Recall $\mathcal{M} = (M, <, +, \cdot, 0, 1, \ldots)$ an o-minimal expansion of a field.

<u>Propn</u> If $X \subseteq M^m$, $Y \subseteq M^n$ def. and $f : X \to Y$ is a definable bijection then

$$\dim X = \dim Y.$$

<u>Proof</u> Say $d = \dim X$, $e = \dim Y$. We show $d \leq e$.

Let $C \subseteq X$ be an $(i_1, \ldots, i_m)$-cell with $d = i_1 + \cdots + i_m$. Composing with a def. bijection $C \to M^d$, we get a definable injective

$$g : M^d \to Y$$

Cell decompose Y, as $Y = C_1 \cup \cdots \cup C_k$. Then

$$M^d = g^{-1}(C_1) \cup \cdots \cup g^{-1}(C_k)$$

So some $g^{-1}(C_i)$ contains an open cell $D \subseteq M^d$, say $D \subseteq g^{-1}(C_1)$, and C_1 is a $(j_1, \ldots, j_n)$-cell

For a contradiction, suppose $d > j_1 + \cdots + j_n = e'$.

We have

$$D \xrightarrow{g} C_1 \xrightarrow{\sim} M^{e'}$$

a def. injective, $h' : D \to M^{e'}$. Then we get

$$h'' : D \to M^{e'} \times \{0\} \subseteq M^d$$

definable and injective. Since D is open, this contradicts a lemma from last time. $\square$

<u>Propn</u> Suppose $X \subseteq M^m \times M^n$ is def., $d \leq n$. Then

$$X(d) = \{ x \in M^m : \dim X_x = d \}$$

is definable and

$$\dim \left(\bigcup_{x \in X(d)} \{x\} \times X_x \right) = \dim X(d) + d.$$

<u>Proof.</u> Let $\mathcal{D}$ be a cell decomposition of $M^m \times M^n$ compatible with X. If $C \in \mathcal{D}$ is an $(i_1, \ldots, i_{m+n})$-cell then πC is an $(i_1, \ldots, i_m)$-cell and C_x is an $(i_{m+1}, \ldots, i_{m+n})$-cell for each $x \in \pi C$. (Here $\pi : M^{m+n} \to M^m$ proj. to first m coordinates.)

So
$$\dim C = \dim \pi C + \dim C_x \quad (*).$$
for each $x \in \pi C$.

Fix $C' \in \pi \mathcal{D}$ and let $C^{(1)}, \dots, C^{(k)} \in \mathcal{D}$ be the cells contained in X whose projection is C'. Then for $x \in C'$
$$X_x = C^{(1)}_x \cup \dots \cup C^{(k)}_x$$
so that
$$\dim X_x = \max \dim C^{(i)}_x$$
$$= \max \{\dim C^{(i)} - \dim C'\} \quad \text{by } (*).$$

Let d be this max. Then $\dim C_x = d$ for all $x \in C'$, so $C' \subseteq X(d)$. So $X(d)$ is a finite union of cells and so is definable.

We have
$$d = \max \dim C^{(i)} - \dim C'$$
$$= \dim \bigcup C^{(i)} - \dim C'$$
$$= \dim \bigcup_{x \in C'} \{x\} \times C^{(i)}_x - \dim C'$$

So
$$\dim \bigcup_{x \in C'} \{x\} \times C^{(i)}_x = d + \dim C'$$

Then taking the union over all $C' \in \pi \mathcal{D}$ with $C' \subseteq X(d)$, we get the result. □

Cor (i) If $X \subseteq M^{m+n}$ is definable, then
$$\dim X = \max_{0 \leq d \leq n} (\dim X(d) + d)$$
$$\geq \dim \pi X.$$

(ii) Suppose $X \subseteq M^n$, $f : X \to M^m$ is def.

Let $X_f(d) = \{x \in M^m : \dim f^{-1}(x) = d\}$

Then $X_f(d)$ is def. and
$$\dim f^{-1}(X_f(d)) = \dim X_f(d) + d$$
and $\dim X \geq \dim f(X)$.

(iii) Let $X \subseteq M^m$, $Y \subseteq M^n$ def. Then

$$\dim X \times Y = \dim X + \dim Y.$$

Left as exercise.

Also an exercise:

Suppose $X, Y \subseteq M^{1+n}$, $Y \neq \emptyset$. Suppose that for each $x \in M$, either $X_x = \emptyset$ or

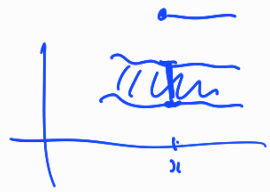
$$\dim X_x < \dim Y_x$$

Then $\dim X < \dim Y$.

Lemma Suppose $X \subseteq M^{1+n}$ is def. Then

$$\{ x \in M : cl(X_x) \neq (cl(X)_x) \}$$

is finite.



Proof. Note that $cl(X_x) \subseteq (cl(X))_x$.

Suppose that the set is infinite, as it is def. it then contains an open interval I. For each $x \in I$, there is an open box $B \subseteq M^n$ s.t.

$$B \cap X_x = \emptyset \quad \& \quad B \cap cl(X)_x \neq \emptyset.$$

The family of open boxes in M^n is def., so by def. choice we get B as a def. function of x. By monotonicity we can assume that B is continuous.

Let $U = \{ (x,y) \in I \times M^n : y \in B(x) \}$

Then U is open in $I \times M^n$, with $U \cap X = \emptyset$ and $U \cap cl(X) \neq \emptyset$, a contradiction. □

Thm Suppose $X \subseteq M^n$ a non empty def. set.
Then $\dim fr X < \dim X$. (Recall $fr X = cl X \setminus X$).

Proof. By induction on n. $n=1$ is ok.
So suppose $X \subseteq M^{n+1}$ and that the result holds for M^n.
For $i = 1, \dots, n+1$, let

$$cl_i(X) = \{ x \in M^{n+1} : x \in cl(X \cap \pi_i^{-1}(\pi_i(x))) \}$$

where $\pi_i : M^{n+1} \to M$ is projection onto the ith coordinate.

Note that $cl_i(X) \subseteq cl(X)$.

Suppose $x \in cl(X) \setminus cl_1(X)$. Then $x = (x_1, x')$ with $x' \in cl(X)_{x_1}$ and $x' \notin cl(X_x)$.

So there are only finitely many possible x_1's by the Lemma. That is, there exist $a_{1,1}, \dots, a_{1,k_1} \in M$ s.t.

$$cl(X) \setminus cl_1(X) \subseteq \bigcup_{j=1}^{k_1} \pi_1^{-1}(a_{1,j}).$$

Permuting coordinates, we get for $i = 2, \dots, n+1$ we get $a_{i,1}, \dots, a_{i,k_i}$ st.

$$cl(X) \setminus cl_i(X) \subseteq \bigcup_{j=1}^{k_i} \pi_i^{-1}(a_{i,j}).$$

So $cl(X) \setminus \bigcup_{i=1}^{n+1} cl_i(X)$ is the contained in the finite set $\{(a_{1,j_1}, \dots, a_{n+1,j_{n+1}}) : j_1 = 1, \dots, k_1, \ \vdots \ , j_{n+1} = 1, \dots, k_{n+1}\}$

So

$$\dim cl(X) \setminus X \leq \max\{\dim cl_i(X) \setminus X, 0\}.$$

We next show that

$$\dim cl_i(X) \setminus X < \dim X$$

for each i. Let $i = 1$, and $a \in M$. Then

$$\dim \left(cl(X \cap \pi_1^{-1}(a)) \setminus X \cap \pi_1^{-1}(a) \right) < \dim X \cap \pi_1^{-1}(a)$$

or $\dim X \cap \pi_1^{-1}(a) = \emptyset$, by the induction hypothesis.

Then note that

$$cl(X \cap \pi_1^{-1}(a)) \setminus X \cap \pi_1^{-1}(a) = (cl_1(X) \setminus X) \cap \pi_1^{-1}(a)$$

So for each $a \in M$ with $X \cap \pi_1^{-1}(a) \neq \emptyset$ we have

$$\dim\left(cl_1(X)\setminus X\right)\cap \pi_1^{-1}(a)$$
$$< \quad \dim X\cap\pi_1^{-1}(a).$$

so that $\dim\left(cl_1(X)\setminus X\right)_a < \dim X_a$.

So that by the exercise,
$$\dim cl_1(X)\setminus X < \dim X.$$

Permuting coordinates, we also get this for $i=2,\dots,n+1$.

So either $\dim cl(X)\setminus X < \dim X$

or $\dim X = 0$.

If $\dim X=0$ then X is finite, so closed, and we're done. $\square$

<u>Dimension via model theory</u>

Still have M an o-minimal expansion of a field.

Recall that if $A\subseteq M$ then the <u>model-theoretic alg. closure</u> of A is the union of all finite A-def. subsets of M, and the <u>definable closure</u> of A is the union of all A-def. singletons.

In our setting, because of the ordering, these coincide, that is
$$\operatorname{acl} A = \operatorname{dcl} A.$$

Easy properties: for $A\subseteq M$, then

- $A \subseteq \operatorname{dcl}(A)$
- $\operatorname{dcl}(\operatorname{dcl}(A)) = \operatorname{dcl}(A)$.
- $\operatorname{dcl} A = \bigcup \{\operatorname{dcl}(\{a_1,\dots a_n\}) : n\in\mathbb{N},\ a_1,\dots,a_n\in A\}$

<u>Thm</u> (Exchange, Pillay & Steinhorn 1986)

Suppose $A\subseteq M$, $a,b\in M$.

If $b\in\operatorname{dcl}(A\cup\{a\})$, $b\notin\operatorname{dcl}A$

then $a\in\operatorname{dcl}(A\cup\{b\})$

Proof. Adding constants for elements of A, we may suppose that $A = \emptyset$. So $b \in dcl(\{a\})$, $b \notin dcl(\emptyset)$.

Since $b \in dcl(\{a\})$, there is an $\emptyset$-def. function f with $a \in \mathrm{dom} f \subseteq M$ s.t. $f(a) = b$.

Since $\mathrm{dom} f$ is $\emptyset$-def. it is a finite union of $\emptyset$-def. points and $\emptyset$-def. open intervals. If a is one of the points, then a is $\emptyset$-def, and hence so is b. So we can assume that $\mathrm{dom} f$ is a $\emptyset$-def. open interval I, with $a \in I$. By monotonicity, we can assume that f is strictly monotonic on I or constant on I. If f is constant on I then $b = f(a) = f(\text{midpoint of } I)$ so that b is $\emptyset$-def. So f is strictly monotonic, then $f^{-1}(b) = a$, so $a \in dcl(\{b\})$. $\square$

This, together with the easy properties, shows that dcl is a pregeometry.

Suppose $A \subseteq M$, $a \in M^n$. We define

$\dim a/_A$ = minimum cardinality of a subtuple a' of a s.t. $a \in dcl(A \cup a')^n$

Call $X \subseteq M$ <u>independent</u> over A if for every $x \in X$, $x \notin dcl(A \cup X \setminus \{x\})$

Then $\dim a/_A$ = cardinality of a maximal independent over A subtuple of a.

Assume $\mathcal{M}$ is sufficiently saturated and that all sets of parameters A, B are small relative to saturation of $\mathcal{M}$.

<u>defn</u> Suppose $X \subseteq M^n$ definable over $A \subseteq M$. Then

$$\dim X = \max\{\dim a/_A : a \in X\}$$

This doesn't depend on A (as long as we choose a small set). Say $A \subseteq B$, $B \subseteq M$ small.

Clearly $\dim_B X \leq \dim_A X$. Say $\dim_A X = h$

Let $a \in X$ witness this (this is called a generic point of X). Suppose $a_1, \dots, a_k$ are independent over A

Using saturation, inductively find $b_1, \dots, b_k$ independent over B s.t. $tp(a_1, \dots, a_k)/A = tp(b_1, \dots, b_k)/B$

Then $(b_1, \dots, b_k)$ extends to $b \in X$, so

$\dim_B X \geqslant h$.

<u>Lemma</u> Suppose that $a \in M^n$, $A \subseteq M$ (small).

Then

$a_1, \dots, a_n$ are independent over A if and only if every A-def. set $X \subseteq M^n$ with $a \in X$ has nonempty interior.

<u>Proof</u>. Suppose $a_1, \dots, a_n$ are dependent over A.

Say $a_n \in dcl(A \cup \{a_1, \dots, a_{n-1}\})$. Then there is a formula ϕ with parameters from A, s.t. a_n is the unique x with $\phi(a_1, \dots, a_{n-1}, x)$.

Let $X = \{x \in M^n : \phi(x_1, \dots, x_n)$ and s.t. x_n is unique with $\phi(x_1, \dots, x_n)\}$

Then X has empty interior and $a \in X$.

The other direction is by induction on n.

$n=1$: If a_1 independent over A, then a_1 is not in any finite A-def set. So if $a_1 \in X$ with X A-def, then X is infinite, so contains an open interval.

Suppose true for M^n. Let $a \in M^{n+1}$, with $a_1, \dots, a_{n+1}$ ind $/A$. Suppose $X \subseteq M^{n+1}$ is def $/A$ with $a \in X$.

By cell decomposition, we can suppose that $X = C$ is a cell defined over A.

If $C = \text{graph} f$, where $f : C' \to M$ is

A-def., then $a_{n+1} = f(a_1, \dots, a_n)$

so $a_{n+1} \in dcl(A \cup \{a_1, \dots, a_n\})$, contradicting

$a_1, \dots, a_{n+1}$ ind $/A$.

So $C = (f, g)_{C'}$ where $f, g : C' \to M$

cts, A-def, $f < g$ (or $f = -\infty$, or $g = +\infty$).

So $(a_1, \dots, a_n) \in C'$. By the inductive hyp.

C' is an open cell, so C is too, so

X has nonempty interior.

$\square$

<u>Propn</u> $X \subseteq M^n$ A-def., $k \leq n$. Then

$\dim X \geq k$

if and only if there is coordinate projection $\pi : M^n \to M^k$ s.t.

$\pi(X)$ has non empty interior.

<u>Corollary</u> $\dim X$ agrees with what we had before.

(Ex).

<u>Proof of propn</u> Suppose $\dim X \geq k$. Let $a \in X$,

assume $a_1, \dots, a_k$ ind $/A$. Let $\pi : M^n \to M^k$ be

projection onto first k coordinates. Then $(a_1, \dots, a_k) \in \pi(X)$

so by Lemma, $\pi(X)$ has nonempty interior.

For the other direction, let $\pi : M^n \to M^k$ be a

projection s.t. $\pi(X)$ has nonempty interior, suppose onto

first k coordinates.

Then $\pi(X)$ is A-def., and has interior, so

contains an open A-def. box, $U = I_1 \times \dots \times I_k$.

Using saturation, inductively find $(a_1, \dots, a_k) \in U$

independent over A. Then take $a \in X$ s.t.

$$\pi(a) = (a_1, \dots, a_k)$$

Then $\dim X \geq k$.

$\square$

Smoothness $\mathcal{M} = (M, <, +, \cdot, 0, 1, \ldots)$ an o-min exp. of a field. Using ordered field structure, we can define differentiability in the usual way.

Thm Suppose $f: (a,b) \to M$ is def., $r \in \mathbb{N}$. Then f is C^r at all but finitely many points.

C^r-cells are defined exactly as cells were, but requiring all fns involved to be C^r.

Thm (C^r cell decomposition). $r \in \mathbb{N}$.

(1) Suppose $X_1, \ldots X_k \subseteq M^n$ are def. Then there is a cell decomposition $\mathcal{D}$ cptble with $X_1, \ldots X_k$ st. each cell $C \in \mathcal{D}$ is C^r.

(2) If $f: X \to M$ is def. $X \subseteq M^n$ then there is cell dec. $\mathcal{D}$ into C^r cells s.t.

$$f|_C \text{ is } C^r \quad \text{for each } C \in \mathcal{D}$$

with $C \subseteq X$.

Le Gal & Rolin gave an o-minimal $\widetilde{\mathbb{R}} = (\bar{\mathbb{R}}, \ldots)$ which does not have C^∞ cell decomposition.