F(a, \cdot)\|_r \leq 1$, for each $a \in I^{n-1}$.

<u>Lemma</u> Suppose $f : I^n \to I$ def, C^1 such that
$$\left|\frac{\partial f}{\partial x_j}(x)\right| \leq 1 \quad \text{for each } x \in I^n, \ j = 2, \dots, n.$$
Then the set of $a \in I$ s.t. $\frac{\partial f}{\partial x_1}(a, \cdot)$ is unbounded, is finite.

(Proof later).

Order $\mathbb{N}^n$ by degree and then lexicographically. Let $\alpha \in \mathbb{N}^n$ be least s.t. $|D^\alpha F| > 1$. (If there is no such α we're finished.) We parametrize F and show that we can increase α. Then finish by induction.

Note that $\alpha_1 \geq 1$. By the lemma above there are only finitely many $a \in I$ s.t. $D^\alpha F(a, \cdot)$ is unbounded. We can use $(II)_{n-1}$ to handle these, and so assume that for each $a \in I$, $D^\alpha F(a, \cdot)$ is bounded.

Let

$$S = \{ x \in I^n : |D^\alpha F(x)| \geq \frac{1}{2} \sup_{x' \in I^{n-1}} D^\alpha F(x_1, x') \}$$

By dep. choice there is a def. $\gamma : I \to S$ with $\gamma_1(t) = t$. Consider the map

$$t \mapsto (\gamma(t), D^{\alpha'} F(\gamma(t)))$$

$$\alpha' = (\alpha_1 - 1, \alpha_2, \dots, \alpha_n)$$

By (II), we get a cellular r-param Φ of this map. Let $\phi \in \Phi$, and consider

$$G(x_1, \dots, x_n) = F(\phi(x_1), x_2, \dots, x_n)$$

Computing, for $\beta < \alpha$, we get

$$|D^\beta G| < \text{const}_{n,r}.$$

When we compute $D^\alpha G$ we get $c_{n,r}$ terms bounded by $c_{n,r}$ plus a term

$$\phi'(x_1)^{\alpha_1} (D^\alpha F)(\phi(x_1), x_2, \dots, x_n).$$

By our definition of S, and γ, we have:

$$| \phi'(x_1)^{\alpha_1} \cdot (D^\alpha F)(\phi(x_1), x_2, \dots, x_n) |$$

$$\leq 2 |\phi'(x_1)|^{\alpha_1} \, |(D^\alpha F)(\gamma(\phi(x_1)))|$$

$$\leq 2 |\phi'(x_1)| \, |(D^\alpha F)(\gamma(\phi(x_1)))|$$

since $\alpha_1 \geq 1$, $|\phi'| \leq 1$.

To bound this, we compute:

$$\frac{\partial}{\partial x_1} \left((D^{\alpha'} F)(\gamma(\phi(x_1))) \right)$$

bounded, since Φ is a cell r-param of $(D^{\alpha'} F)(\gamma(t))$.

$$= \phi'(x_1) \cdot (D^\alpha F)(\gamma(\phi(x_1)))$$

hence bounded.

$$+ \phi'(x_1) \cdot \sum_{j=2}^{n} \frac{\partial \gamma_j}{\partial t}(\phi(x_1)) \cdot \underbrace{(D^{\alpha^{(j)}} F)(\gamma(\phi(t)))}_{\text{bounded since } \alpha^{(j)} < \alpha}$$

bounded

bounded as Φ is a cellular r-param of γ

$$\alpha^{(j)} = \alpha' + (0, \dots, 0, 1, 0, \dots, 0)$$

So $\quad |\delta'(x_1)(D^\alpha f)(\gamma(\delta(x_1)))| \leq c_{n,r}$

Using linear substitution, we finish the proof. $\square$

Have to prove Lemma on derivatives. For this, we need:

<u>Lemma</u> Suppose $f : M \times I \to I$ a def. family of C^1 functions. Then there is $c > 0$ s.t. for all $a \in M$, $B > 0$

$$\mu\left(\{t \in I : |f_a'(t)| > B\}\right) < \frac{c}{B}$$

sum of lengths of intervals.

<u>Proof.</u> The sets $\{t \in I : f_a'(t) > B\}$ and $\{t \in I : f_a'(t) < -B\}$ are finite unions of intervals, with the number of intervals bounded uniformly in a & B. Each of these intervals has length $< 1/B$ as $f_a : I \to I$ (ex: using mean value thm). $\square$

<u>Lemma</u> Suppose $f : I^n \to I$ def, C^1 such that

$$\left|\frac{\partial f}{\partial x_j}(x)\right| \leq 1 \quad \text{for each } x \in I^n, \ j = 2, \dots, n.$$

Then the set of $a \in I$ s.t. $\frac{\partial f}{\partial x_1}(a, \cdot)$ is unbounded, is finite.

Pf/. If not this set contains an interval, and we can assume $\frac{\partial f}{\partial x_1}(a, \cdot)$ is unbounded for all $a \in I$.

By choice, smooth monotonicity, we get a C^1 def. $\gamma : (0, \infty) \times I \to I^{n-1}$ s.t. for all $B > 0$ and $t \in I$,

$$\left|\frac{\partial f}{\partial x_1}(t, \gamma_B(t))\right| > B.$$

Applying the previous lemma to the coordinates of γ_B and to $f(t, \gamma_B(t))$

there is a constant c s.t.

outside a set of measure $\leq C/B$ we have

$$\left| \frac{d}{dt} F(t, \gamma_B(t)) \right| \leq B/3$$

&

$$\max_{j=1,\dots,n-1} |\gamma_{B,j}'(t)| \leq B/3n.$$

So outside this set we have:

$$\frac{B}{3} \geq \left| \frac{d}{dt} F(t, \gamma_B(t)) \right|$$

$$= \Big| \underbrace{\frac{\partial F}{\partial x_1}(t, \gamma_B(t))}_{> B} + \underbrace{\sum_{j=2}^{n} \frac{\partial F}{\partial x_j}(t, \gamma_B(t)) \cdot \gamma_{B,j-1}'(t)}_{\leq B/3} \Big|$$

This is a contradiction for $B > C$.

$\square$

The diophantine part (following Habegger 'Diophantine approx. on definable sets')

Suppose $q \in \overline{\mathbb{Q}}$, let $P \in \mathbb{Z}[X]$ be the unique irred. polynomial with $P(q) = 0$ having coprime coefficients and leading coefficient $a_0 \geq 1$.

Define

$$H(q) = \Big(a_0 \prod_{\substack{z \in \mathbb{C} \\ P(z)=0}} \max\{1, |z|\} \Big)^{1/\deg P}$$

E.g. $q = a/b$, $P = bX - a$

$$H(q) = b \cdot \max\{1, |a/b|\}$$
$$= \max\{|a|, b\}$$

This height has nice properties:

eg $\left\{ q \in \overline{\mathbb{Q}} : \begin{array}{l} H(q) \leq H \\ [\mathbb{Q}(q) : \mathbb{Q}] \leq e \end{array} \right\}$ is finite

$$H(q + q') \leq 2 H(q) H(q')$$
$$H(q \cdot q') \leq H(q) H(q')$$
$$H(1/q) = H(q)$$

References: Bombieri - Gubler 'Heights in diophantine geometry'

Masser 'Auxiliary polynomials in number theory'.

Given $X \subseteq \mathbb{R}^n$, $e \geq 1$, $H \geq 1$, we put

$$X(e, H) = \left\{ q \in X \cap \bar{\mathbb{Q}} : \begin{array}{l} H(q) \leq H \\ [\mathbb{Q}(q):\mathbb{Q}] \leq e \end{array} \right\}.$$

Propn (Habegger). Suppose $k, n, e \geq 1$, $k < n$, $d \geq (e+1)n$ are integers. Then there exist $c, \varepsilon > 0$, $r \in \mathbb{N}$, with the following property. Suppose $\phi : (0,1)^k \to (0,1)^n$ has $|D^\alpha \phi| \leq 1$ $\forall \alpha \in \mathbb{N}^k$, $|\alpha| \leq r$, $X = \operatorname{Im} \phi$.

Then for $H \geq 1$ the set $X(e, H)$ is contained in the union of at most cH^ε algebraic hypersurfaces of degree $\leq d$. Moreover $\varepsilon \to 0$ as $d \to \infty$.