O-minimality & The Pila–Wilkie Thm.

Plan 1) o-minimality, monotonicity, cell decomposition, dimension th.

2) Pila–Wilkie

3) Either applications of Pila–Wilkie, or examples of o-minimal structures.

Setting: $(M, <)$ a dense linear order without endpoints.

defn An <u>open interval</u> in $(M, <)$ is a set of one the following forms:

(i) $(a,b) = \{x \in M : a < x < b\}$ $\quad a, b \in M$, $a < b$

(ii) $(-\infty, a) = \{x \in M : x < a\}$ $\quad a \in M$

(iii) $(a, +\infty) = \{x \in M : a < x\}$ $\quad a \in M$

Similarly define closed intervals, etc. Intervals of form (i) are <u>bounded</u>.

Note that we require endpoints in M. So, e.g., in $(\mathbb{Q}, <)$ the set $\{x \in \mathbb{Q} ; x < \sqrt{2}\}$ is not an interval.

We give M the order topology, and M^n the product topology.

defn Suppose $\mathcal{M} = (M, <, \ldots)$ is an expansion of $(M, <)$. We say $\mathcal{M}$ is <u>o-minimal</u> if every $\mathcal{M}$-definable (with parameters) subset of M is a finite union of points and open intervals.

Rmks

- If we only asked for convex sets in place of open intervals, we get 'weak o-minimality'.

- Note that finite unions of points and intervals are definable. So o-minimality says we only define the subsets of M that are already definable in $(M,<)$.

The name o-minimality comes from order-minimality. definition is due to Pillay & Steinhorn, following work by van den Dries.

Reference: - van den Dries 'Tame topology & o-minimal structures'.

- 'Definable sets in ordered structures I, II'

(Knight). Pillay & Steinhorn

<u>Examples</u> - $\overline{\mathbb{R}} = (\mathbb{R}, <, +, \cdot, 0, 1)$ is o-minimal (Tarski)

- $\mathbb{R}_{exp} = (\overline{\mathbb{R}}, \exp)$ is o-minimal (Wilkie)

Fix $\mathcal{M}$ o-minimal. Definable means $\mathcal{M}$-definable, with parameters from M.

<u>The Monotonicity Theorem</u> (Pillay & Steinhorn, 1986)

Suppose $f: I \to M$ is a definable function on an open interval $I \subseteq M$. Then there are $a_1 < \dots < a_k$ in I such that on each interval (a_i, a_{i+1}), $i = 0, \dots, k$, $I = (a_0, a_{k+1})$, f is either constant, or continuous and strictly monotonic.

Moreover, if f is definable over A, some $A \subseteq M$ then $a_1, \dots, a_k$ are also definable over A.

Observation: If $X \subseteq M$ is definable and infinite then X contains an open interval.

Sketch of proof of Monotonicity Thm. (with parameters).

We need three lemmas, in which $J \subseteq I$ is an open interval.

<u>Lemma 1</u> There is an open interval $J' \subseteq J$ on which f is either constant or injective.

Lemma 2 If f is injective on J, then there is an open interval $J' \subseteq J$ on which f is strictly monotonic.

Lemma 3 If f is strictly monotonic on J then there is an open interval $J' \subseteq J$ on which f is continuous.

proof of Lemma 1 Suppose that there is some $y \in M$ s.t. $f^{-1}(y) \cap J$ is infinite. Then there is an open interval $J' \subseteq J$ on which f is constant.

So we can assume that if $y \in M$ Then $f^{-1}(y) \cap J$ is finite. So $f(J)$ is infinite, so contains some interval (a,b), $a<b$. Define

$$g : (a,b) \to J$$
$$y \mapsto \min \{ x \in J : f(x) = y \}$$

g is injective so there is an open interval $J' \subseteq g((a,b))$. Then f is injective on J'. $\square$ Lemma 1.

Proof of monotonicity. Let

$X = \{ x \in I$: There are $a, b \in I$, $a<b$, $x \in (a,b)$ s.t. f is constant on (a,b) or f is cts & str. inc. on (a,b) or f is cts & str. dec on (a,b) $\}$.

Then X is definable, hence $I \setminus X$ is definable. Suppose $I \setminus X$ is infinite. Successively applying the Lemmas, gives a contradiction.

Hence $I \setminus X$ is finite. Hence we may assume $I = X$.

Partitioning again we can assume:

(i) for all $x \in I$, there exist $a<b$ $x \in (a,b)$ s.t. f is constant on (a,b)

or (ii) — " — ______ f is cts & str. inc on (a,b)

or (iii) — " — ______ f is cts & str. dec on (a,b).

We do (c). Fix $x_0 \in I$, $f(x_0) = c$.

$$d = \sup \{ x \in I : \forall y \in (x_0, x) \; f(y) = c \}$$

If $d \in I$, by (c) f is constant around d, hence has value c around d, contradicting $d = \sup$.

So $f(x) = c$ on $\{ x \in I : x \geq x_0 \}$

Similarly on $\{ x \in I : x < x_0 \}$.

So f is constant on I. $\square$

Exercises:

1) If $f : I \to M$ is mct. and $a \in I$ then

$\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$ exist in $M \cup \{ \pm \infty \}$

Similarly at end points.

2) If $f : [a,b] \to M$ is cts & mct, $a < b$ in M

then f takes a min and a max in $[a,b]$.

Thm (finiteness) Suppose $A \subseteq M^2$ and that for each $x \in M$, the fibre A_x is finite.

($A_x = \{ y : (x,y) \in A \}$). Then there is a natural number N s.t. $\# A_x \leq N$ for all $x \in M$.

Proof.

Call a point $(a,b) \in M^2$ normal if there is an open box $I \times J$ with $(a,b) \in I \times J$ s.t.

- $(I \times J) \cap A = \emptyset$

- $(a,b) \in A$ and $(I \times J) \cap A = \text{graph } f$

for some continuous $f : I \to M$.

Call $(a, +\infty)$ normal if there is an open interval I with $a \in I$ and a $b \in M$ s.t.

$$(I \times (b, +\infty)) \cap A = \emptyset.$$

Similarly define $(a, -\infty)$ being normal.

Note that $\{ (a,b) \in M^2 : (a,b) \text{ is normal} \}$ is definable. (Similarly for $\pm\infty$).

Define $f_1, f_2, \dots$ by:

$$\operatorname{dom} f_n = \{ x \in M : \# A_x \geq n \}$$

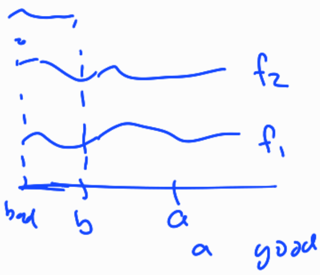
$$f_n(x) = n^{\text{th}} \text{ element of } A_x$$

So each f_i is definable.

Fix $a \in M$. Let $n \geq 0$ maximal s.t. $f_1, \dots, f_n$ are defined and continuous on an open interval around a.

Call a good if $a \notin cl(\operatorname{dom} f_{n+1})$

bad if $a \in cl(\operatorname{dom} f_{n+1})$



Let $\mathcal{G} = \{ a : a \text{ is good} \}$

$\mathcal{B} = \{ a : a \text{ is bad} \}$.

We want $\mathcal{G}$ definable.

Show that if $a \in \mathcal{B}$ then there is a least $b \in M \cup \{\pm\infty\}$ s.t. (a,b) is not normal.

For $a \in \mathcal{B}$, put:

$$\lambda(a,-) = \begin{cases} \lim_{x \to a^-} f_{n+1}(x) & \text{if } f_{n+1} \text{ is defined on } (t,a) \text{ some } t \\ +\infty & \text{o.w.} \end{cases}$$

$$\lambda(a,0) = \begin{cases} f_{n+1}(a) & \text{if } a \in \operatorname{dom} f_{n+1} \\ +\infty & \text{o.w.} \end{cases}$$

$$\lambda(a,+) = \begin{cases} \lim_{x \to a^+} f_{n+1}(x) & \text{if } f_{n+1} \text{ is defined on } (a,t) \text{ some } t \\ +\infty & \text{o.w.} \end{cases}$$

Put $\beta(a) = \min \{ \lambda(a,-), \lambda(a,0), \lambda(a,+) \}$.

Then check that $\beta(a)$ is the least $b \in M \cup \{\pm\infty\}$ s.t. (a,b) is not normal.

If $a \in \mathcal{G}$ then (a,b) is normal for all $b \in M \cup \{\pm\infty\}$

So $\mathcal{B} = \{ a \in M : \exists b \in M \cup \{\pm\infty\} \ (a,b) \text{ not normal} \}$

is definable.

If $a \in \mathcal{G}$ then $\# A_x$ is constant on an open interval around a.

Then show if $\mathcal{B}$ is finite, then the conclusion.

Suppose $\mathcal{B}$ is infinite. Note that $\beta : \mathcal{B} \to M$ is definable. Define

$$\mathcal{B}_+ = \{ a \in \mathcal{B} : \exists y (y > \beta(a) , (a,y) \in A \}$$

$$\mathcal{B}_- = \{ a \in \mathcal{B} : \exists y (y < \beta(a) , (a,y) \in A \}$$

Both definable. Since $\mathcal{B}$ is infinite, so is at least one

$$\mathcal{B}_+ \cap \mathcal{B}_- , \ \mathcal{B}_+ \setminus \mathcal{B}_- , \ \mathcal{B}_- \setminus \mathcal{B}_+ , \ \mathcal{B} \setminus (\mathcal{B}_+ \cup \mathcal{B}_-)$$

is infinite. In each case, get a contradiction using monotonicity theorem.

$\square$

Cell decomposition

defn For a sequence $(i_1, \dots, i_n)$ with $i_j \in \{0,1\}$, we define $(i_1, \dots, i_n)$-cells in M^n as follows:

(i) A 0-cell in M is a point $\{a\}$ $a \in M$

A 1-cell in M is an open interval in M.

(ii) Suppose $(i_1, \dots, i_n)$-cells are defined.

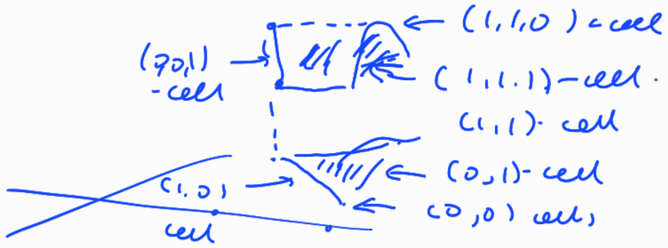
A $(i_1, \dots, i_n, 0)$-cell is a def. set of the form graph f where $f : C \to M$ is a cts def. fn. on an $(i_1, \dots, i_n)$-cell $C \subseteq M^n$

An $(i_1, \dots, i_n, 1)$-cell is a def set of the form

$(f,g)_C = \{(x,y) \in C \times M : f(x) < y < g(x)\}$

where $f, g : C \to M$ are cts, def,

C an $(i_1, \dots, i_n)$-cell or $f \equiv -\infty$ or $g \equiv +\infty$.



If C is an $(i_1, \dots, i_n)$-cell, then there is a projection $\pi : M^n \to M^k$ $k = i_1 + \dots + i_n$ s.t. $\pi|_C$ is a homeomorphism to its image.

<u>defn</u> A <u>cell-decomposition</u> of M is a finite set

$$\{(-\infty, a_1), \dots, (a_k, \infty), \{a_1\}, \dots, \{a_k\}\}$$

where $k \geq 0$, $a_1 < \dots < a_k$ in M

A <u>cell-decomposition</u> of M^{n+1} is a finite partition $\mathcal{D}$ of M^{n+1} into cells such that

$$\{\pi(C) : C \in \mathcal{D}\}$$

is a decomposition of M^n.

A cell-decomposition $\mathcal{D}$ of M^n is compatible with $X \subseteq M^n$ if for each cell $C \in \mathcal{D}$, $C \cap X = \emptyset$ or $C \subseteq X$.

<u>Cell decomposition thm</u> (Knight – Pillay – Steinhorn, 1986)

$n \in \mathbb{N}$.

$(\mathrm{I})_n$ Suppose $X_1, \dots, X_k \subseteq M^n$ def. Then there is a cell decomposition of M^n cptble with each X_i.

$(\mathrm{II})_n$ If $f : X \to M$ is def. then there is cell-decomposition $\mathcal{D}$ of M^n cptble with X s.t. $f|_C$ is cts for each $C \in \mathcal{D}$.

Moreover, if $X_1, \dots, X_n$ or f are definable over A then we can take cells in $\mathcal{D}$ to be definable over A.

We prove C.O.T assuming $M = \mathbb{R}$, and assuming <u>uniform finiteness</u>. Suppose $X \subseteq \mathbb{R}^{n+1}$ is def. with X_x finite for each $x \in \mathbb{R}^n$. Then there is $N \in \mathbb{N}$ s.t. $\# X_x \leq N$ for all $x \in \mathbb{R}^n$.

The proof we give is due to van den Dries.

<u>Proof of cell decomposition</u>

By induction on n. $(\mathrm{I})_1$ follows from definition.

$(\mathrm{II})_1$ follows from monotonicity.

We show (a) $(\mathrm{I})_1, \ldots, (\mathrm{I})_n, (\mathrm{II})_1, \ldots, (\mathrm{II})_{n-1}$

$$\Rightarrow (\mathrm{II})_n$$

& (b) $(\mathrm{I})_1, \ldots, (\mathrm{I})_n, (\mathrm{II})_1, \ldots, (\mathrm{II})_n$

$$\Rightarrow (\mathrm{I})_{n+1}.$$

First we do (a). Suppose $f : X \to M$ is def. By $(\mathrm{I})_n$ we may assume X is a cell.

If X is not open, we can use projection and the inductive hyp. to conclude.

So suppose X is an open cell. Let

$$X' = \{ x \in X : f \text{ is cts at } x \}$$

So X' is definable.

Suppose we know X' is dense in X. Applying $(\mathrm{I})_n$ we get a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n$ cptble with $X \setminus X'$ and with X'. If $C \in \mathcal{D}$ is an open cell & $C \subseteq X$, then by density $C \cap X' \neq \emptyset$, so $C \subseteq X'$, and so $f|_C$ is cts.

For nonopen cells we can proceed as above.

So we show that X' is dense in X.

Suppose $B \subseteq X$ is an open box. We show that there is a point in B at which f is cts.

May assume that if $B' \subseteq B$ is an open box then f takes infinitely many values on B' (using (I)$_n$).

We construct a sequence B_n of open boxes in B, and I_n of open intervals, with length $I_n < 1/n$

$cl(B_{n+1}) \subseteq B_n$, $f(B_n) \subseteq I_n$.

Then by compactness $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$, and at a point in the intersection, f is cts.

To find I_1, write $f(B) = \bigcup_{p \in \mathbb{N}} J_p \cup F$

where J_p are open intervals of length < 1, and F is a finite set. So

$$B = \bigcup_{p \in \mathbb{N}} f^{-1}(J_p) \cap B \cup \bigcup_{r \in F} f^{-1}(r) \cap B$$

By (I)$_n$, each $f^{-1}(J_p) \cap B$ and each $f^{-1}(r) \cap B$ is a finite union of cells.

So B is a countable union of cells, each contained in one of these sets. By Baire Category one of these cells must be open, and this cell must be contained in $f^{-1}(J_p) \cap B$ for some p.

Put $I_1 = J_p$, let $B_1 \subseteq f^{-1}(J_p) \cap B$ be an open box with $cl(B_1) \subseteq B$. So $f(B_1) \subseteq I_1$.

Then once $I_1, \ldots, I_n$, $B_1, \ldots, B_n$ constructed, repeat to get I_{n+1}, B_{n+1}.

$\square$

Now assume (I)$_1, \ldots,$ (I)$_n$, (II)$_1, \ldots$ (II)$_n$.

We want to prove (I)$_{n+1}$.

<u>Propn 1</u> Suppose $\mathcal{D}_1, \mathcal{D}_2$ are cell decompositions of $\mathbb{R}^{n+1}$. Then there is a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n+1}$ s.t. $\mathcal{D}$ is compatible with all the cells in $\mathcal{D}_1 \cup \mathcal{D}_2$.

(say $\mathcal{D}$ refines $\mathcal{D}_1$ and $\mathcal{D}_2$).

(Exercise)

If $A \subseteq \mathbb{R}$ is def. we define its type $\tau(A)$ as follows:

Let $a_1 < \cdots < a_l$ be the points in the boundary of A. Put $a_0 = -\infty$, $a_{l+1} = +\infty$. Put

$$\tau_{2i+1} = \begin{cases} 1 & \text{if } (a_i, a_{i+1}) \subseteq A \\ -1 & \text{if } (a_i, a_{i+1}) \subseteq \mathbb{R} \setminus A \end{cases}$$

$$\tau_{2i} = \begin{cases} 1 & \text{if } a_i \in A \\ -1 & \text{if } a_i \notin A \end{cases}$$

$i = 1, \ldots, l$.

Then $\tau(A) = (\tau_1, \ldots, \tau_{2l+1}) \in \{\pm 1\}^{2l+1}$

E.g. $\tau([1,2] \cup \{3\}) = (-1, -1, +1, +1, -1, +1, -1)$

$= \tau([9, 10] \cup \{73\})$

Using uniform finiteness, we get

Propn 2 Suppose $X \subseteq \mathbb{R}^{n+1}$. Then

$$\{\tau(X_x) : x \in \mathbb{R}^n\}$$

is finite, and for each $l \in \mathbb{N}$, $\tau \in \{\pm 1\}^{2l+1}$ the set $\{x \in \mathbb{R}^n : \tau(X_x) = \tau\}$ is definable.