Singular Foliations

We consider an n-dimensional analytic manifold M (real or complex)

An **analytic distribution** $\mathcal{D}$ on M is a coherent subsheaf of the sheaf of sections of TM.

At each point p, the stalk $\mathcal{D}_p$ is generated by a finite set of germs of vector fields $\{X_1, \ldots, X_k\}$.

A **(singular) foliation** is an analytic distribution $\mathcal{F}$ which is *involutive*

Namely,

$$\forall X, Y \in \mathcal{F}_x: \qquad [X, Y] \in \mathcal{F}_x$$

For $p \in M$, let $T_p\mathcal{F} \subset T_pM$ denote the subspace $\{X_1(p), \ldots, X_k(p)\}$ (where $\{X_i\}$ generates the stalk).

Note that $p \rightarrow \dim T_p\mathcal{F}$ is an upper semi-continuous function.

The *dimension* of $\mathcal{F}$ is generic dimension of $T_p\mathcal{F}$

A leaf of $\mathcal{F}$ is a maximal connected immersed submanifold $L \subset M$ such that

$$\forall p \in L: \qquad T_pL = T_p\mathcal{F}$$

Integrability Theorem (Sussman): There exists a leaf of $\mathcal{F}$ through each point $p \in M$.

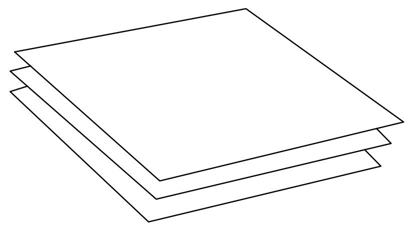
Classical Frobenius Theorem: Let $p \in M$ be such that $\mathcal{F}$ locally defines a **subbundle of the tangent bundle** TM (i.e. $T\mathcal{F}$ is locally of constant dimension d).

Then, there exists local coordinates $(x_1, \ldots, x_n)$ such that

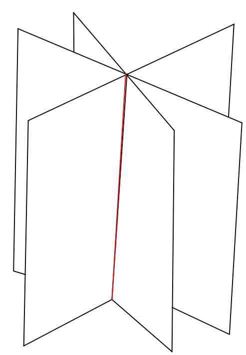
The leafs of $\mathcal{F}$ are locally given by

$$x_{d+1} = \cdots = x_n = \text{const}$$

where $d = \dim T_p\mathcal{F}$.



Singular example (with degeneracy of the rank): $\mathcal{D}$ is generated by $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$



In this course, we will be mostly interested in **foliations by curves**

In this context, we can assume the subsheaf $\mathcal{D}$ to be locally generated by a single vector field.

A singular **foliation by curves** $\mathcal{F}$ on M is defined by a collection $\{(U_i, \partial_i)\}_{i\in I}$ where

1) $(U_i)_{i\in I}$ is an open covering of M

2) ∂_i is an analytic vector field in U_i

Such that, for each $i, j \in I$, we have

$$\partial_i = \varphi_{ij}\,\partial_j$$

for some non-zero analytic function $\varphi_{ij} \in \mathcal{O}^\star(U_i \cap U_j)$.

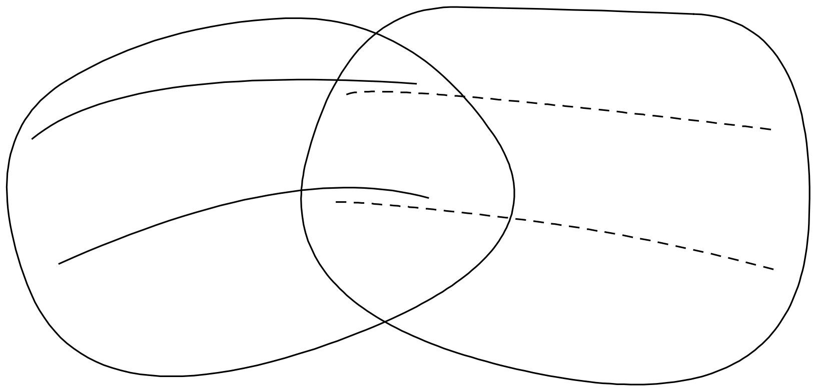
Each ∂_i will be called a *local generator* of $\mathcal{F}$.

More generally, each vector field ∂ with domain an open set $U \subset M$ is a local generator if

$$\partial|_{U_i \cap U} = \varphi_i\,\partial_i$$

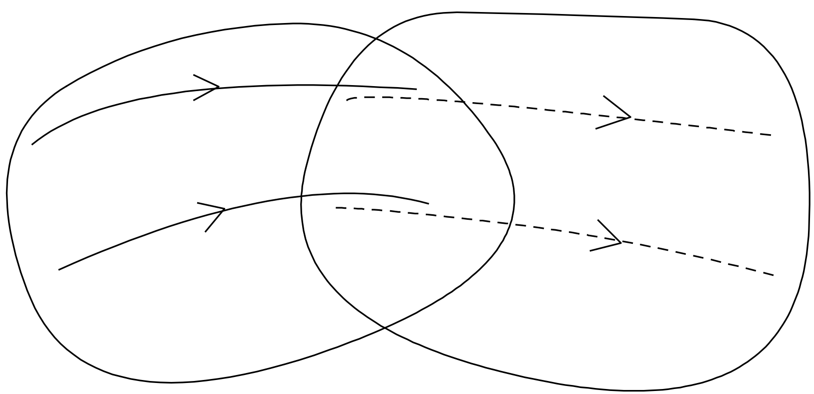
for some $\varphi_i \in \mathcal{O}^\star(U_i \cap U)$.

Remark: In general, we cannot expect to have a single global generator for a foliation.



We authorize reparametrizations of time
for the solution curves

In the real analytic setting, we usually demand that $\varphi_{ij} > 0$.



In local coordinates $x = (x_1, \ldots, x_n)$, each local generator can be written

$$\partial = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$$

with $a_1, \ldots, a_n$ analytic functions.

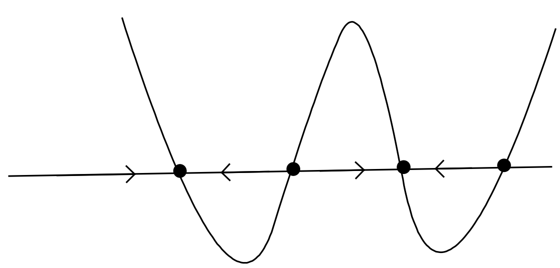
The **singular set** of $\mathcal{F}$ is the locally defined by the vanishing locus of the ideal generated by $(a_1, \ldots, a_n)$

$$\mathrm{Sing}(\mathcal{F}) = Z(a_1, \ldots, a_n)$$

Some simple examples...

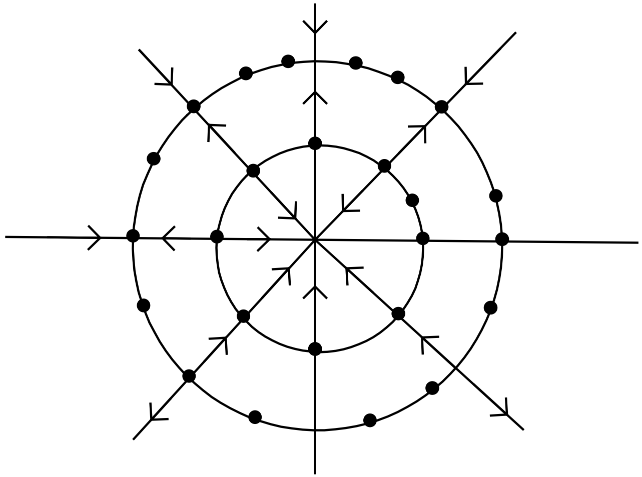
Example 1:

$$\partial = f(x) \frac{\partial}{\partial x}$$



Example 2:

$$\partial = f(x^2 + y^2)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$



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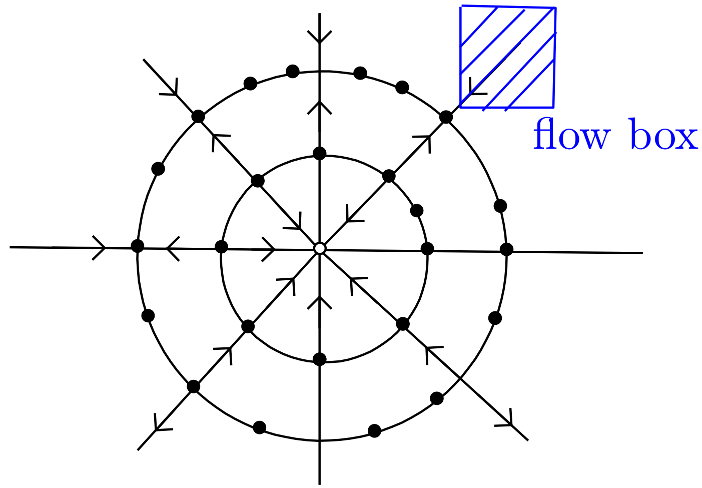
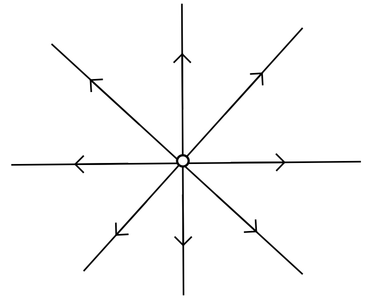


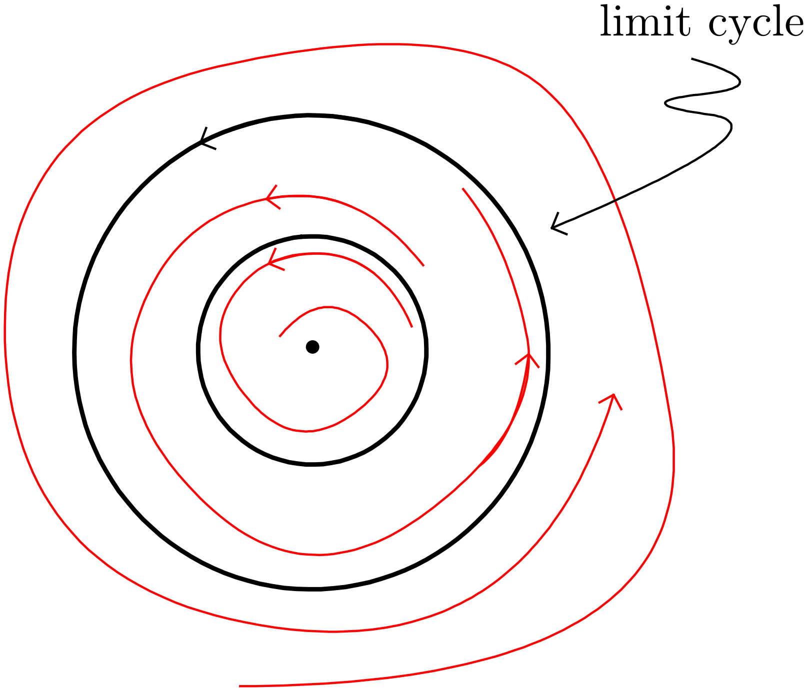
In these examples, $\mathrm{Sing}(\mathcal{F})$ is a codimension one analytic subset.

We could potentially consider the so-called **saturated** foliation $\mathcal{F}^{\mathrm{sat}}$, defined by $\frac{1}{f}\partial$



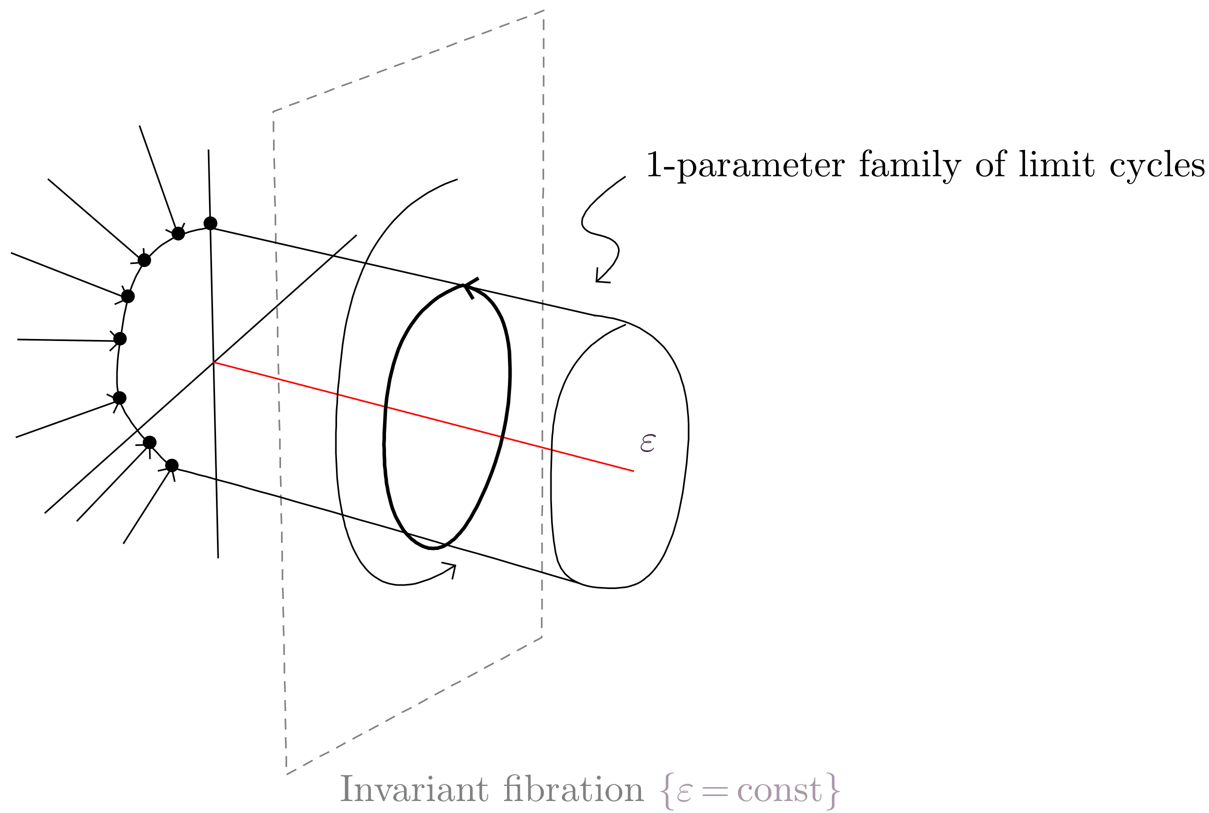
Example 3:

$$\partial = f(x^2+y^2)\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)$$



Example 4: ("singular perturbation problems") $\mathbb{R}^3$ with coordinates (x, y, ε)

$$\partial = f(x^2+y^2)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \varepsilon\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$$



Basic goals (in decreasing degrees of ambition)

1) Classify foliations analytically

2) Classify foliations C^k or topologically

3) Determine the asymptotic behaviour of a typical leaf.

4) Obtain statistical information: e.g. invariant/ergodic transverse measures.

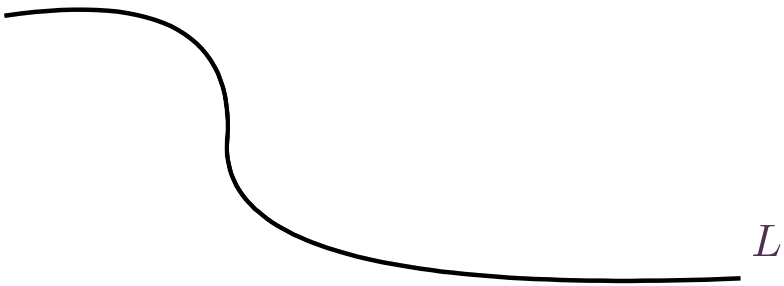
Local description: The foliation is locally trivial on $M \setminus \mathrm{Sing}(\mathcal{F})$.

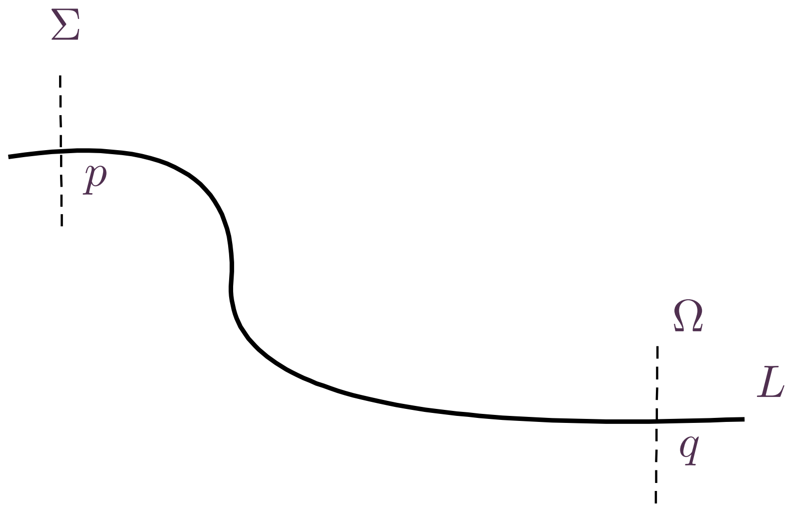
We would like to understand the foliation in the **vicinity** of its singular points.

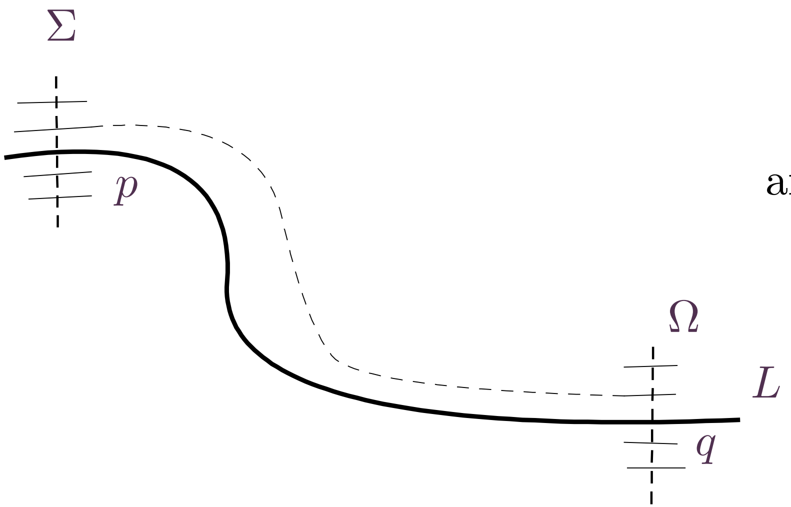
Thom: *The singularities are the* ***organizing centers*** *of the dynamics* .

As a first step, we would like to describe the transverse behaviour of the foliation by looking at its so-called

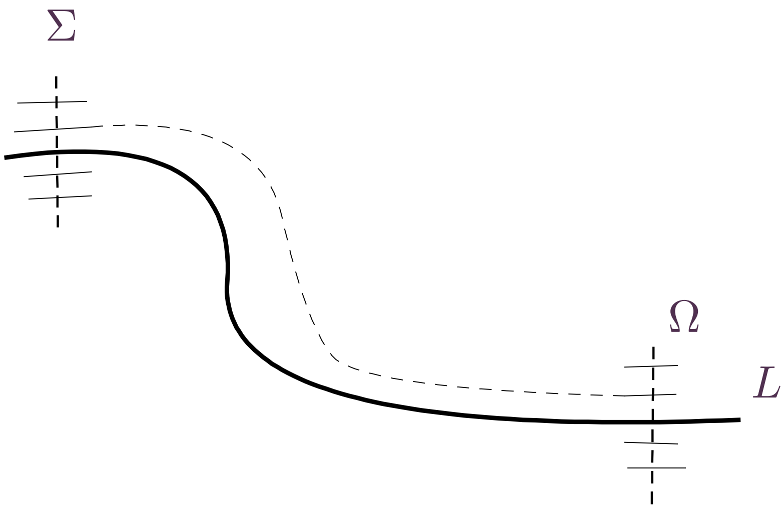
Holonomy Groupoid


L


Σ
p
Ω
L
q



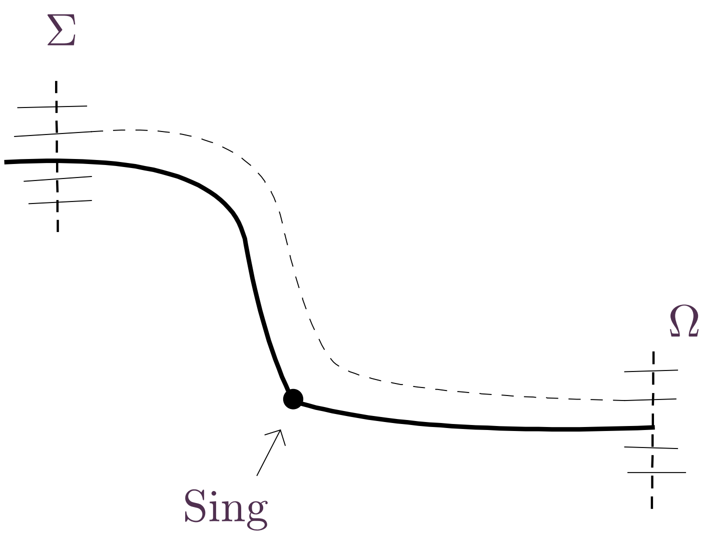
any path $p \to q$ on L can be lifted to nearby leafs



$$\mathrm{hol}\colon (\Sigma, p) \to (\Omega, q)$$

$$\mathrm{hol} \in \mathrm{Diff}^{\omega}(\Sigma \to \Omega)$$

Adding a singularity on the path…



$\mathrm{hol} \notin \mathrm{Diff}^{\omega}(\Sigma, \Omega)$

In general, there is an intrinsic **multivaluedness** for such map.

This is a very well-studied problem for foliations in surfaces.

It is in the heart of the Hilbert's XVIth's problem.

(see the course of Patrick…)

Elementary germs - and some words about classical normal forms... (over $\mathbb{C}$)

A germ of vector field ∂ at $p \in M$ defines a derivation of the local ring $(\mathcal{O}, \mathfrak{m}) = (\mathcal{O}_p, \mathfrak{m}_p)$.

Namely, in local coordinates $x = (x_1, \ldots, x_n)$ we can write

$$\partial = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$$

with $a_1, \ldots, a_n \in \mathcal{O}$ and ∂ defines a linear $\mathbb{C}$-endomorphism of $\mathcal{O}$ by

$$f \longmapsto \partial f = a_1 \frac{\partial f}{\partial x_1} + \cdots + a_n \frac{\partial f}{\partial x_n}$$

which moreover satisfies the Leibniz rule $\partial(fg) = (\partial f)\, g + f(\partial g)$. We note $\partial \in \mathrm{Der}(\mathcal{O})$.

The germ is **singular** if $a_1, \ldots, a_n$ vanish at p (i.e. $a_1, \ldots, a_n \in \mathfrak{m}$)

This is equivalent to require that

$$\partial(\mathfrak{m}) \subset \mathfrak{m}, \qquad \text{where } \mathfrak{m} = (x_1, \ldots, x_n)\mathcal{O}$$

(i.e. that $\partial \in \mathrm{End}_{\mathbb{C}}(\mathcal{O})$ stabilizes the maximal ideal)

Non-singular case: Assume that $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$.

Flow-box Theorem Then, there exists local analytic coordinates $(f, g_1, \ldots, g_{n-1})$ such that

$$\partial f = 1 \quad \text{and} \quad \partial g_1 = \cdots = \partial g_{n-1} = 0$$

i.e. $\partial = \frac{\partial}{\partial f}$.

Proof. Choose a local coordinate $f \in \mathfrak{m}$ such that $\partial f = u$ (unit).

Let us assume that $u = 1$ to simplify.

We complete f to a local system of coordinates $(f, g_1, \ldots, g_{n-1})$,

and consider the linear operator $\mathcal{O} \to \mathcal{O}$ given by

$$\Phi = I - f\partial + \cdots + (-1)^n \frac{f^n}{n!} \partial^n + \cdots$$

Notice that, for all $h \in \mathcal{O}$,

$$\partial(\Phi h) = \partial \sum_{n \geqslant 0} (-1)^n \frac{f^n}{n!} \partial^n h = 0$$

Therefore $f, \Phi(g_1), \ldots, \Phi(g_{n-1})$ is the required new coordinate system.

Singular case: Assume that now that $\partial(\mathfrak{m}) \subset \mathfrak{m}$.

Then, (by Leibniz' rule) $\partial(\mathfrak{m}^{k+1}) \subset \mathfrak{m}^{k+1}$ for each $k \in \mathbb{N}$, and ∂ induces an sequence of endomorphism $\{\partial_k\}_k$ on the jet spaces

$$J^k = \mathcal{O}/\mathfrak{m}^{k+1}$$

which is compatible with projections $\pi_{kl}\colon J^k \to J^l$ $(k > l)$.

By considering the inverse limit (under Krull completion), of the classical Jordan decompositions of the finite dimensional endomorphisms ∂_k, we obtain a unique **Jordan decomposition**

$$\partial = \partial_s + \partial_n, \qquad [\partial_s, \partial_n] = 0$$

where

- ∂_s is semi-simple
- ∂_n is asymptotically nilpotent (i.e. nilpotent restricted to each jet space).

Moreover, ∂_s and ∂_n are derivations of $\hat{\mathcal{O}} = \varprojlim J^k$ (see Jean Martinet - Exposé Bourbaki'81).

By the semi-simplicity of ∂_s, we have direct sum decompositions

$$\forall k \in \mathbb{N}: \quad J^k = \bigoplus_{\alpha \in \mathbb{C}} \mathrm{Gr}_\alpha(J^k, \partial_s)$$

where $\mathrm{Gr}_\alpha(J^k, \partial) = \{ f \in J^k \mid \partial f = \alpha f \}$.

with the compatibility condition

$$\forall k > l: \qquad \pi_{kl}(\mathrm{Gr}_\alpha(J^k, \partial_s)) = \mathrm{Gr}_\alpha(J^l, \partial_s)$$

derived from the commutative diagram

$$\begin{array}{ccc} J^k & \xrightarrow{\partial_k} & J^k \\ \pi_{k,k-1} \downarrow & & \downarrow \pi_{k,k-1} \\ J^{k-1} & \xrightarrow{\partial_{k-1}} & J^{k-1} \end{array}$$

Definition. A germ of vector field ∂ is ***elementary*** if:

- either $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ (i.e. in appropriate local coordinates $\partial = \frac{\partial}{\partial x}$)
- Or $\partial(\mathfrak{m}) \subset \mathfrak{m}$ and

$$\partial_s \neq 0$$

Poincaré-Dulac normalisation: (over $\mathbb{C}$) Suppose that $\partial(\mathfrak{m}) \subset \mathfrak{m}$. Then, there exists formal coordinates $(x_1, \ldots, x_n)$ which diagonalize the semi-simple part of ∂, namely such that

$$\partial_s = \sum_i \lambda_i x_i \frac{\partial}{\partial x_i}$$

In these coordinates, each eigenspace of the direct sum decomposition

$$\hat{\mathcal{O}} = \bigoplus_{\alpha \in \mathbb{C}} \mathrm{Gr}_\alpha(\hat{\mathcal{O}}, \partial_s)$$

is generated (over $\mathbb{C}$) by the monomials $x^k = x_1^{k_1} \ldots x_n^{k_n}$ such that $\langle k, \lambda \rangle = \alpha$.

What can we say about ∂_n?

The set of diagonal vector fields

$$L(\mu) = \sum_{i=1}^{n} \mu_i x_i \frac{\partial}{\partial x_i}, \qquad \mu \in \mathbb{C}^n$$

forms an abelian Lie $\mathbb{C}$-subalgebra, i.e. $[L(\mu), L(\lambda)] = 0$.

We say that it is a **maximal toral subalgebra** of $\mathrm{Der}(\mathcal{O})$.

Writing $\partial = \partial_s + \partial_n$, and assuming $\partial_s = L(\lambda)$ (as in the Theorem), the commutativity relation

$$[\partial_s, \partial_n] = 0$$

implies that ∂_n can be expanded as

$$\partial_n = \sum_k x^k L(\mu_k)$$

where k ranges over the subset $\mathbb{Z}^n \setminus \{0\}$ such that $\langle \lambda, k \rangle = 0$. These are the **resonant monomials**.

Example. (1:1) saddle. Consider a vector field having an initial expansion (in arbitrary coordianates)

$$\partial = (x + \dots)\frac{\partial}{\partial x} - (y + \cdots)\frac{\partial}{\partial y}$$

Then, $\operatorname{Spec}(\partial|_{J^1}) = \{1, -1\}$ and the resonant monomials are $(x y)^k$, $k \in \mathbb{Z}$.

The Poincaré-Dulac Theorem says that, up to a formal change of coordinates, we can write

$$\partial = \underbrace{\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)}_{\partial_s} + \underbrace{\sum_{k \geqslant 1} (x y)^k \left(a_k\, x\frac{\partial}{\partial x} + b_k\, y\frac{\partial}{\partial y}\right)}_{\partial_n}$$

where $u = x y$ is the generator of the subring $\ker(\partial_s)$. By further reductions, we can write

$$(1+F)\left(\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) + \frac{u^n}{1+\rho u^n}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right) \quad \text{or} \quad (1+F)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

for some $F \in \mathbb{C}[[u]]$ of order $\geqslant 1$, $n \geqslant 1$ and $\rho \in \mathbb{C}$.

Application: Integrability of Poincaré-Dulac normal forms

$\partial = (1+F)\Big(\Big(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\Big) + \frac{u^n}{1+\rho u^n}\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)\Big)$

Up to *reparametrization of time*, we can assume that $F = 0$.

Consider the new variables $u = xy, \qquad v = x/y$ and get

$$\partial(u) = 2\frac{u^{n+1}}{1+\rho u^n}, \qquad \partial(v) = 2v$$

which is a fully integrable system.

The corresponding differential system is given by

$$\left(\frac{1}{u^{n+1}} + \rho\frac{1}{u}\right)du = \frac{dv}{v}$$

and, by direct integration,

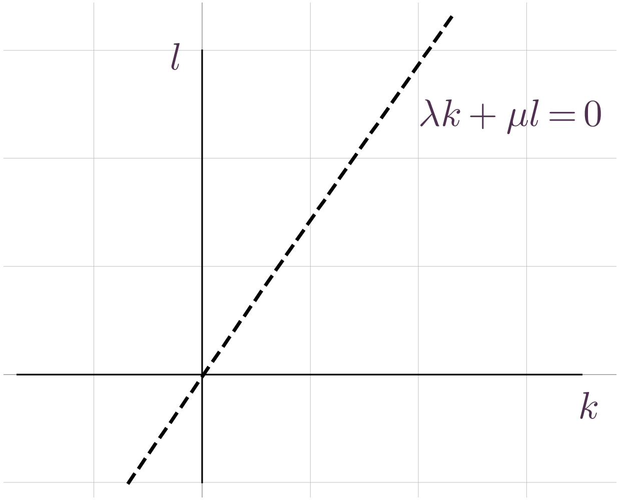
$$I = \frac{1}{nu^n} + \rho \ln u - \ln v$$

is a first integral of the vector field (namely, $\partial I = 0$). It is an element of $\mathbb{R}_{\mathrm{an,exp}}$.

Example: $(\lambda\colon \mu)$-saddle.

$$\partial = (\lambda x + \dots)\,\frac{\partial}{\partial x} - (\mu y + \cdots)\frac{\partial}{\partial y}$$

Then, $\operatorname{Spec}(\partial|_{J^1}) = \{\lambda, -\mu\}$

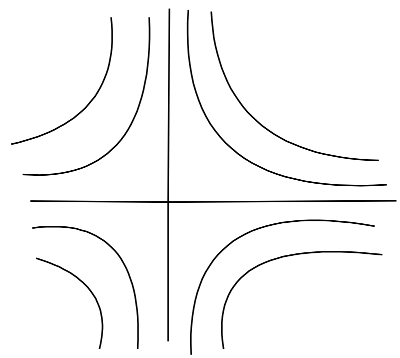


If $\lambda/\mu \notin \mathbb{Q}$ then the Poincaré-Dulac normal form is

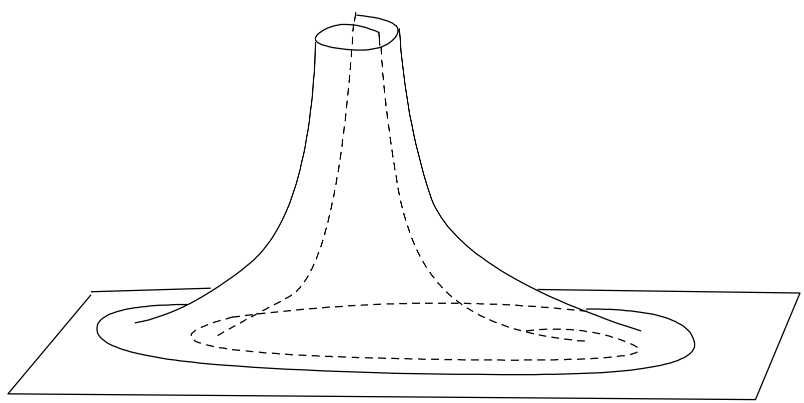
$$\partial = \lambda x\frac{\partial}{\partial x} - \mu y\frac{\partial}{\partial y}$$

and the first integral is simply $I = x^{\mu}y^{\lambda}$.

Two saddles $(\lambda\colon\mu)$ and $(\lambda'\colon\mu')$ have exactly the same topological phase portrait over $\mathbb{R}^2$



but they are completely different over $\mathbb{C}^2$ for $\lambda/\mu \neq \lambda'/\mu'$.



$\{y=0\}$

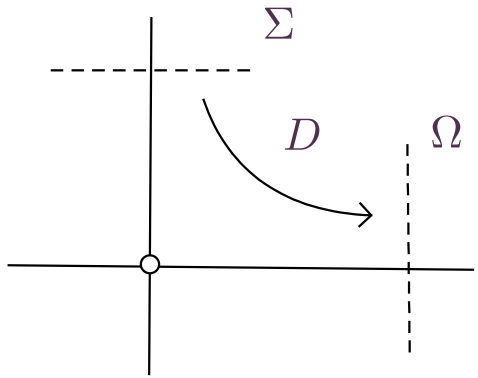
Over $\mathbb{C}^2$: There are several **rigidity phenomena**

E.g. Some analytic invariants are topologically determined (for instance, linearizability).

Transverse behaviour of the foliation in the vicinity of a saddle point.

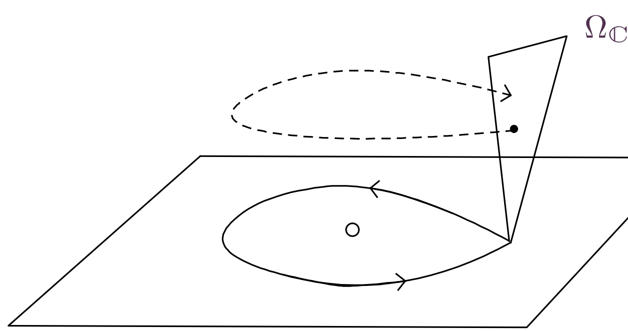
There are two holonomy maps of interest:

1)



Corner transition map

2) In the complex setting…



“The” Holonomy map

We can recover the (orbital) analytic class of the saddle from the analytic class of one of these maps (once we fix the ratio μ/λ)

Definition: Two germs of vector fields

$$\partial, \tilde{\partial} \in \mathrm{Der}(\mathcal{O})$$

(seen as derivations of the local ring)

are **analytically conjugated** if there exists an automorphism

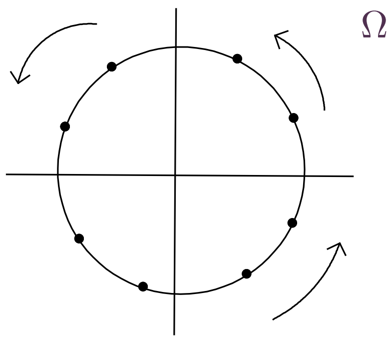
$$\varphi \in \mathrm{Aut}(\mathcal{O})$$

(i.e. an $\mathbb{C}$-endomorphism of the local ring such that $\varphi(fg) = \varphi(f)\varphi(g)$) such that

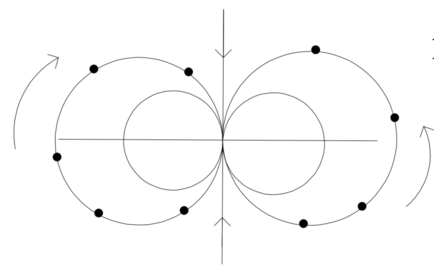
$$\varphi^{-1} \partial \varphi = \tilde{\partial}$$

Definition: Two germs of vector fields $\partial, \tilde{\partial}$ are **orbitally analytic equivalent** if there exists a unit $u \in \mathbb{C}\{x\}$ such that ∂ is analytically conjugated to $u\,\tilde{\partial}$.

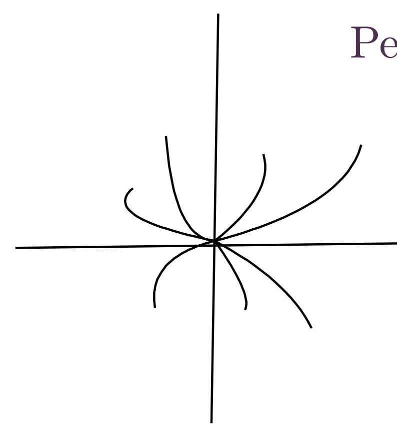
Dynamics of the complex holonomy map as an element of $\mathrm{Diff}(\mathbb{C}, 0)$



rotation



parabolic fixed point

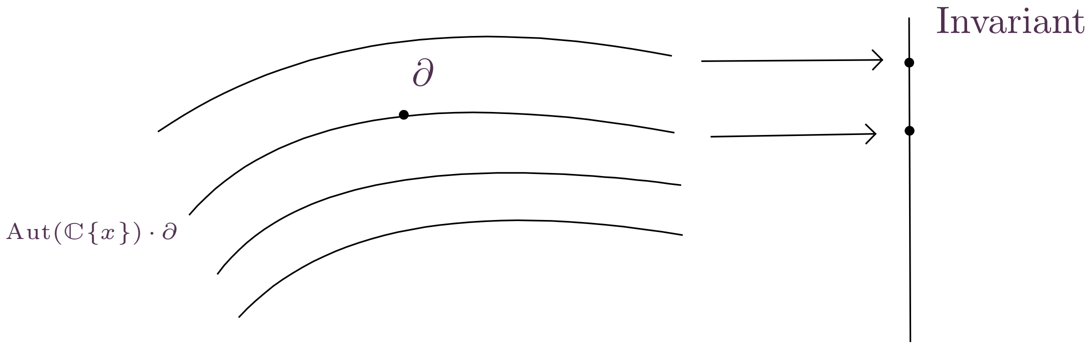


$\mathrm{Perez - Marco's}$
hedgehogs

Classification Problem: "Describe" the orbits of the action of $\mathrm{Aut}(\mathbb{C}\{x\})$ on $\mathrm{Der}(\mathbb{C}\{x\})$ by conjugation

$$(\varphi, \partial) \longmapsto \varphi \cdot \partial = \varphi^{-1} \partial \varphi$$

I.e. local analytic changes of coordinates.



$$\partial \sim \tilde{\partial} \Longleftrightarrow \mathrm{Invariant}(\partial) = \mathrm{Invariant}(\tilde{\partial})$$

The problem is reasonably well-understood for **elementary singularities in dimension two** (modulo some very hard *small divisor problems*) see e.g. Dulac,Ecalle,Ilyashenko,Martinet,Ramis,Yoccoz and Perez Marco,... works.

This problem is much less understood for vector fields higher dimensions.

What about the local transverse behaviour in the vicinity of non-**elementary** singularities?

Example: (Cerveau-Moussu 1988) The cuspidal singularity

$$\partial = 2y\,\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y} + \Delta$$

$$\text{"Almost" first integral.} \quad f(x,y) = y^2 - x^3$$

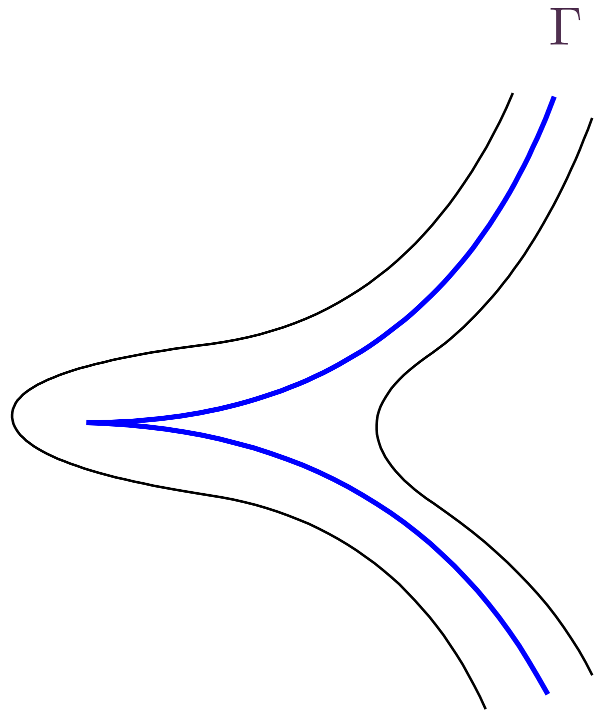
$$\partial_s = 0, \qquad \mathrm{Jac}_{(0,0)} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

For **Δ of (2,3)-quasi homogeneous order $\geqslant 2$**, there exists a local analytic coordinate change such that, up to division by a unit,

$$\partial = 2y\,\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y} + r(x,y)\left(2x\,\frac{\partial}{\partial x} + 3\,y\frac{\partial}{\partial x}\right), \qquad r \in \mathfrak{m}$$

$\partial(f) = 6rf$.

The cusp $\Gamma = \{f = 0\}$ is an invariant curve.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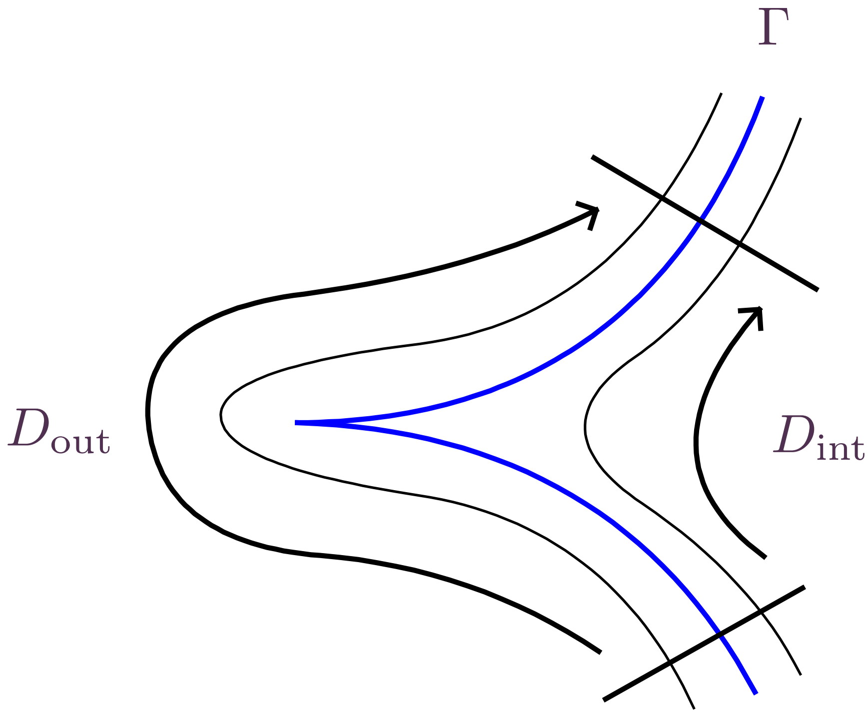
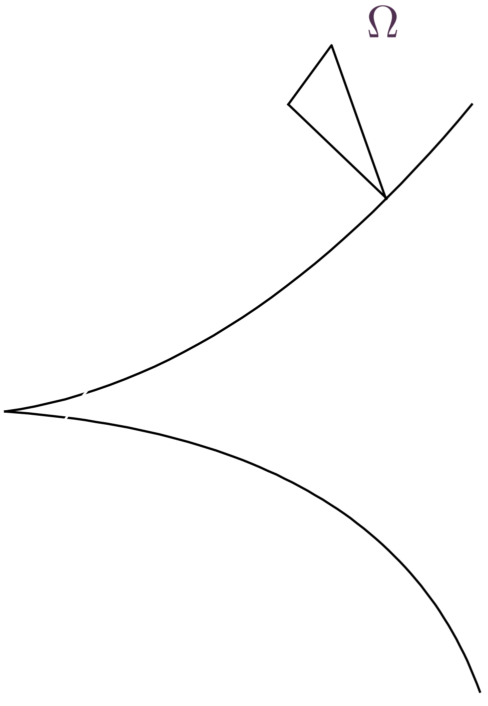
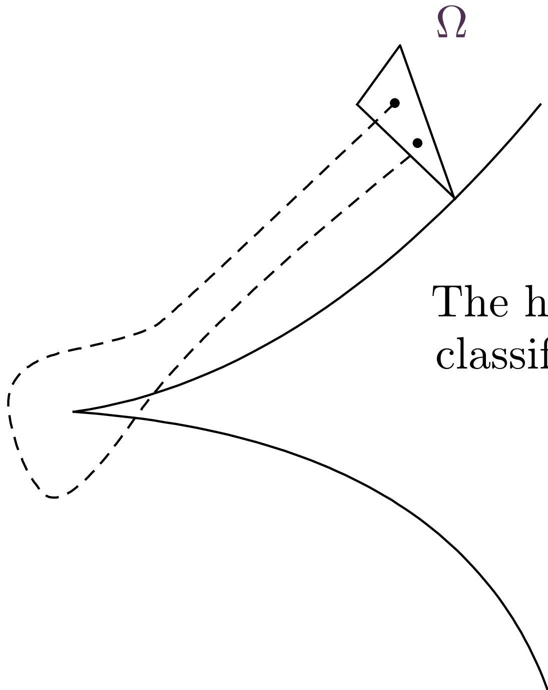


There are two **distinct** corner transition maps.

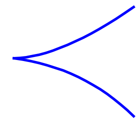

Ω



The holonomy map **does not** classify the singularity

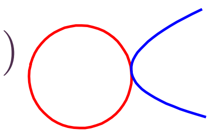
Resolution of the cuspidal foliation. We consider the dual 1-form to simplify

$$d(y^2 - x^3)$$



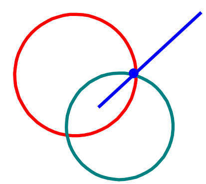
Blow-up 1: $x \to x\,, \qquad y \to x\,y$

$$d(x^2(y^2 - x))$$



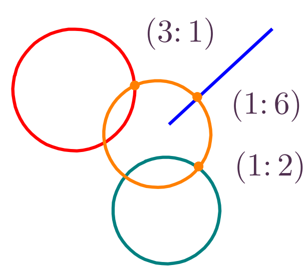
Blow-up 2: $x \to x\,y\,, \qquad y \to y$

$$d(x^2 y^3 (y - x))$$

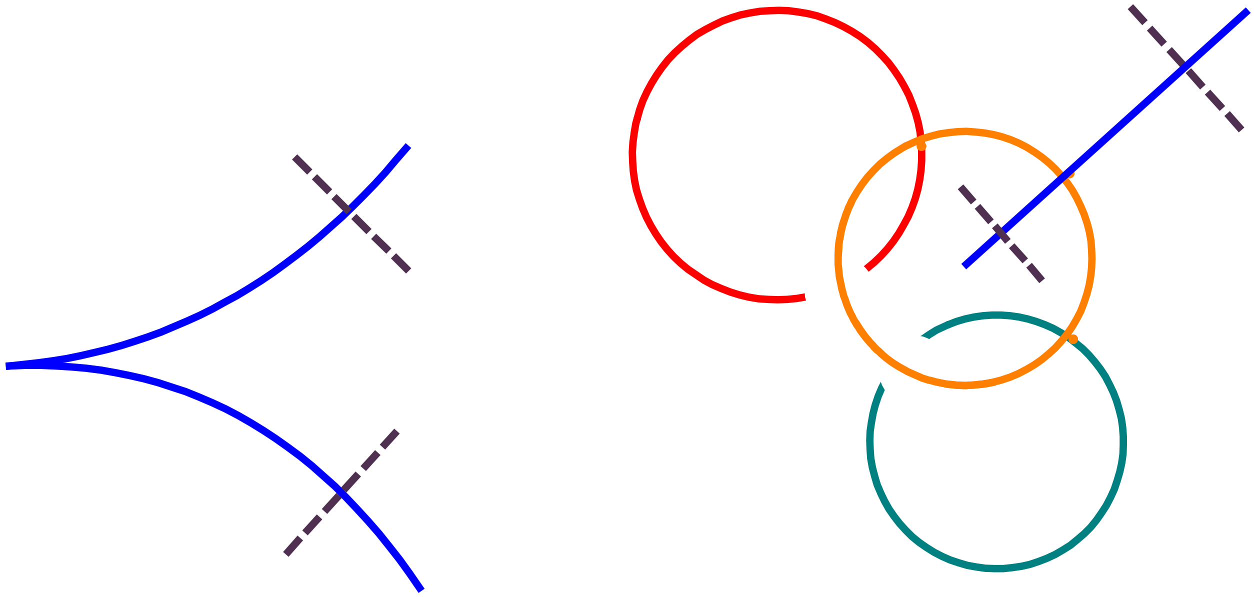


Blow-up 3: $x \to x\,, \qquad y \to x\,y$

$$d(x^6 y^3 (y - 1))$$



All singularities are now elementary saddles.



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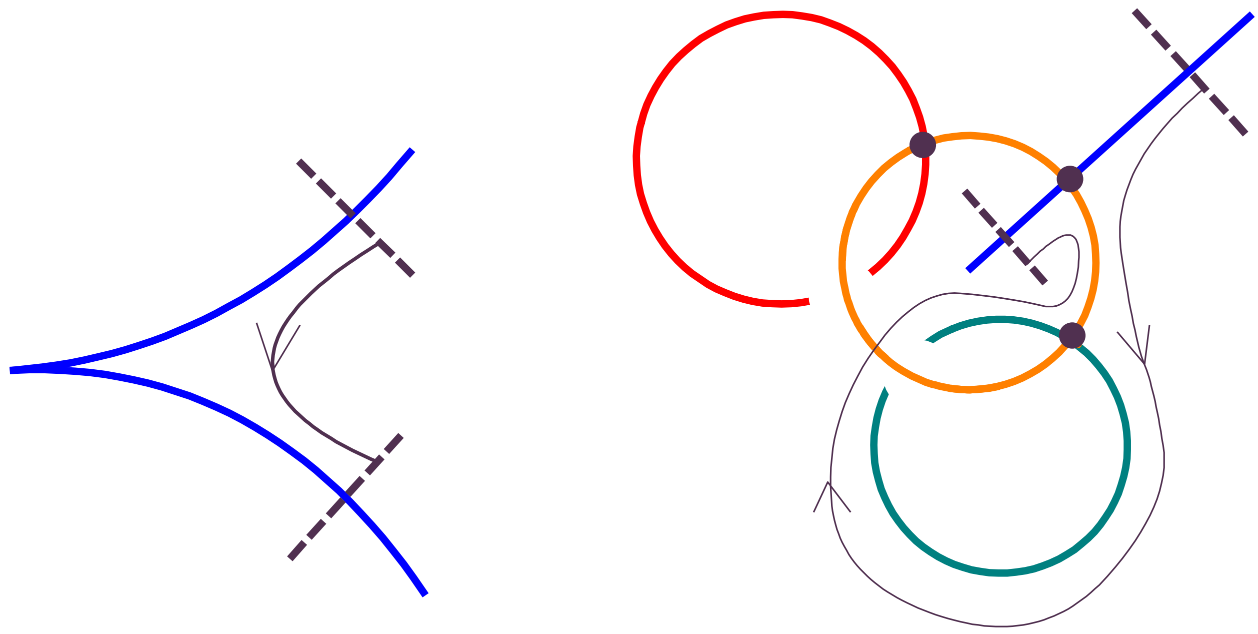
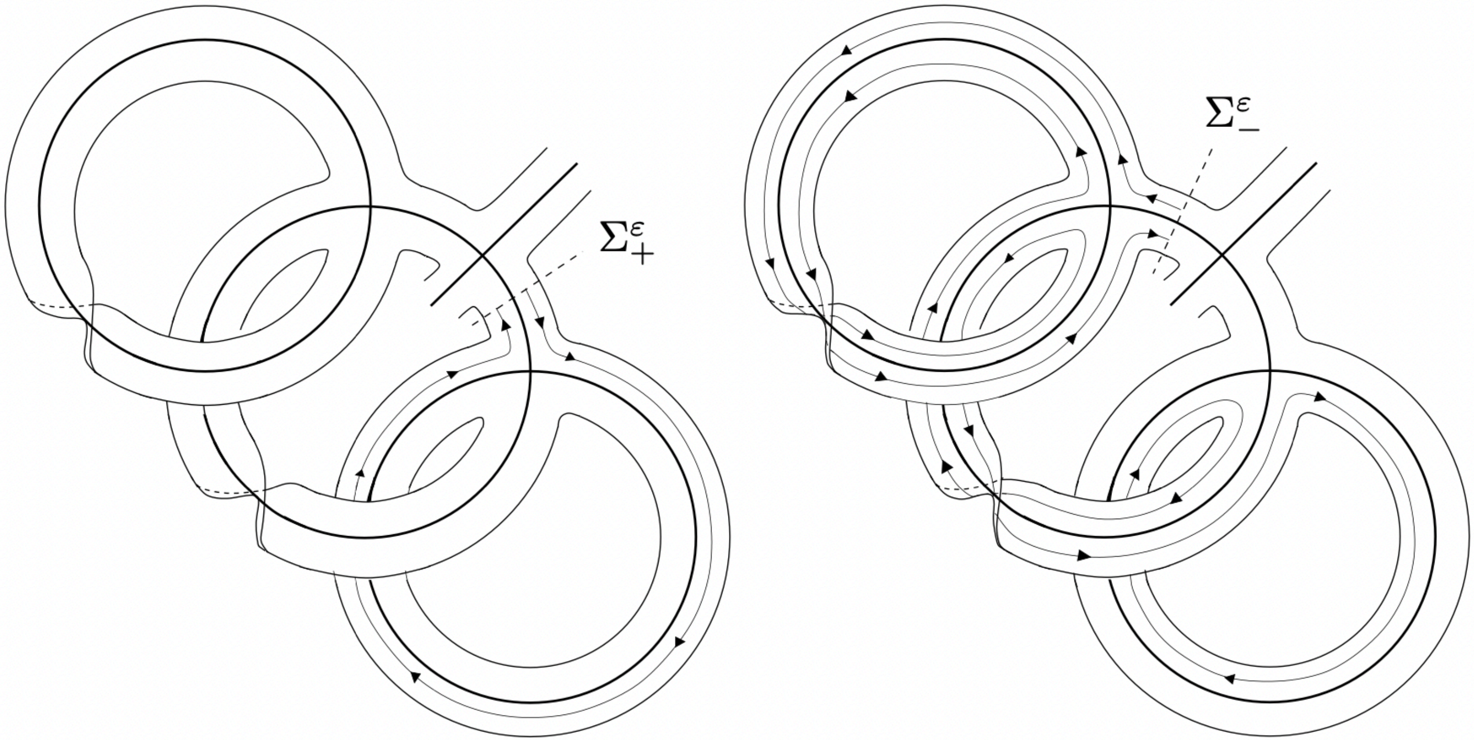
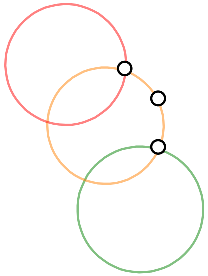
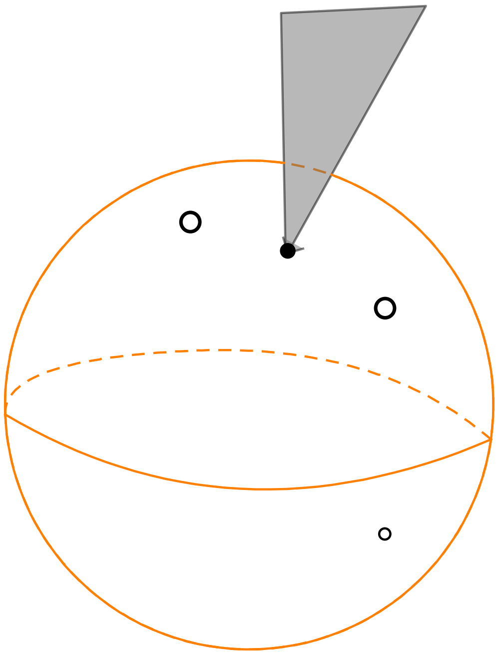


The foliation is now organized in a neighborhood of the exceptional divisor..


Σ_+^ε
Σ_-^ε

Can we recover the analytic moduli from the transverse behaviour?





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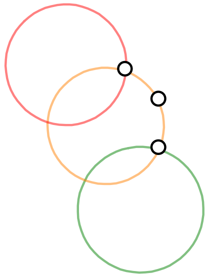

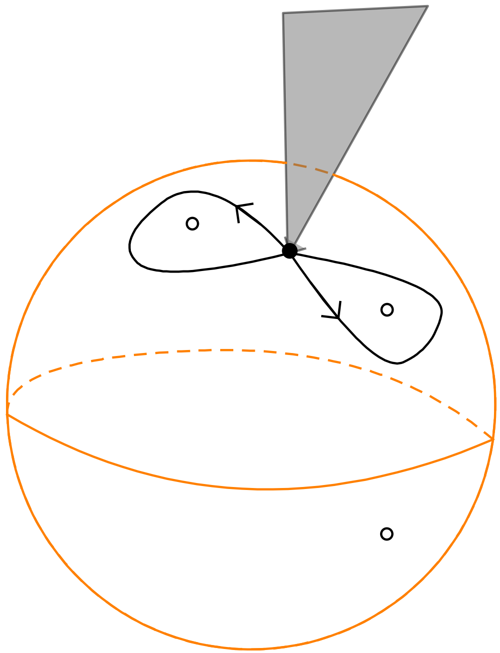

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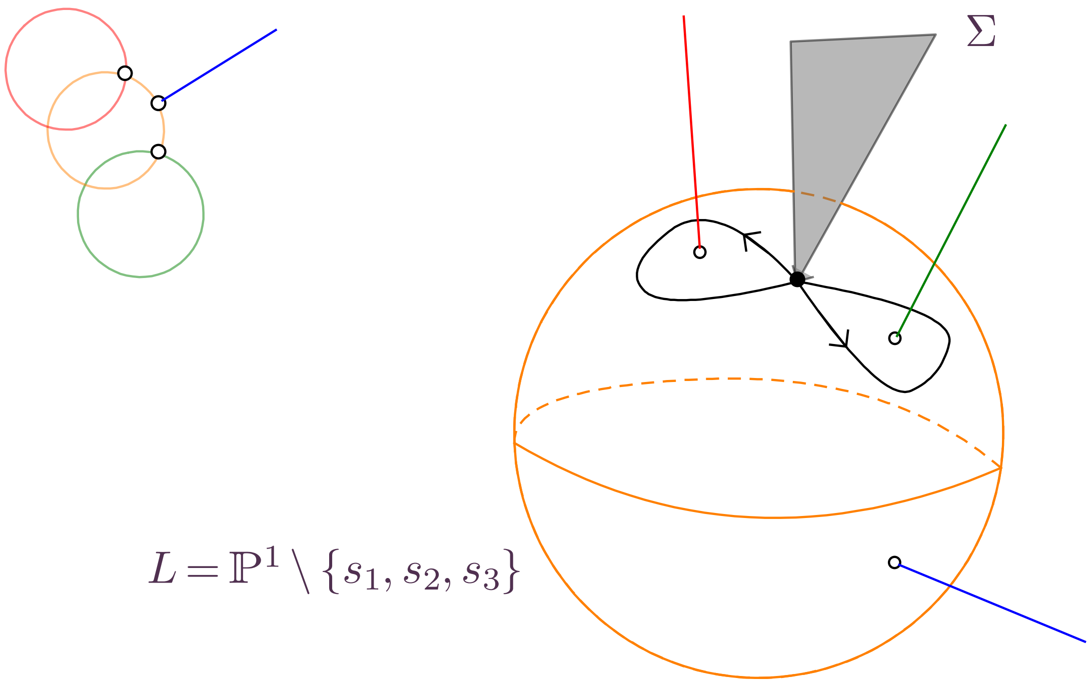


(Moussu) The vanishing holonomy $\mathrm{Hol}(\mathcal{F}, L) = \langle f, g \in \mathrm{Diff}(\mathbb{C}, 0) \mid \ f^2 = g^3 = \mathrm{id}\rangle$ characterizes the analytic class of the germ of foliation.

Nilpotent locus for foliations by curves

The *nilpotent locus* of a foliated manifold is the subset $\operatorname{Nilp}(M, \mathcal{F})$ of points where $\mathcal{F}$ is not elementary.

Claim: $\operatorname{Nilp}(M, \mathcal{F})$ is an analytic (or algebraic) subset of M.

(in fact, $p \in \operatorname{Nilp}(M, \mathcal{F}) \iff \partial(\mathfrak{m}_p) \subset \mathfrak{m}_p$ and $\partial_1 \in \operatorname{End}_{\mathbb{C}}(\mathfrak{m}_p / \mathfrak{m}_p^2)$ is a nilpotent endomorphism, for ∂ some arbitrarily chosen local generator).

Alternatively,

$$p \in \operatorname{Nilp}(M, \mathcal{F}) \iff \forall k \in \mathbb{N} \exists n \in \mathbb{N} : (\partial_k)^n = 0$$

where $\partial_k : J^k \to J^k$ is the induced derivation on the k^{th} jet.

Suppose that $(M, \mathcal{F})$ is further equipped with a normal crossings divisor E.

Definition: We say that $\mathcal{F}$ is **adapted** to E each irreducible component is invariant by $\mathcal{F}$.

More precisely, for each point $p \in M$, consider

• ∂ a local generator of $\mathcal{F}$, and

• f an equation for a local irreducible component of E,

Then

$$\forall i \in \mathbb{N} \quad : \quad \partial(\langle f^i \rangle) \subset \langle f^i \rangle$$

We further say that $\mathcal{F}$ is **tightly adapted** to D if there exists an index i such that

$$\partial(\langle f^i \rangle) \not\subset \langle f^{i+1} \rangle$$

In other words, for $E=(x_1 \ldots x_k=0)$,

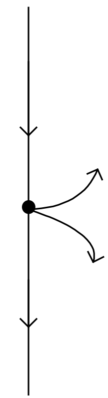
$$\partial=\sum_{i=1}^{k} a_i\left(x_i\frac{\partial}{\partial x_i}\right)+\sum_{i=k+1}^{n} a_i\frac{\partial}{\partial x_i}$$

with $a_1,\ldots,a_n\in\mathbb{C}\{x\}$ such that $\langle a_1,\ldots,a_n\rangle\not\subset\langle x_i\rangle$, for each $i=1,\ldots,k$.

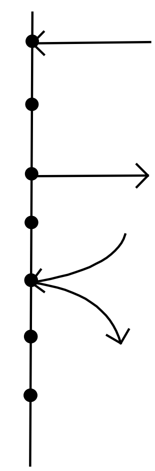
Example: $E=(x=0)$

$$\partial = a\,x\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y}$$

with $\langle a,b\rangle \not\subset \langle x\rangle$



$b\neq 0$: The generic point on the divisor is non-singular



$b=0$: The generic point on the divisor is an elementary singularity

(The singular set of the foliation can have codimension one components)

$$\mathcal{F}\text{ is tightly adapted to } E \Longleftrightarrow \text{no irreducible component of } E \text{ lies on } \mathrm{Nilp}(M,\mathcal{F})$$

The problem of elimination of the nilpotent locus

A singularly foliated manifold is a triple $(M, E, \mathcal{F})$ formed by a manifold M, equipped with

- A normal crossings divisor E and
- A singular foliation by curves $\mathcal{F}$ which is tightly adapted to E.

such that $\mathrm{Nilp}(M, \mathcal{F})$ has codimension greater or equal than two.

Problem: For each relatively compact subset $M_0 \subset M$, find a finite sequence of blowing-ups

$$(M_0, E_0, \mathcal{F}_0) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_n} (M_n, E_n, \mathcal{F}_n)$$

such that:

1) The center C_i of π_i has normal crossings with E_i and is contained in $\mathrm{Nilp}(M_i, \mathcal{F}_i)$

2) $\mathrm{Nilp}(M_n, \mathcal{F}_n) = \emptyset$.

How to compute the transform of a foliation by blowing-up?

via local generators, In local coordinates

$$x_1 \rightarrow x_1, \quad x_2 \rightarrow x_1 x_2 \quad \ldots \quad x_n \rightarrow x_1 x_n$$

It is easier to compute the strict transform of the **logarithmic basis** $\left\{x_1 \frac{\partial}{\partial x_1}, \ldots, x_n \frac{\partial}{\partial x_n}\right\}$.

$$x_1 \frac{\partial}{\partial x_1} \longrightarrow x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_1} - \cdots - x_n \frac{\partial}{\partial x_n}$$

$$x_2 \frac{\partial}{\partial x_2} \rightarrow x_2 \frac{\partial}{\partial x_2}, \quad \ldots \quad , \quad x_n \frac{\partial}{\partial x_n} \rightarrow x_n \frac{\partial}{\partial x_n}$$

(or via de dual basis of logarithmic one-forms $\left\{\frac{dx_1}{x_1}, \ldots, \frac{dx_n}{x_n}\right\}$)

Example: $(\lambda\colon\mu)$ – linear saddle, $\lambda,\mu>0$

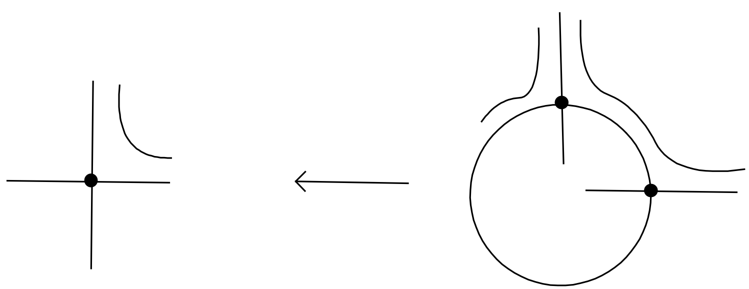
$$\lambda x\frac{\partial}{\partial x}-\mu y\frac{\partial}{\partial y}\qquad(\lambda\colon\mu)$$

Under the substitution $x\to x,\ y\to xy$

$$\lambda\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)-\mu y\frac{\partial}{\partial y}\qquad(\lambda\colon\lambda+\mu)$$

Under the substitution $x\to xy,\ y\to y$

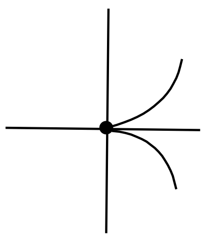
$$\lambda x\frac{\partial}{\partial x}-\mu\left(y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}\right)\qquad(\lambda+\mu\colon\mu)$$



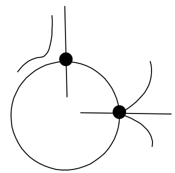
We can never get rid of saddle points...

Example: node

$$x\frac{\partial}{\partial x} + \rho y \frac{\partial}{\partial y} \qquad , \quad \rho > 0$$



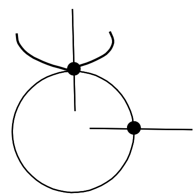
$\rho > 1$



$\tilde{\rho} = \rho - 1$

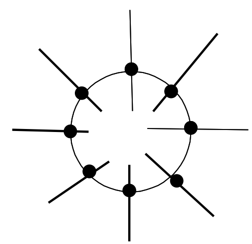
$$\rho = \rho_0 + \frac{1}{\rho_1 + \frac{1}{\cdots}}$$

$\rho < 1$



$\tilde{\rho} = \frac{1}{\rho} - 1$

$\rho = 1$



(dicritical situation)

We can never get rid of a node if $\rho \notin \mathbb{Q}$.

Example: saddle-nodes

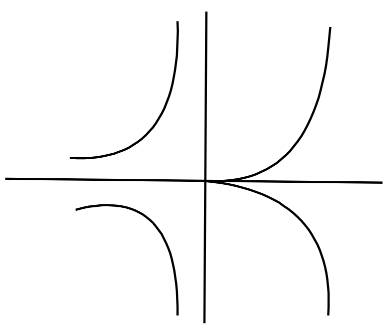
$$x^k\, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad k \geqslant 1$$

After m directional blowing-ups: $x \to x$, $y \to x\,y$

$$x^k\left(x\frac{\partial}{\partial x} - m\,y\frac{\partial}{\partial y}\right) + y\frac{\partial}{\partial y}$$

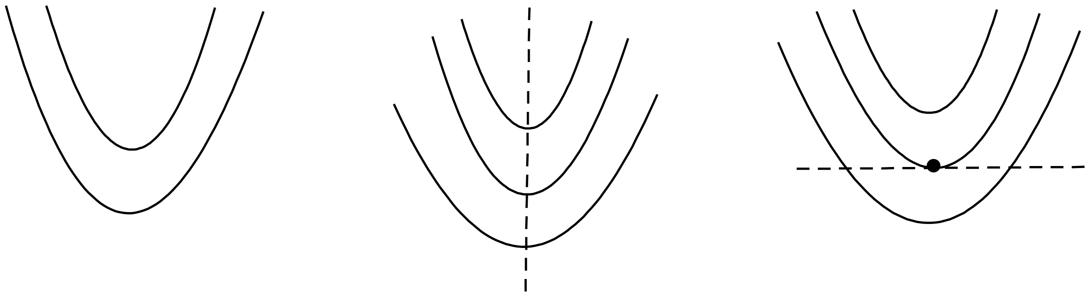
This model is completely stable. It is a final model.

$$\text{First integral} \qquad h = (x^m y)\exp\left(\frac{1}{k x^k}\right)$$



Blowing-up centers with tangencies with the foliation can create non-elementary points.

$$\partial = \frac{\partial}{\partial x} + x^k \frac{\partial}{\partial y}, \qquad k \geqslant 1$$



In logarithmic basis:

$$x^{-1}\left(x\frac{\partial}{\partial x}\right) + x^k y^{-1}\left(y\frac{\partial}{\partial y}\right)$$

$$\text{Center } (x=0): \qquad \tilde{\partial} = x\,\partial = x\frac{\partial}{\partial x} + x^{k+1}\frac{\partial}{\partial y}$$

$$\text{Center } (y=0): \qquad \tilde{\partial} = y\partial = y\frac{\partial}{\partial x} + x^k\left(y\frac{\partial}{\partial y}\right) \quad \text{(nilpotent singularity)}$$

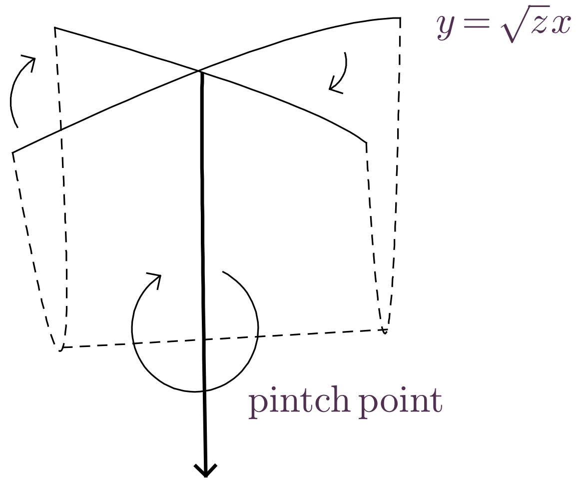
Theorem of Bendixson-Seidenberg. The elimination of nilpotent points holds for singularly foliated **surfaces**.

But… It is false for $\dim M \geqslant 3$.

Example of Sanz and Sancho-Salas:

$$\partial = \left(y\frac{\partial}{\partial x} + x\,z\frac{\partial}{\partial y} \right) + \beta z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) - z\left(-x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z} \right)$$

is tangent to the Whitney umbrella $W = y^2 - zx^2$.



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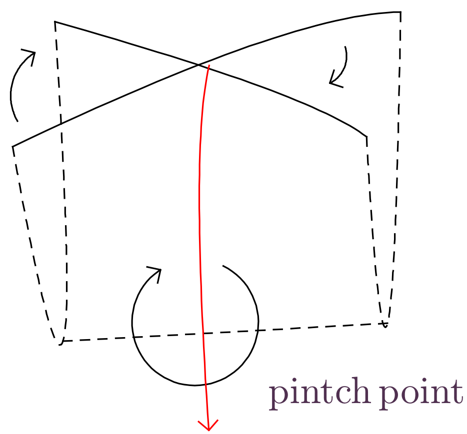
with $\beta \notin \frac{1}{2}\mathbb{Z}_{>0}, \quad \lambda \in \mathbb{C}^{\star}$.



Formal expansion of the "handle"

$$y = \tau(z) = \sum \tau_n z^n, \quad \tau_n \sim \lambda\,(n!)^2$$

$$x = \xi(z) = \sum \xi_n z^n, \quad \xi_n \sim \lambda\,(n!)^2$$

We cannot take the handle as a blowing-up center because it is non-analytic.

Weighted blowing-up

Fix some $\omega \in (\mathbb{Z}_{>0})^n$ and consider the orbits of the action of $\mathbb{C}^\star$ on $\mathbb{C}^n \setminus \{0\}$ by

$$(t, x) \longmapsto t \cdot x = t^\omega x = (t^{\omega_1} x_1, \ldots, t^{\omega_n} x_n)$$

The orbit space is the so-called *weighted projective space*

$$\pi\colon \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{P}_\omega^{n-1}$$

$$x \rightarrow \text{orbit through}\, x$$

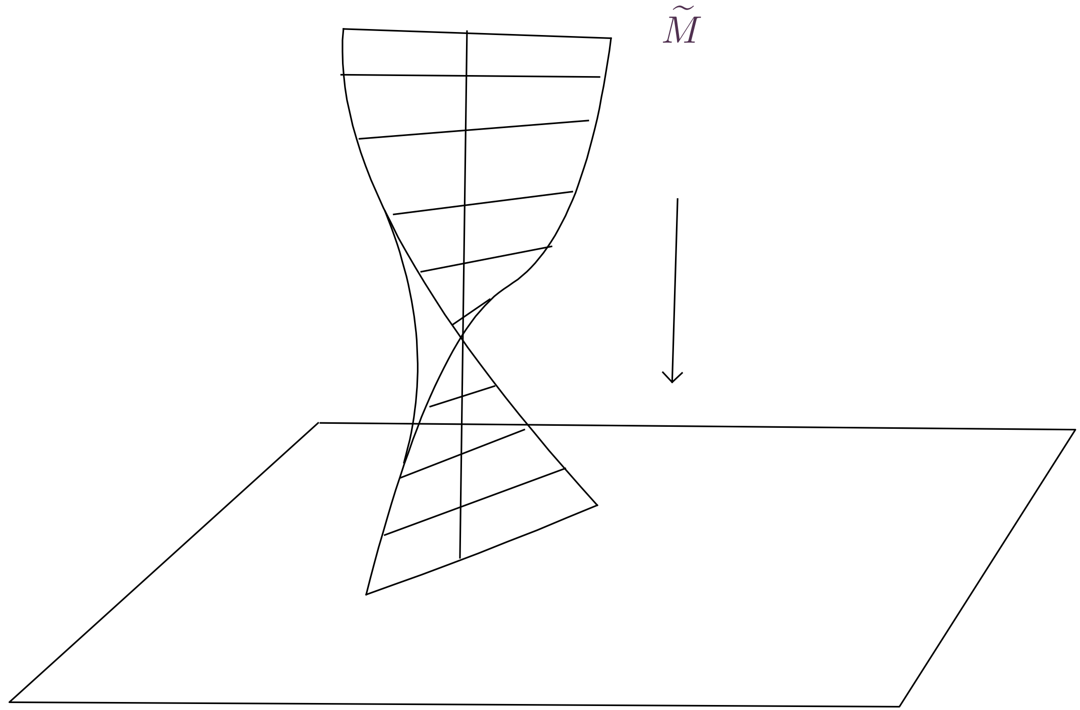
We consider the graph of the quotient mapping as a subset of $\mathbb{C}^n \times \mathbb{P}_\omega^{n-1}$

$$\mathrm{Graph}(\Phi) \subset \mathbb{C}^n \times \mathbb{P}_\omega^{n-1}$$

The blowed-up space is its Zariski-closure

$$\widetilde{M} = \overline{\mathrm{Graph}(\Phi)}^{\,\mathrm{Zar}}$$

and the projection $\pi\colon \widetilde{M} \rightarrow \mathbb{C}^n$ is the weighted blowing-up of the origin in $\mathbb{C}^n$.


$\tilde{M}$

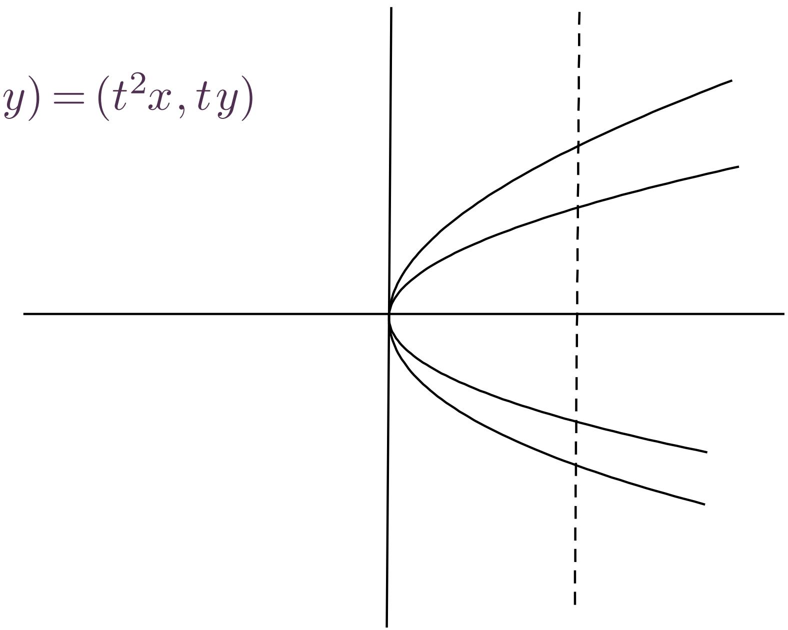
Structure of $\mathbb{P}^{n-1}_\omega$: The hyperplanes $\{x_i = 1\}$ are slices for the torus action modulo the action of a group of symmetries.

Example

$\omega = (2, 1)$

slice $\{x = 1\}$

$t \cdot (x, y) = (t^2 x, t y)$



$y \sim -y$

$\mathbb{Z}/2\mathbb{Z}$

We have to take into account the quotient by $\mathbb{Z}/2\mathbb{Z}$.

The charts of a weighted-blowing up

The x_1-directional chart is given by

$$\begin{array}{ccc} x_1 & \to & y_1^{\omega_1} \\ x_2 & \to & y_1^{\omega_1} y_2 \\ \vdots & & \vdots \\ x_n & \to & y_1^{\omega_n} y_n \end{array}$$

We interpret $(y_{1,..}, y_n)$ as an **orbifold chart** on $\widetilde{M}$. Namely the affine space $\mathbb{C}^n$ equipped with an action of the cyclic group $\mathbb{Z}/\omega_1\mathbb{Z}$, defined by

$$y_1 \to \xi y_1, \qquad \text{For } 2 \leqslant k \leqslant n: \quad y_k \longrightarrow \xi^{-\omega_k} y_k$$

where ξ is a ω_1^{th}-primitive root of unity. The other charts are defined analogously.

The glueing of these charts equipps $\widetilde{M}$ with the structure of an **orbifold**.

Orbifolds (in one slide) (cf. Moerdijk, Mrcun - *Introduction to foliations and Lie groupoids*)

Let M be a paracompact Hausdorff space.

An *orbifold chart* on M is given by triple (U, G, ϕ) where U is a connected open subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$), G is a finite subgroup of $\mathrm{Diff}(U)$ and $\phi\colon U \to M$ is an open map

which induces a homeomorphism $U/G \to \phi(U)$.

An *embedding* $\lambda\colon (V, H, \psi) \hookrightarrow (U, G, \phi)$ between orbifold charts on M is an embedding $\lambda\colon V \to U$ such that $\phi \circ \lambda = \psi$ (this induces an injective homomorphism $H \to G$).

Two orbifold charts (U, G, ϕ) and (V, H, ψ) on M are *compatible* if for any $z \in \phi(U) \cap \psi(v)$ there exists an orbifold chart (W, K, θ) defined near z and embeddings

$$(W, K, \theta) \hookrightarrow (U, G, \phi), \quad (W, K, \theta) \hookrightarrow (V, H, \psi)$$

An orbifold atlas on M is a collection $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$ of pairwise compatible orbifold charts such that $\{\phi(U_i)\}_{i \in I}$ forms an open cover of M.

An **orbifold** is a pair $(M, \mathcal{U})$ where M is paracompact Hausdorff topological space and $\mathcal{U}$ is a maximal orbifold atlas on M.

A sub-variety $Y \subset M$ is a **sub-orbifold** if for each point $p \in Y$ there exists a local chart (U, G, ϕ) such that $\phi^{-1}(Y \cap U)$ is a G-invariant submanifold of U.

Important: 1) The local group actions are part of the structure.

“Remember the group”

2) The underlying topological space can be a singular.

Example: $X = \mathbb{C}^2/G, \quad G = \mathbb{Z}/2\mathbb{Z}$

$$(x, y) \longrightarrow (-x, -y)$$

$X = \mathrm{Spec}\,\mathbb{C}[x, y]^G$ (ring of invariants)

$$\mathbb{C}[x, y]^G = \mathbb{C}[x^2, xy, y^2]$$

$$X = \mathrm{spec}\,\mathbb{C}[u, v, w]/(v^2 - uw)$$

X is the quadratic cone.

General idea: The weighted blowing-up allows to take into account some *natural* quasi-homogeneous filtration of the initial object.

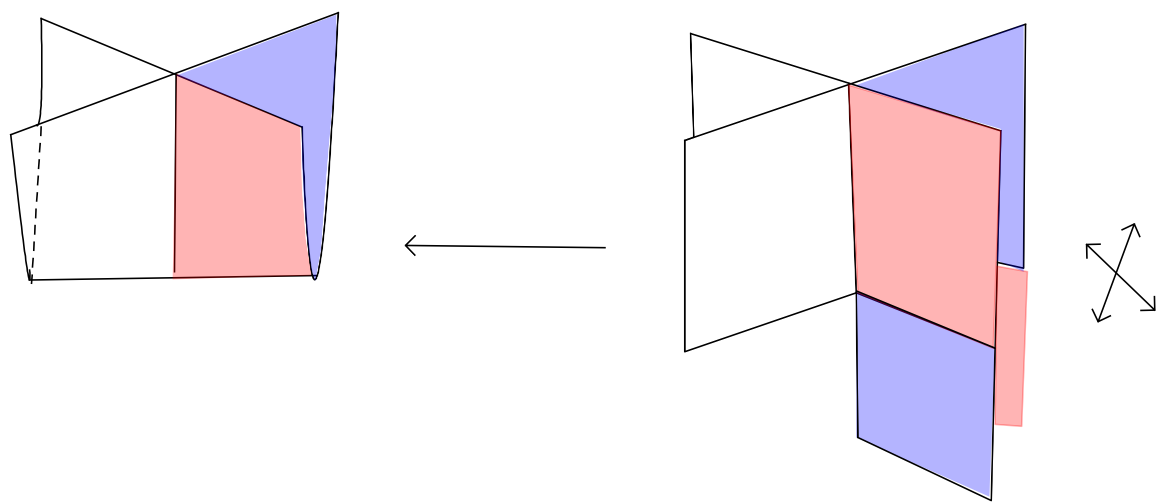
Example: Let us blow-up the origin in $\mathbb{C}^3$ with weight $\omega = (1, 2, 2)$ and look at the pull-back of the Whitney umbrella $w = y^2 - zx^2$

In the z-directional chart we obtain

$$x \to zx, \quad y \to z^2 y, \quad z \to z^2$$

and $w = z^4(y^2 - x^2)$ becomes a normal crossings divisor.

This is the orbifold chart $(\mathbb{C}^3, \mathbb{Z}/2\mathbb{Z}, \phi)$, where the action is $(x, y, z) \to (-x, y, -z)$



Over $\mathbb{R}$: We can alternatively work in the category of **manifold with corners**

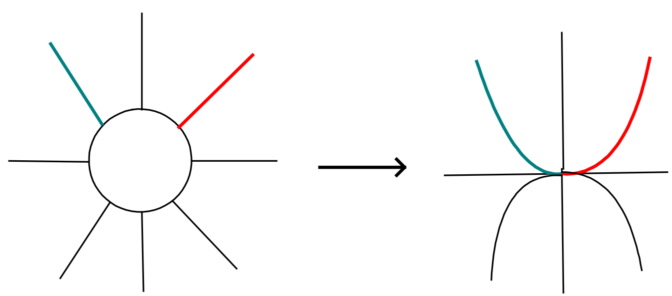
The **spherical blowing-up** of $\mathbb{R}^n$ at the origin with weight ω is the real analytic map

$$\Phi\colon \mathbb{R}_{\geqslant 0} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n$$

given by $\Phi(t, \bar{x}) = t^{\omega}\bar{x}$. The exceptional divisor is the **boundary**

$$\text{boundary}(\mathbb{R}_{\geqslant 0} \times \mathbb{S}^{n-1}) = \{0\} \times \mathbb{S}^{n-1}$$

In general, we require the blowing-up center to have normal crossings with the boundary.



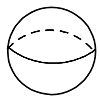
(advantage: stay in the category of smooth manifolds)

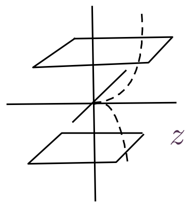
(drawback: we "forget the group" and potentially loose information about the local symetries)

(c.f. Melrose's "Analysis on manifolds with corners" - online)

Example: Spherical blowing-up of the (real) Whitney umbrella

$$\Phi\colon \mathbb{R}_{\geqslant 0} \times \mathbb{S}^2 \longrightarrow \mathbb{R}^3$$



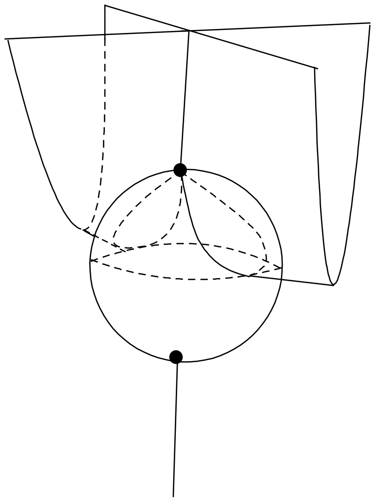


Two z-directional "slices":

$\{z>0\}$-chart: $x \to zx, \quad y \to z^2 y, \quad z \to z^2$: $\quad f = z^4(y^2 - x^2)$

$\{z<0\}$-chart: $x \to z^2 x, \quad y \to z^2 y, \quad z \to -z^2$: $\quad f = z^4(y^2 + x^2)$

$\{x>0\}$-chart: $x \to \pm x, \quad y \to x^2 y, \quad z \to x^2 z$: $\quad f = x^4(y^2 - z)$



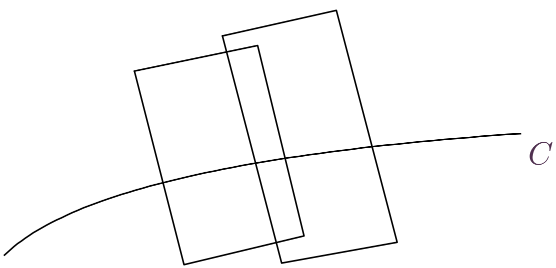
Weighted blowing-up along global centers

If we consider the torus action

$$(t, x) \longmapsto t \cdot x = t^{\omega} x = (t^{\omega_1} x_1, \ldots, t^{\omega_k} x_k, x_{k+1}, \ldots, x_n)$$

Then the above construction leads to a local blowing-up with center $C = Z(x_1, \ldots, x_k)$.

We need to understand how to glue-up these local actions in order to obtain globally defined blowing-up with center C.



Existence of global Weighted blowing-ups

A weighted blowing-up of a point $p \in M$ is fully determined by a **quasi-homogeneous filtration** of the local ring. Namely a filtration

$$\mathcal{O}_p = \mathcal{O}_0 \supset \mathcal{O}_1 \supset \mathcal{O}_2 \supset \cdots \qquad \mathcal{O}_k \cdot \mathcal{O}_l \subset \mathcal{O}_{k+l},$$

such that in appropriate coordinates $(x_1, \ldots, x_n)$, we have $x_1 \in \mathcal{O}_{\omega_1}, .., x_n \in \mathcal{O}_{\omega_n}$.

In other words, $\mathcal{O}_k$ is the subring of functions of quasi-homogeneous weight $\geqslant k$.

In order to define a quasi-homogeneous blow-up along a submanifold (suborbifold) $C \subset M$, we need to require the existence of a global trivialization of C

Such that the diffeomorphisms between the transition charts respects the local quasi-homogeneous filtration. This is a non-trivial topological restriction.

More abstractly: This amounts to the existence of a *global weighted filtration of the structure sheaf*. Namely a sequence of nested of ideal sheafs

$$\mathcal{O} = F_0 \supset F_1 \supset \cdots$$

such that $F_i F_j \subset F_{i+j}$ and such that, for each point p on the support, the stalk of this filtration coincides with a quasi-homogeneous filtration as defined above.

Example: $C = Z(x, y) \subset \mathbb{C}^3$

$$\omega = (1, \beta, 0) \in \mathbb{Z}^3$$

$$\beta > 1$$

All automorphisms of the form

$$x \rightarrow x + \rho y^m, \quad y \rightarrow y + \xi x^l, \qquad l \geqslant \beta$$

preserve the $(1, \beta, 0)$-filtration of $\mathbb{C}[x, y, z]$.

More generally, all automorphisms obtained by integrating the Lie algebra (over $\mathbb{C}$) generated by

$$\left\{ x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^l\frac{\partial}{\partial y}, y^m\frac{\partial}{\partial x} \quad \middle| \quad m \geqslant 1, l \geqslant \beta \right\}$$

Weighted blowing-up of vector fields

$$x_1 \to x_1^{\omega_1}, \quad \text{For } 2 \leqslant k \leqslant n: \quad x_k \to x_1^{\omega_k} x_k$$

Transformation of the logarithmic basis

$$x_1 \frac{\partial}{\partial x_1} \longrightarrow \frac{1}{\omega_1}\left(x_1 \frac{\partial}{\partial x_1} - \omega_2 x_2 \frac{\partial}{\partial x_2} - \cdots - \omega_n x_n \frac{\partial}{\partial x_n} \right)$$

$$x_k \frac{\partial}{\partial x_k} \longrightarrow x_k \frac{\partial}{\partial x_k}$$

Example: $\partial = x\frac{\partial}{\partial x} + n\, y\frac{\partial}{\partial y}, \quad n \in \mathbb{Z}_{>0}$.

$$x \to x, \quad y \to x^n y$$

$$\partial = x \frac{\partial}{\partial x}$$

The solution curves of ∂ are precisely the orbits of the torus action $t \cdot (x, y) = (t x, t^n y)$.

Example: weighted resolution of the cuspidal singularity

$$\partial = 2y\,\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y} + \Delta$$

Based on the quasi-homogeneity the almost first integral $y^2 - x^3$, we consider the blow-up with weight $(2,3)$.

We write ∂ in the logarithmic basis

$$\partial = 2x^{-1}y\left(x\,\frac{\partial}{\partial x}\right) + 3x^2y^{-1}\left(y\frac{\partial}{\partial y}\right) + \Delta$$

In the x-chart: $x \rightarrow x^2, y \rightarrow x^3y$: (Using the assumption of the $(2,3)$-order of Δ)

$$\partial = x\,y\left(x\,\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y}\right) + 3x\,y^{-1}\left(y\frac{\partial}{\partial y}\right) + x^2\Delta = x\left(x\,y\,\frac{\partial}{\partial x} + 3(1-y^2)\frac{\partial}{\partial y}\right) + x^2\Delta$$

The divisor $\{x=0\}$ is contained in the nilpotent locus. We factor out x and write

$$\partial_1 = x\,y\,\frac{\partial}{\partial x} + 3(1-y^2)\frac{\partial}{\partial y} + \Delta_1$$

In the y-chart: $x \to y^2 x$, $y \to y^3$:

The original cuspidal foliation

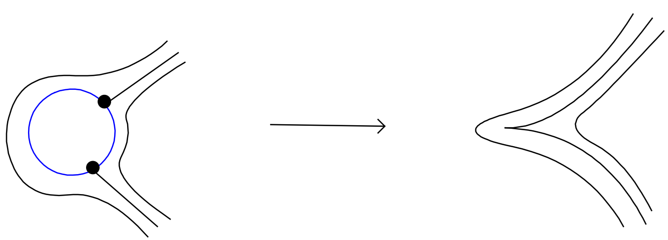
$$\partial = 2x^{-1}y\left(x\frac{\partial}{\partial x}\right) + 3x^2y^{-1}\left(y\frac{\partial}{\partial y}\right) + \Delta$$

transforms into

$$\partial = 2x^{-1}y\left(x\frac{\partial}{\partial x}\right) + x^2y\left(y\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial x}\right) + y^2\Delta = y\left(2(1-x^3)\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y}\right) + y^2\Delta$$

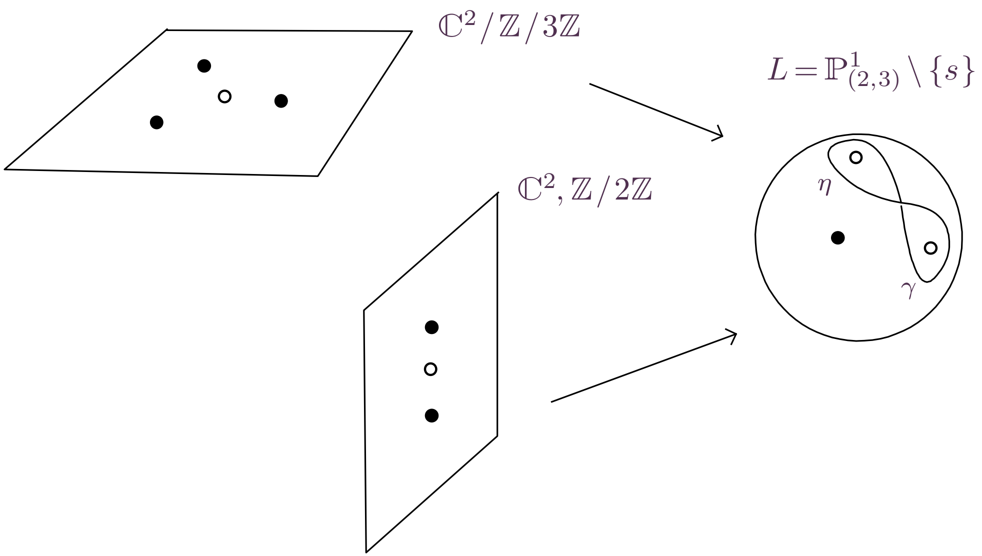
and, factoring out y, we obtain

$$\partial_2 = 2(1-x^3)\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + \Delta_2$$



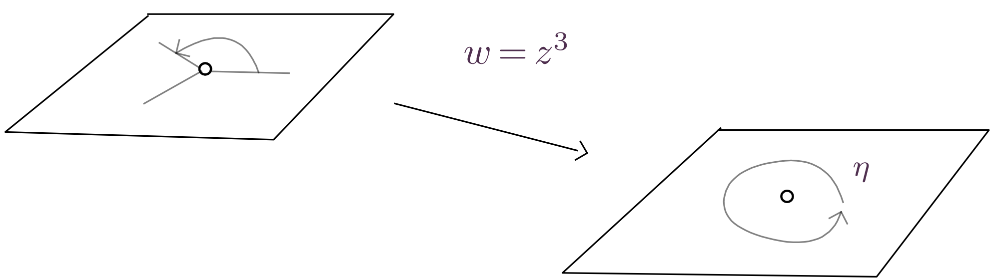
The resulting perturbation Δ is of quadratic order along E (does not change the eingen-values at the singular point)

Local symmetries of the foliated orbifold



The fundamental group of the (orbi-)leaf L is

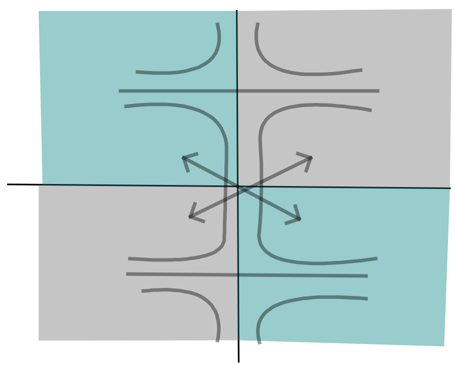
$$\pi_1(L) = \{\gamma, \eta, \rho |\, \gamma^2 = \eta^3 = 1, \, \rho = \gamma\eta\}$$



$$\partial_1 = x\,y\,\frac{\partial}{\partial x} + 3(1-y^2)\frac{\partial}{\partial y} \qquad \circlearrowleft \quad \mathbb{Z}/2\mathbb{Z}$$

$$g \cdot x = -x, \quad g \cdot y \to -y$$

$$g \cdot \partial_1 = -\partial_1$$

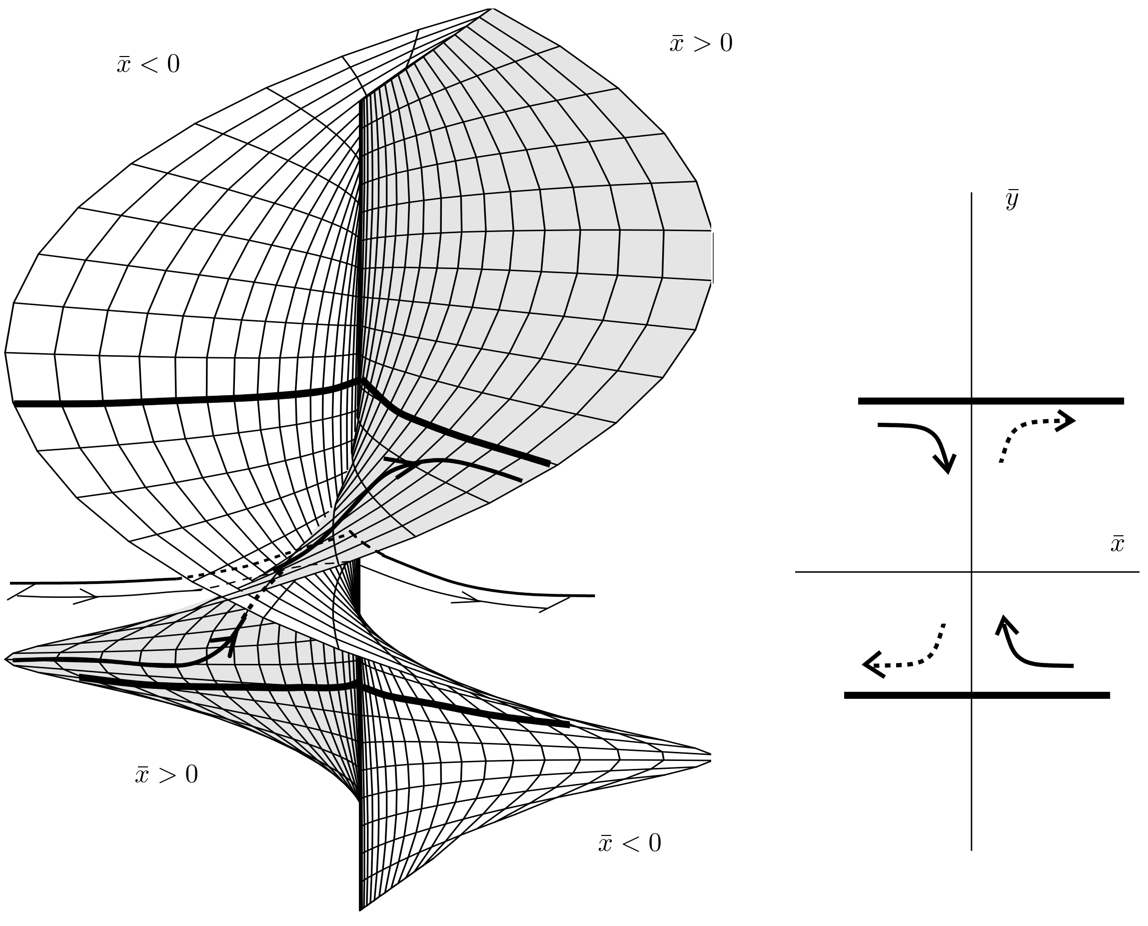


Other chart

$$\partial_2 = 2(1-x^3)\frac{\partial}{\partial x} - x^2 y \frac{\partial}{\partial y}$$

$$g \cdot x = \xi^{-2}\,x, \quad g \cdot y = \xi\,y, \qquad (\xi^3 = \mathrm{id})$$

$$g \cdot \partial_2 = \xi^2 \partial_2$$



$\bar{x} < 0$
$\bar{x} > 0$
$\bar{x} > 0$
$\bar{x} < 0$
$\bar{y}$
$\bar{x}$

Elimination of nilpotent points in dimension two - Classical proof

Van der Essen's proof (c.f. Ilyashenko-Yakovenko's book) We write $\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$

Suppose that the germ is singular. We can assume that $a, b \in \mathbb{C}\{x,y\}$ have no common factor and consider

$$m(0) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(a,b)} \geqslant 1, \quad \mu(0) = \min_k \{(J^k a, J^k b) \neq (0,0)\}$$

($m(0)$ is the *local intersection multiplicity* of the curves $Z(a)$ and $Z(b)$ at 0)

After a blowing-up, the *Noether's formula* give,

$$\sum m(\tilde{p}_j) = m(0) - l^2 + l + 1$$

where $\{\tilde{p}_j\}$ are the singular points of the blowed-up vector field and

$$l = \begin{cases} \mu(a,b) & \text{if } \partial \text{ is non-dicritic} \\ \mu(a,b)+1 & \text{if } \partial \text{ is dicritic} \end{cases}$$

- If $l(0) \geqslant 2$ then $m(\tilde{p}_j) < m(p)$
- If $l(0) = 1$ then this is a special case which has to be treated separately...

Example of "special case".

$$y\frac{\partial}{\partial x}+x^M\frac{\partial}{\partial y}$$

$\mu=1, m=M\geqslant 3$

$x\to x\,,\quad y\to x\,y$

$$x\,y\frac{\partial}{\partial x}+(x^{M-1}-y^2)\frac{\partial}{\partial y}$$

$\mu=2, m=M+1$

The "invariant" increases and this case needs to be treates separately...

Using weighted blowing-ups (modified version of a proof by M.Pelletier).

Initial setup: $(M, E, \mathcal{F})$, where M is a two-dimensional real analytic manifold with corners,

$$\text{boundary}(M) = E$$

is a normal crossings divisor and $\mathcal{F}$ is a foliation tangent to E such that

$\text{Nilp}(M, \mathcal{F})$ is of codimension two (i.e. consists of isolated points).

Definition: The **local desingularization strategy** at a point $p \in \text{Nilp}(M, \mathcal{F})$ is the choice of a quasi-homogeneous filtration of the local ring.

which will define the blowing-up…

Intermezzo: **The Newton polyhedron of a germ of vector field**

Let us **fix** local coordinates $(x_1,\dots,x_n)$. We can write $\partial = a_1\frac{\partial}{\partial x_1}+\cdots+a_n\frac{\partial}{\partial x_n}$.

Instead, We expand ∂ is the logarithmic basis $\left\{x_1\frac{\partial}{\partial x_1},\dots,x_n\frac{\partial}{\partial x_n}\right\}$ as

$$\partial = b_1x_1\frac{\partial}{\partial x_1}+\cdots+b_nx_n\frac{\partial}{\partial x_n},$$

where each $b_i = x_i^{-1}a_i$ has potentially a pole along $(x_i=0)$.

We can reorder the expansion and write the monomial expansion

$$\partial = \sum_{k\in\mathbb{Z}^n} x^k L(\mu_k)$$

where, we recall, each $L(\mu)=\sum \mu_i\, x_i\frac{\partial}{\partial x_i}$ is a diagonal vector field, i.e. an element of the $\mathbb{C}$-maximal toral subalgebra

$$\mathfrak{t} = \left\langle x_1\frac{\partial}{\partial x_1},\dots,x_n\frac{\partial}{\partial x_n}\right\rangle$$

defined by $(x_1,\dots,x_n)$.

$$\partial = \sum_{k \in \mathbb{Z}^n} x^k L(\mu_k)$$

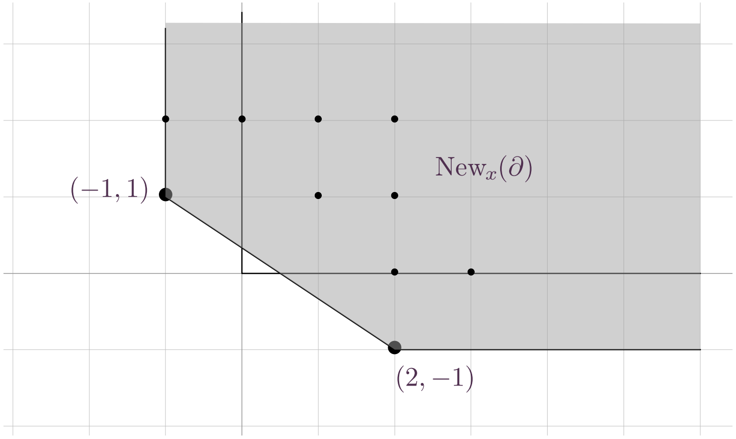
The **support** of ∂ with (respect to x) is defined by $\mathrm{supp}_x(\partial) = \{k \mid \mu_k \neq 0\}$ and

$$\mathrm{New}_x(\partial) = \mathrm{conv}(\mathrm{supp}_x(\partial)) + \mathbb{R}^n_{\geqslant 0}$$

is the **Newton polyhedron** of ∂ (with respect to the coordinates x).

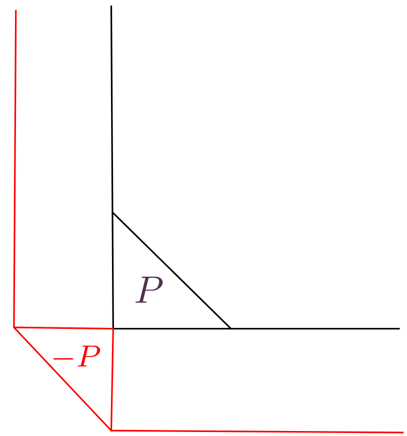
Example: (cuspidal case) $\partial = 2y\,\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y} + \Delta$

$$\partial = 2\,x^{-1}y\left(x\frac{\partial}{\partial x}\right) + 3\,x^2y^{-1}\left(y\frac{\partial}{\partial y}\right) + \Delta$$



Remarks: 1) $\mathrm{New}_x(\partial)$ is always contained in the convex region

$$\mathcal{N} = -\underbrace{(\{k \in \mathbb{N}_{\geqslant 0} | \quad |k| \leqslant 1\})}_{P} + \mathbb{R}^n_{\geqslant 0}$$



2) The hypersurface $(x_i = 0)$ is invariant by ∂ if and only if $\mathrm{supp}_x(\partial) \subset \{k \colon k_i \geqslant 0\}$.

3) The hypersurface $(x_i = 0)$ is tightly invariant by ∂ if and only if

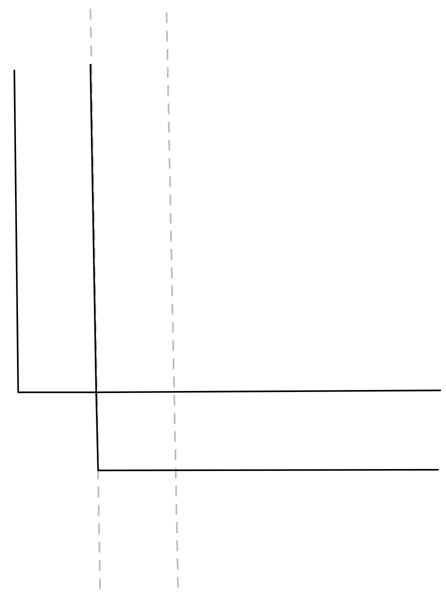
$$\mathrm{supp}_x(\partial) \subset \{k \colon k_i \geqslant 0\} \quad \wedge \quad \mathrm{supp}_x(\partial) \cap \{k \colon k_i = 0\} \neq \emptyset$$

Example. $\partial = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$

$(x=0)$ invariant $\Longleftrightarrow \partial(\langle x\rangle) \subset \langle x\rangle \Longleftrightarrow a \in \mathbb{C}\{x, y\} \Longleftrightarrow [(k,l) \in \mathrm{supp}(\partial) \Longrightarrow k \geqslant 0]$

$(x=0)$ not tightly invariant $\Longleftrightarrow (\partial(\langle x\rangle) \subset \langle x\rangle^2$

$\Longleftrightarrow (ax, bxy) \subset \langle x\rangle^2 \Longleftrightarrow [(k,l) \in \mathrm{supp}(\partial) \Longrightarrow k \geqslant 1]$

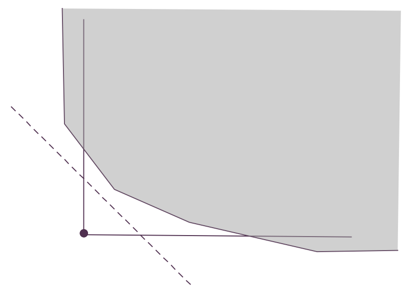


Very classical idea (see Newton, I. 1676):

The resolution of singularities should correspond to a combinatorial game based on the Newton polyhedron.

Can we recognize a "final situation" (a.k.a. an elementary germ) by looking at $\mathrm{New}_x(\partial)$?

Proposition: $\partial \in \mathrm{Der}(\mathcal{O})$ is a nilpotent germ if and only if there exists a local system of coordinates $x = (x_1, \ldots, x_n)$ such that $0 \notin \mathrm{New}_x(\partial)$.



Proof: Assume that $0 \notin \mathrm{New}_x(\partial)$. Then there exists a nonzero $\omega \in \mathbb{Q}^n_{\geqslant 0}$ and $\alpha \in \mathbb{Q}_{>0}$ such that

$$\mathrm{New}_x(\partial) \subset H = \{\langle \omega, \cdot \rangle \geqslant \alpha\}$$

(indeed, if some $\omega_i < 0$ then for $v \in \mathrm{supp}_x(\partial)$, $\langle \omega, v + t e_i \rangle \to -\infty$ as $t \to +\infty$).

We can assume that $\omega \in \mathbb{Z}^n_{\geqslant 0} \setminus \{0\}$ and consider the quasi-homogeneous graduation of $\mathcal{O}$ associated to the torus action $\lambda \colon \mathbb{C}^\star \to \mathrm{Aut}(\mathcal{O})$

$$\lambda(t) \cdot x_i = t^{\omega_i} x_i, \qquad i = 1, \ldots, n$$

(or, equivalently, the graduation associted to the infinitesimal semisimple generator $\delta = \sum \omega_i x_i \frac{\partial}{\partial x_i}$). This action is diagonalizable and we have a direct sum decomposition

$$\mathcal{O} = \bigoplus_{\alpha} \mathrm{Gr}_\alpha(\mathcal{O}, \lambda) = \bigoplus_{\alpha} \mathrm{Gr}_\alpha(\mathcal{O}, \delta)$$

where $\mathrm{Gr}_\alpha(\mathcal{O}, \lambda) = \{f \colon \lambda(t) \cdot f = t^\alpha f\} = \{f \colon \delta(f) = \alpha f\}$ is the module of ω-quasi homogeneous germs of degree α.

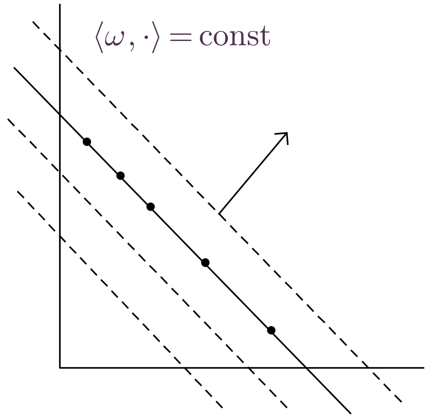
This induces an action of $\mathbb{C}^\star$ on $\mathrm{Der}(\mathcal{O})$ given by conjugation

$$\lambda(t) \cdot \partial = \lambda(t)\, \partial \lambda(t)^{-1}$$

and equally induces a direct sum decomposition $\mathrm{Der} = \bigoplus_\alpha \mathrm{Gr}_\alpha(\mathrm{Der}, \lambda)$.

And, naturally $\partial \in \mathrm{Gr}_\alpha$, $f \in \mathrm{Gr}_\beta \Longrightarrow \partial f \in \mathrm{Gr}_{\alpha+\beta}$.

$$\partial \in \mathrm{Gr}_{\alpha}(\mathrm{Der}, \lambda) \Longleftrightarrow \mathrm{supp}_x(\partial) \subset \{k \colon \langle \omega, k \rangle = \alpha\}$$



By the above hypothesis, our original derivation satisfies

$$\mathrm{supp}_x(\partial) \subset \{k \colon \langle \omega, k \rangle \geqslant \alpha\} \Longrightarrow \partial \in \mathrm{Gr}_{\geqslant \alpha}(\mathrm{Der}, \lambda)$$

Since this is a filtration, $\partial^2 \in \mathrm{Gr}_{\geqslant 2\alpha}$,.., $\partial^r \in \mathrm{Gr}_{\geqslant r\alpha}$ for all $r \geqslant 1$.

and if if $f \in \mathrm{Gr}_{\geqslant \beta}(\mathcal{O}, \lambda)$ then $\partial^r(f) \in \mathrm{Gr}_{\geqslant r\alpha + \beta}(\mathcal{O}, \lambda)$.

As a consequence, for $\mathfrak{m} = \langle x_1, \ldots, x_m \rangle$ the maximal ideal, for each s there exists a $r \geqslant 1$ such that

$$\partial^r(\mathfrak{m}^s) \subset \mathfrak{m}^{s+1}$$

$(\text{because for } k \in \mathbb{Z}^n_{\geqslant 0}, |k| \geqslant \langle \omega, k \rangle / \max\{\omega_i\})$. Hence, ∂ is nilpotent.

Reciprocally, assume that ∂ is nilpotent. Then, $\partial(\mathfrak{m}) \subset \mathfrak{m}$ and $\partial_S = 0$. There exists a local coordinate system such that $\partial|_{J^1} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{pmatrix}$, i.e. such that

$$\partial(x_i) = \varepsilon_i \, x_{i+1} \qquad (\mathrm{mod}\, \mathfrak{m}^2)$$

where $\varepsilon_i \in \{0, 1\}$. In other words, in the logarithmic basis, we obtain

$$\partial = \sum_{i \leqslant n-1} \varepsilon_i x_{i+1} x_i^{-1} \left(x_i \frac{\partial}{\partial x_i} \right) + R$$

where R is a derivation with of degree $\geqslant 1$ with respect the usual homogeneous filtration associated to the derivation $h = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} = L(\mathbf{1})$, with weights $\mathbf{1} = (1, \ldots, 1)$.

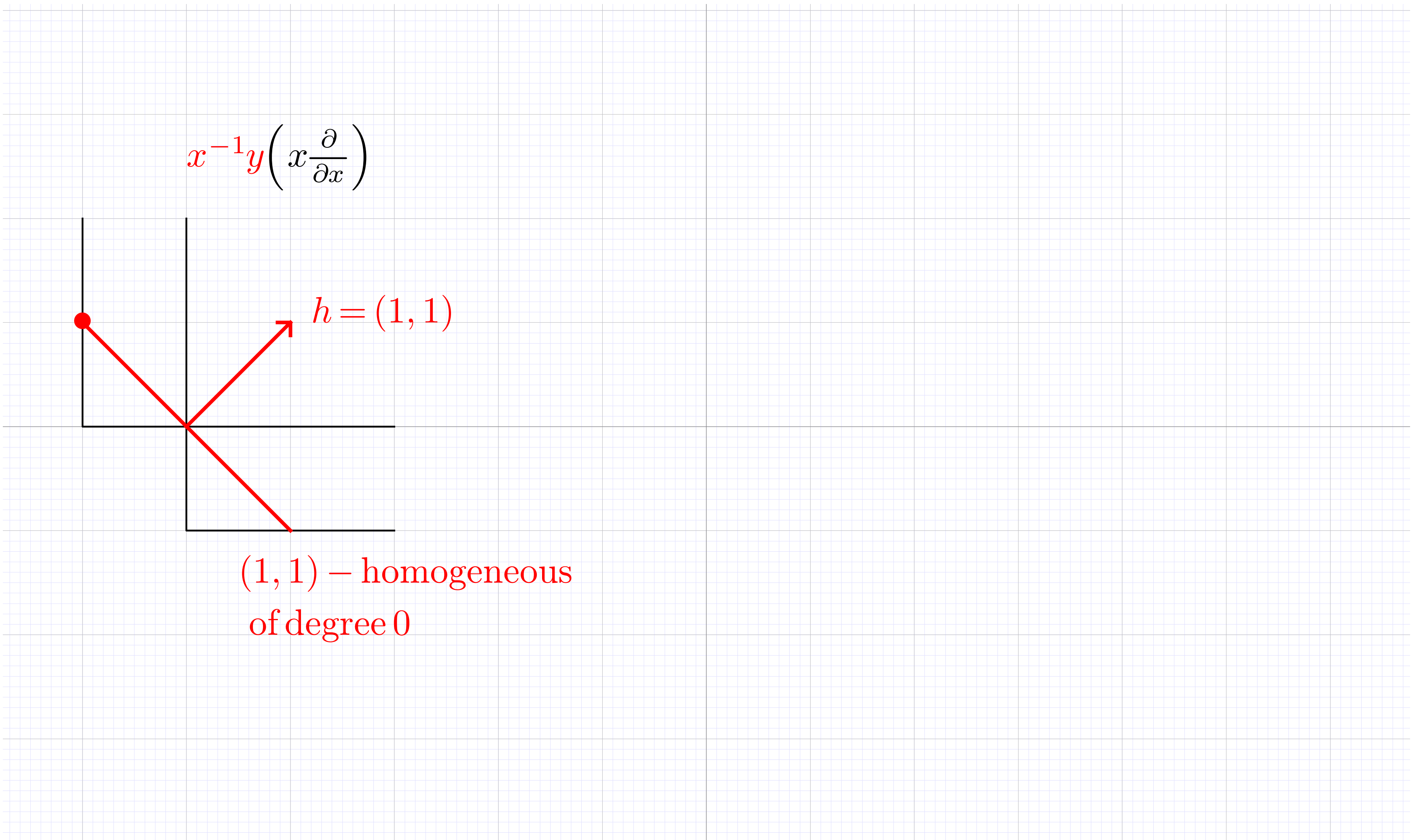
We now consider the weight-vector $\rho = (-n/2, \ldots, n/2)$, or any other rational vector satisfying.

$$\langle \mathbf{1}, \rho \rangle = 0, \qquad \langle \rho, e_{i+1} - e_i \rangle > 0, \qquad e_i = (0, \ldots, 1, \ldots 0)$$

Then, for all sufficiently small $\varepsilon \in \mathbb{Q}_{>0}$, the semi-simple derivation $\omega = h + \varepsilon L(\rho)$ defines a half-space which separates $\mathrm{New}_x(\partial)$ from 0.

(because for $|k| \geqslant 2$, $\langle \omega, k \rangle \geqslant 2 - n\varepsilon |k|$, and $\mathrm{New}_x(\partial)$ has finitely many vertices)

Geometrically, we have used… **The hinge method**



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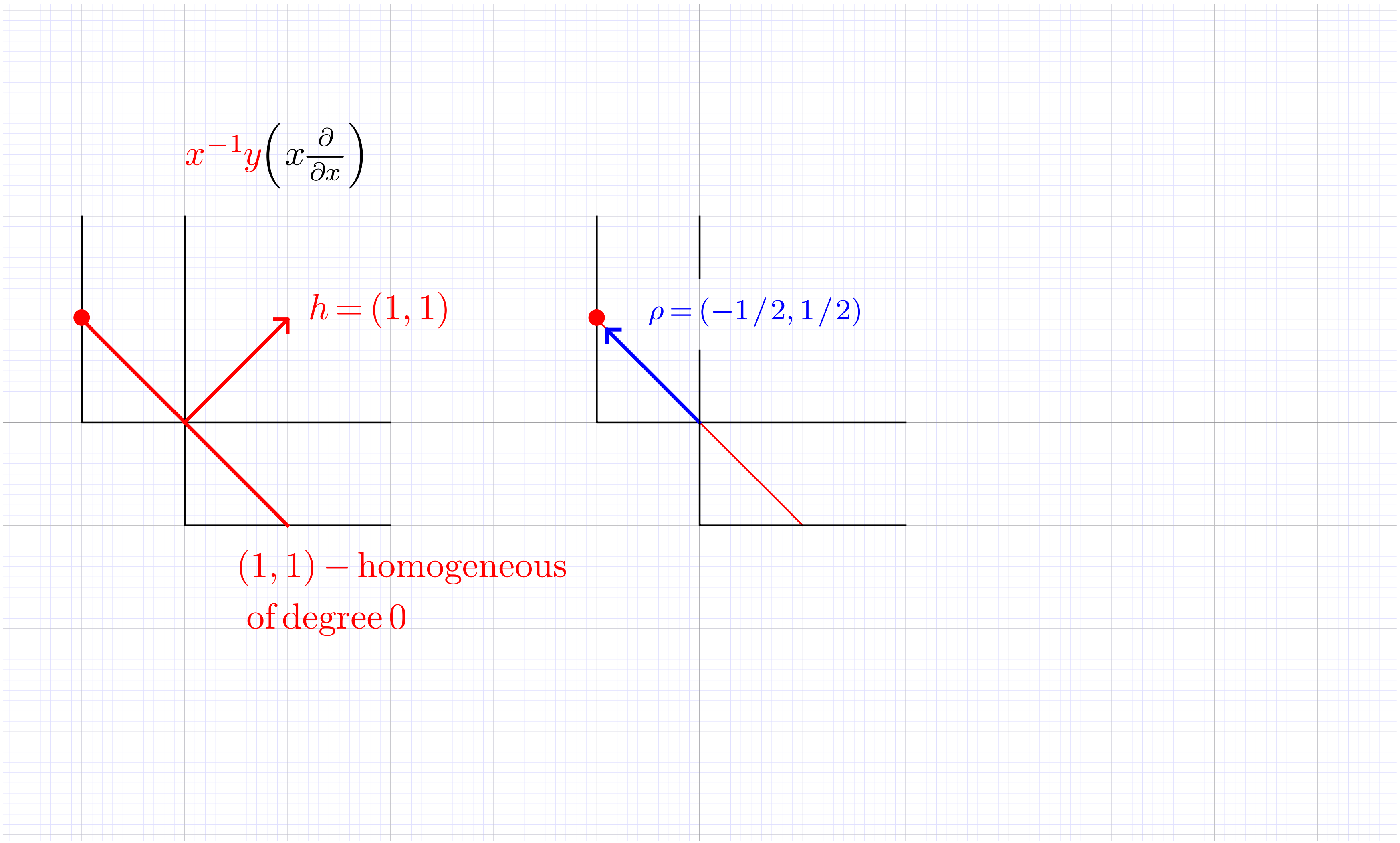

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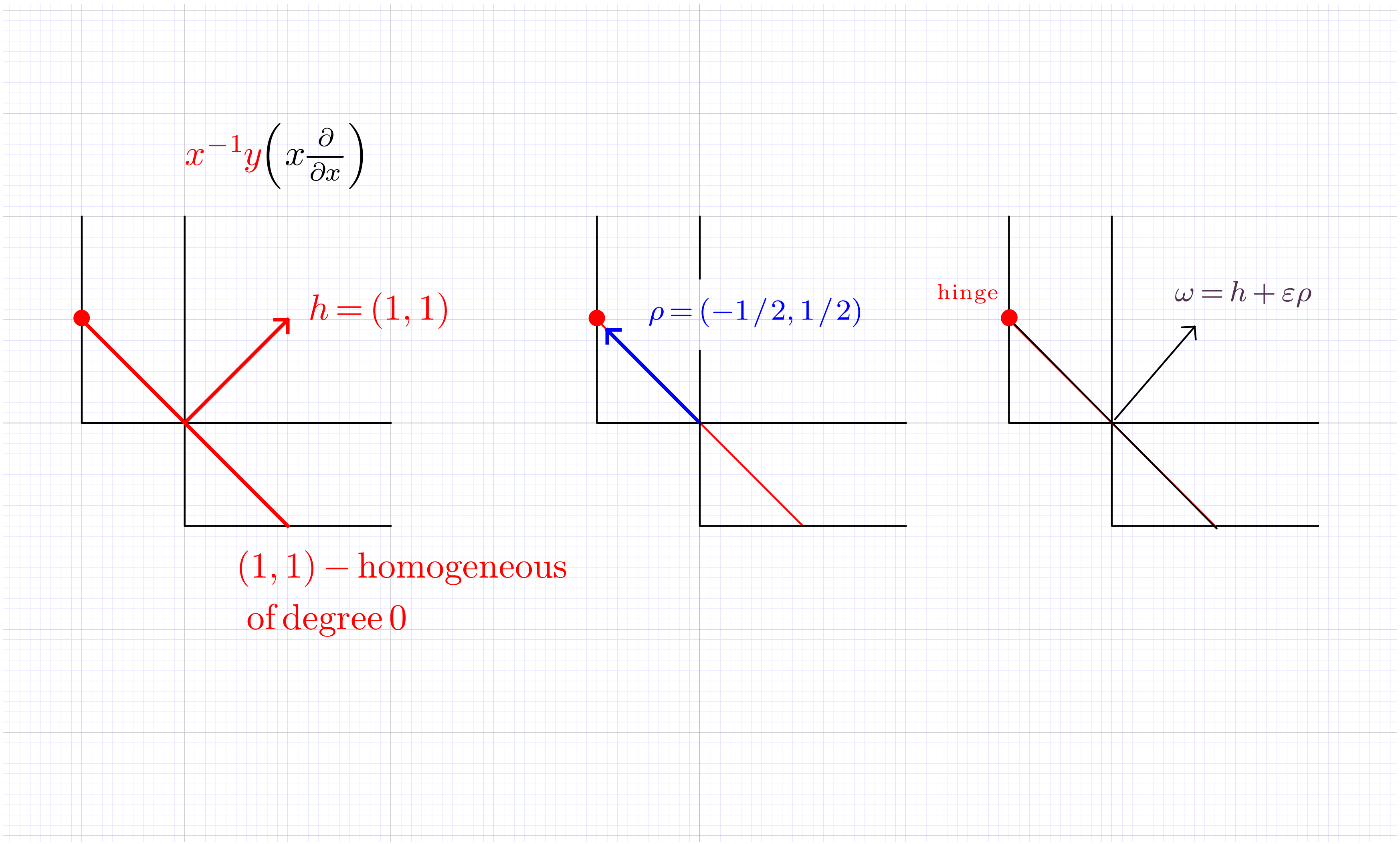

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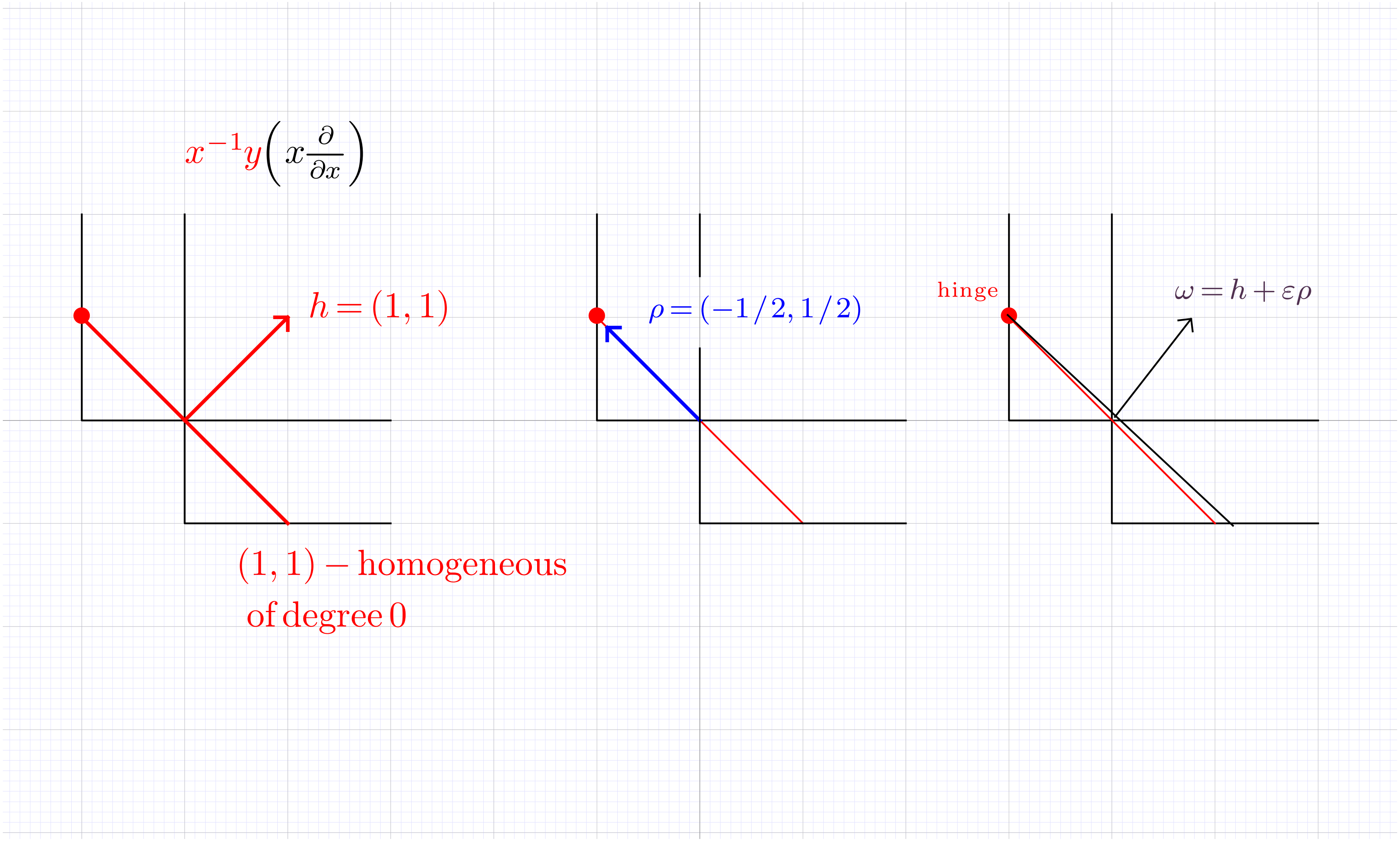

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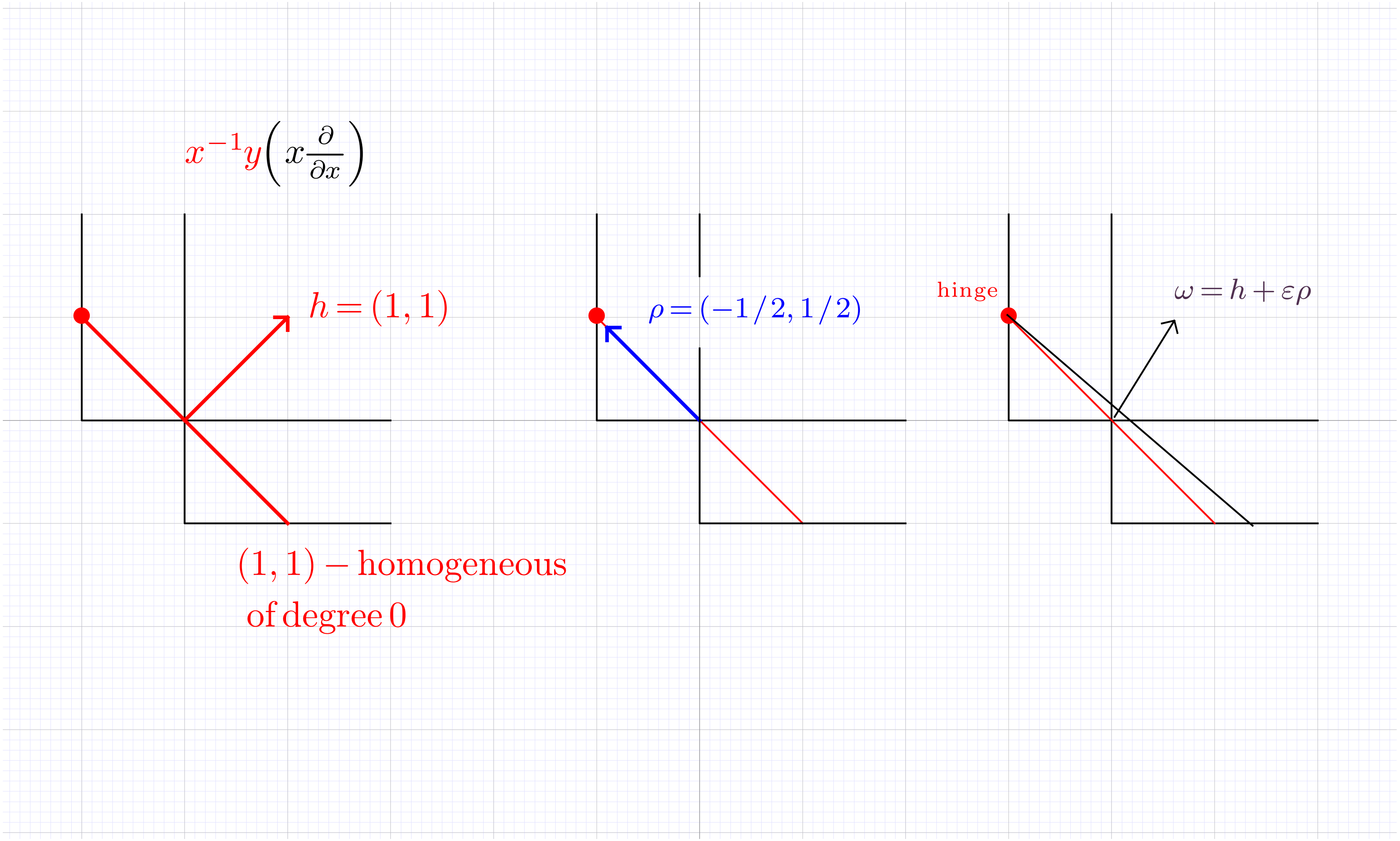

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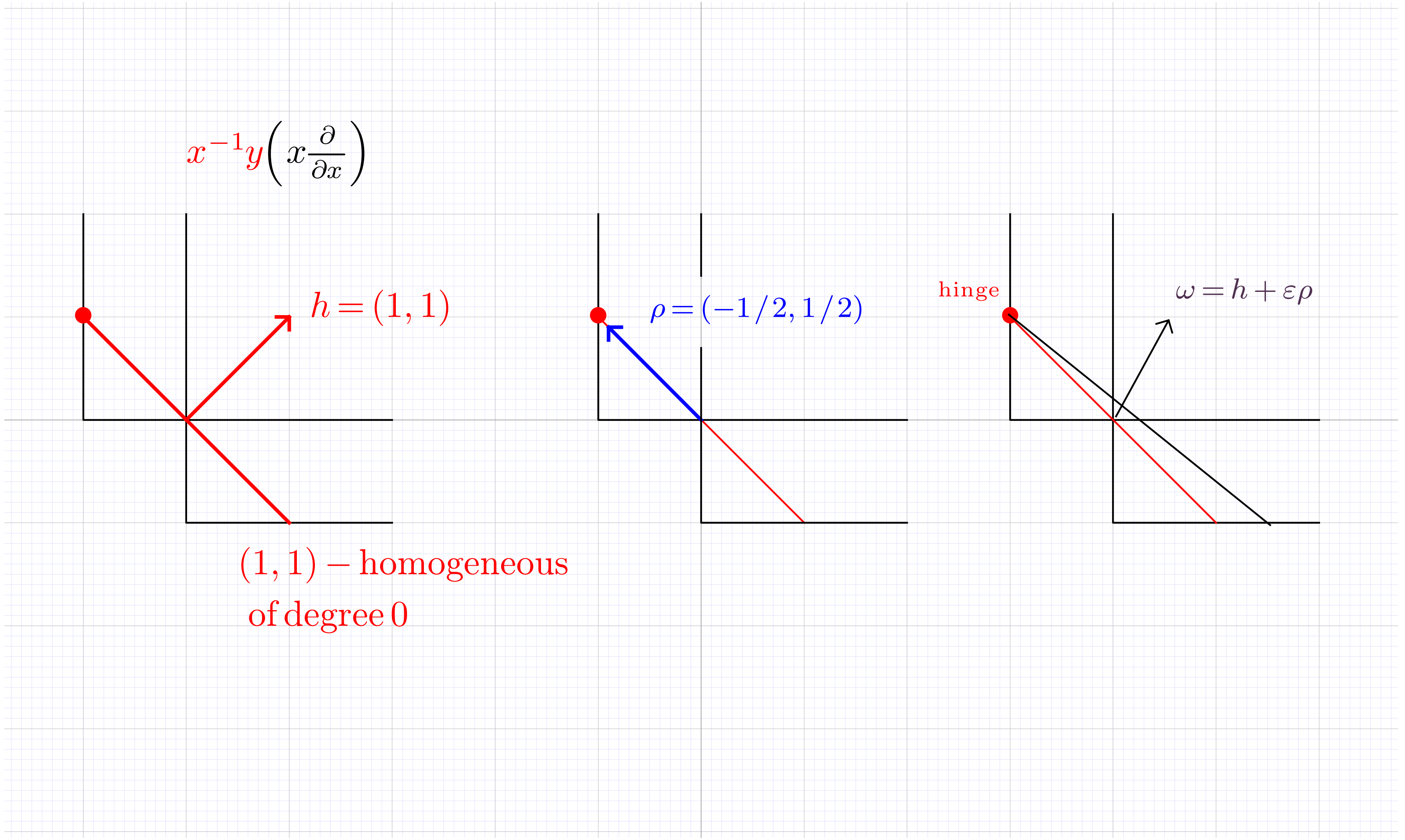

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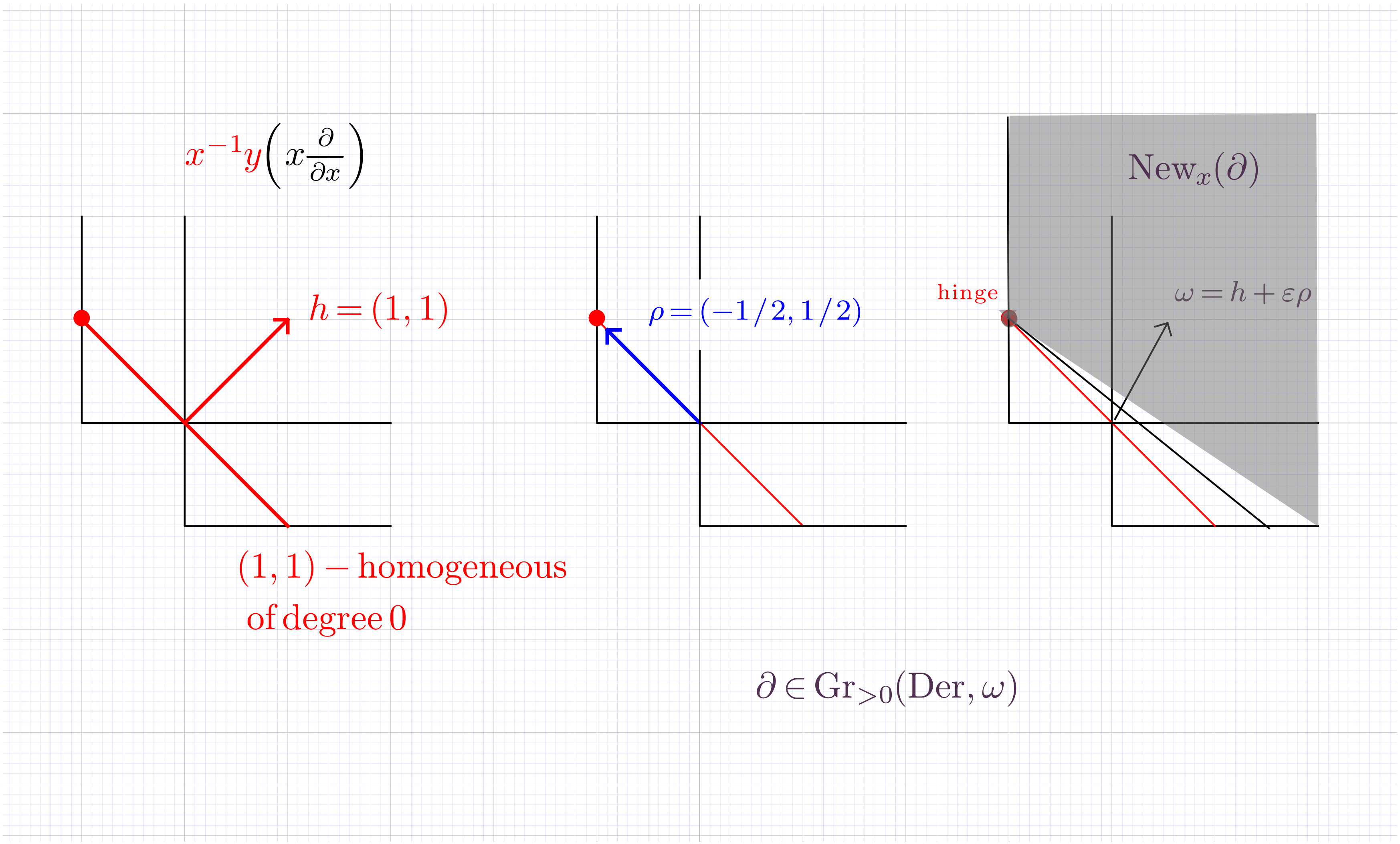


Alternative proof of one of the implications of the Theorem

Claim: Suppose that ∂ is elementary (i.e. not-nilpotent). Then, for all choices of coordinate systems $(x_1, \ldots, x_n)$,

$$0 \in \mathrm{New}_x(\partial)$$

Indeed, the hypothesis means that either $\boxed{\partial(\mathfrak{m}) \not\subset \mathfrak{m}}$ or that $\boxed{\partial(\mathfrak{m}) \subset \mathfrak{m} \text{ and } \partial_s \neq 0}$.

Consider the second case. Then we can find a nonzero $f \in \hat{\mathfrak{m}}$ such that

$$\partial(f) = u f$$

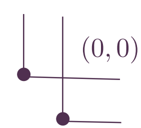
for some unit $u \in \hat{\mathcal{O}}$.

Let Gr be the graduation defined by an arbitrary one-parameter group λ, with positive weights (i.e. such that $\hat{\mathfrak{m}} \subset \mathrm{Gr}_{\geqslant 0}$). Then $f \in \mathrm{Gr}_{\geqslant \alpha}$ and $\partial \in \mathrm{Gr}_{\geqslant \beta}$ implies that $\partial(f) \in \mathrm{Gr}_{\geqslant \alpha + \beta}$.

By the above choice of f, we conclude that $\beta = 0$ (because $u \in \mathrm{Gr}_{\geqslant 0} \setminus \mathrm{Gr}_{\geqslant 1}$)

The case $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ is even easier. In fact, $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ if and only if

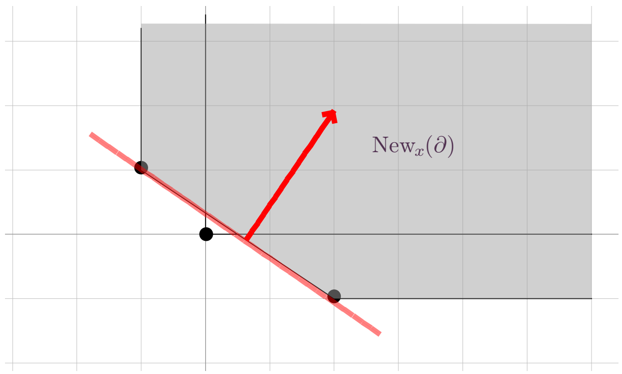
$$\exists i \in \{1, \ldots n\}: \quad -e_i = (0, \ldots, -1, \ldots, 0) \in \mathrm{New}_x(\partial)$$



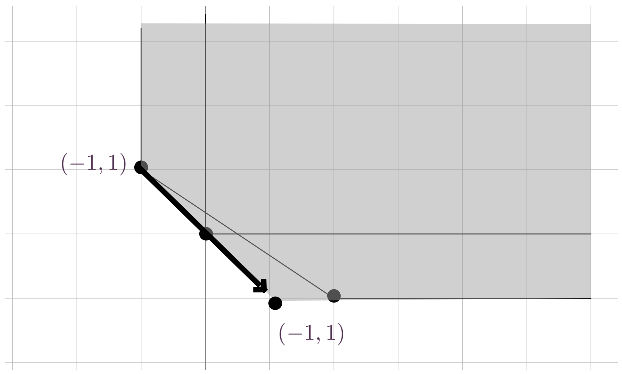
Example: $\partial = y\,\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$. The graduation defined by the one parameter group

$$t\cdot(x,y) = (t^2x, t^3y)$$

is such that $\partial \in \mathrm{Gr}_{\geqslant 1}$. (write $x^{-1}y\left(x\frac{\partial}{\partial x}\right) + x^2y^{-1}\left(y\frac{\partial}{\partial y}\right)$ and $x^{-1}y, x^2y^{-1} \in \mathrm{Gr}_1$)



If we make the coordinate change $y \to y + \xi x$, the action on the polygon corresponds to a "sliding" of the vertices along the $(1,-1)$ direction.



In these new coordinates, $0 \in \mathrm{New}_{(x,y)}(\partial)$.

Back to the proof in dimension two

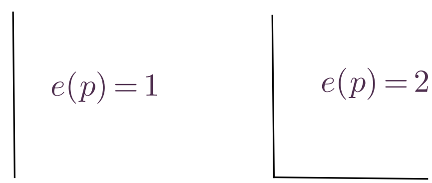
Initial setup: $(M, E, \mathcal{F})$, where M is a two-dimensional real analytic manifold with corners,

$$\text{boundary}(M) = E$$

is a normal crossings divisor and $\mathcal{F}$ is a foliation tangent to E

Notation: $0 \leqslant e(p) \leqslant 2$ is the number of local irreducible componets of E at $p \in M$.

Definition: A coordinate system (x, y) at $p \in E$ is **adapted** if locally $E = (x = 0)$ or $E = (x y = 0)$.

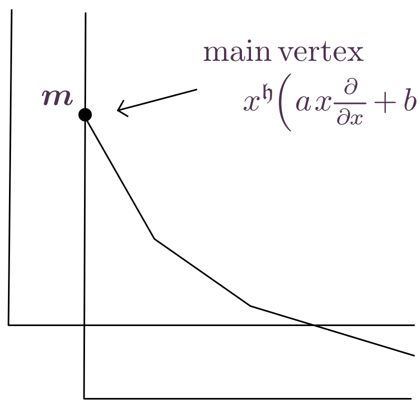


Inclusion into the divisor: We can always assume that $\text{Nilp}(M, \mathcal{F}) \subset E$ by eventually blowing-up these points with an arbitrary weight.

To simplify, we will assume that $e(p) = 1$ for all points $p \in \text{Nilp}(M, \mathcal{F})$.

(otherwise it suffices to slightly modify the invariant by including $e(p)$ lexicographically).

Suppose that $p \in E$. In adapted coordinates (x, y), the Newton polygon has the form

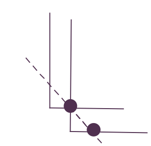


Invariance of $(x=0)$ implies that $\partial \in \mathrm{Gr}_{\geqslant 0}\left(\cdot, x\frac{\partial}{\partial x}\right)$ (i.e. $\partial(\langle x\rangle) \subset \langle x\rangle$)

Claim: $p \in \mathrm{Nilp}(M, \partial) \Longleftrightarrow \mathfrak{h} \geqslant 1$ (which is equivalent to say that $0 \notin \mathrm{New}_{(x,y)}(\partial)$)

Indeed, if $0 \in \mathrm{New}(\partial)$ then the initial (1,1)-homogeneous part of ∂ would be either

$$b\frac{\partial}{\partial y} \quad (\text{case } h=-1), \quad \text{or} \quad ax\frac{\partial}{\partial x}+by\frac{\partial}{\partial y}+cxy^{-1}\left(y\frac{\partial}{\partial y}\right) \quad (\text{case } h=0)$$



where for some constants a, b, c such that $(a, b) \neq (0, 0)$.

- In the first case, $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$.
- In this second case, it is obvious that $\mathrm{Spec}(\partial_s) = \{a, b\} \neq 0$.

Definition: $\mathfrak{h}_x(\partial) := \mathfrak{h}$ will be called the **height of the main vertex**.

Claim: $\mathfrak{h}$ is does not depend on the choice of (adapted) coordinates.

In fact, the group of local automorphisms (preserving $x=0$) has the form

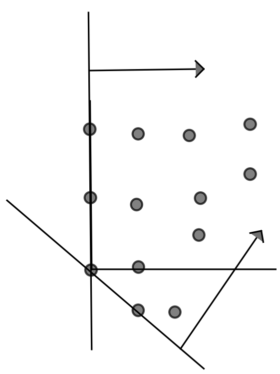
$$x \to x\, f(x,y), \quad y \to g(x,y)$$

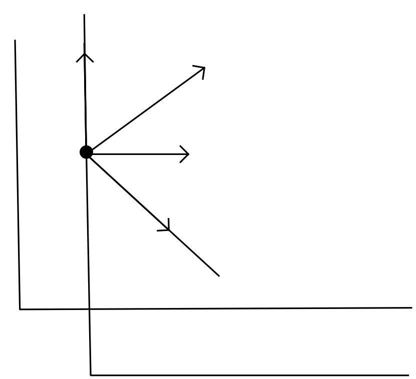
f unit, $\partial g / \partial y(0,0) \neq 0$. Its Lie algebra is generated by vector fields with support in

$$x^k y^l\, x\frac{\partial}{\partial x}, \quad x^u y^v\, y\frac{\partial}{\partial y}, \quad k+l \geqslant 0, u+v \geqslant 0$$

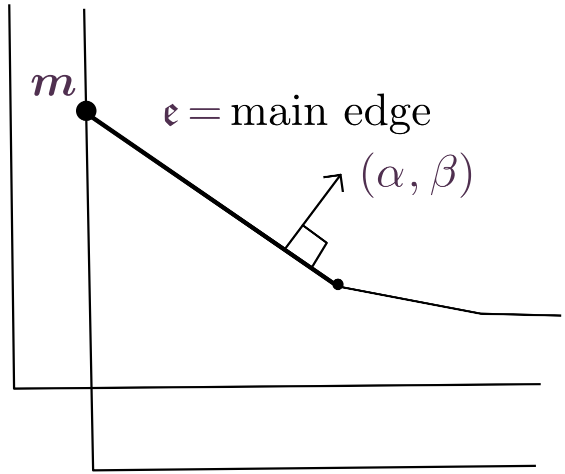
$k,l \geqslant 0$ and $v \geqslant -1$. This Lie algebra lies in $\mathrm{Gr}_{\geqslant 0}\left(\cdot, x\frac{\partial}{\partial x}\right) \cap \mathrm{Gr}_{\geqslant 0}\left(\cdot, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$.

Hence, the main vertex is preserved.





Final touch (and essential idea to generalize to dim.3)



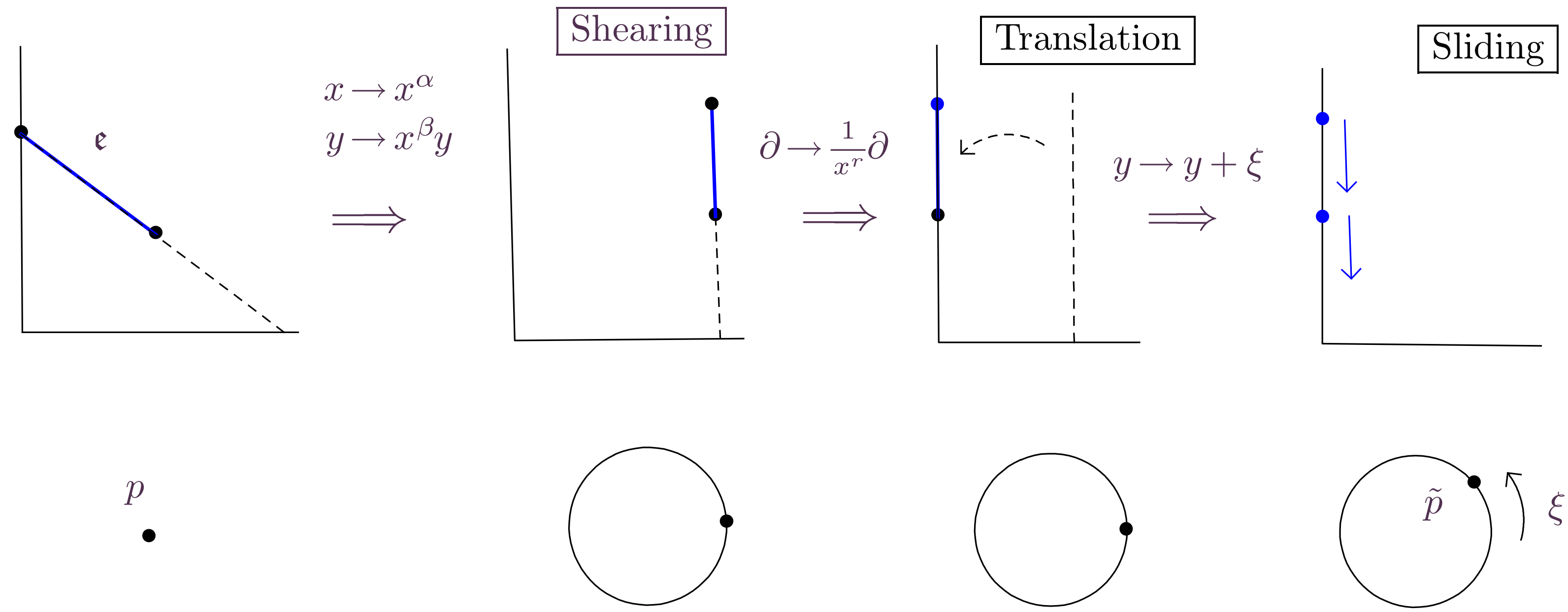
Definition: The main edge of $\mathrm{New}(\partial)$ is the edge $\mathfrak{e}$ determined by the intersection

$$\mathrm{New} \cap \{(i, j) \colon j = \mathfrak{h} - 1/2\}$$

Important: Notice that $\mathfrak{e} = \mathfrak{e}_x(\partial)$ potentially depends on the choice of coordinates.

Let $\mathrm{wt}(\mathfrak{e}) = \alpha x \frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial x}$ denote the irreducible weight-vector determined by $\mathfrak{e}$.

Action of the blowing-up with weight (α, β) on the polygon



We would like to prove that, for each $\tilde{p} \in \Phi^{-1}(p)$, $\tilde{\mathfrak{h}} \leqslant \mathfrak{h} - 1$ (i.e. the invariant decreases)

This is obvious for $\tilde{p} = (1:0) \in \mathbb{P}^1$... But...

We can have a full compensation phenomena in the “sliding phase”.

Example: $\partial = (y+\xi x^k)^{\mathfrak{h}}\Big(\lambda\Big(x\frac{\partial}{\partial x} - \xi k x^k \frac{\partial}{\partial y}\Big) + \mu(y+\xi x^k)\frac{\partial}{\partial y}\Big)$, $(\lambda,\mu)\neq 0$, $\mathfrak{h}, k \geqslant 1, \xi \neq 0$

(up to a unit, this is the **unique** family where full compensation happens)

$\partial \in \mathrm{Gr}_{k\mathfrak{h}}(\mathrm{Der},\omega)$, where $\omega = x\frac{\partial}{\partial x} + k y\frac{\partial}{\partial y}$ (i.e. $[\omega,\partial] = k\mathfrak{h}\,\partial$)

Blow-up: $x \to x, \quad y \to x^k y$

$$(y+\xi x^k)^{\mathfrak{h}} \longrightarrow x^{k\mathfrak{h}}\,(y+\xi)^{\mathfrak{h}}$$

$$x\frac{\partial}{\partial x} \longrightarrow \left(x\frac{\partial}{\partial x} - k y \frac{\partial}{\partial y}\right), \qquad x^k\frac{\partial}{\partial y} \longrightarrow \frac{\partial}{\partial y}$$

$$(y+\xi x^k)\frac{\partial}{\partial y} \longrightarrow (y+\xi)\frac{\partial}{\partial y}$$

$$\tilde{\partial} = x^{-k\mathfrak{h}}\partial = (y+\xi)^{\mathfrak{h}}\left(\lambda x\frac{\partial}{\partial x} + (\mu - k\lambda)(y+\xi)\frac{\partial}{\partial y}\right)$$

Translation $y \to y - \xi$

$$\tilde{\partial} = y^{\mathfrak{h}}\left(\mu x\frac{\partial}{\partial x} + (\lambda - k\mu) y \frac{\partial}{\partial y}\right) \Longrightarrow \boxed{\tilde{\mathfrak{h}} = \mathfrak{h}}$$

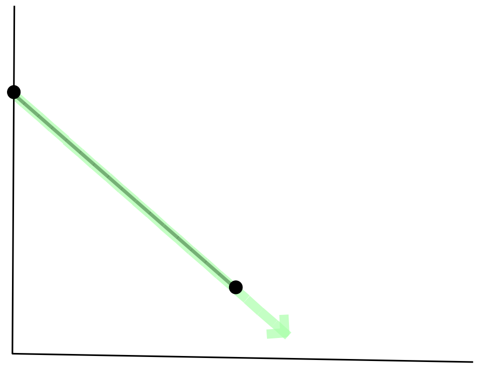
How to prevent this? The main edge $\mathfrak{e}$ should be stable.

Definition. We say that $\mathrm{New}_{(x,y)}(\partial)$ is **edge-unstable** if there exists a polynomial change of coordinates of the form

$$y \to y + \xi x^{\frac{\beta}{\alpha}} =: y_1$$

such that $\mathrm{New}_{(x,y_1)}(\partial) \cap \mathfrak{e} = \{\boldsymbol{m}\}$. Otherwise, we say that $\mathrm{New}_{(x,y)}(\partial)$ are **edge-stable**.

Notice that $\mathrm{New}_{(x,y)}(\partial)$ is always edge-stable if $\beta/\alpha \notin \mathbb{Z}_{\geqslant 1}$.



The above map slides the monomials in the direction of the main edge.

Theorem (Local resolution) Suppose that $\mathrm{New}_{(x,y)}(\partial)$ is edge stable, and let

$$\Phi\colon \tilde{M} \to M$$

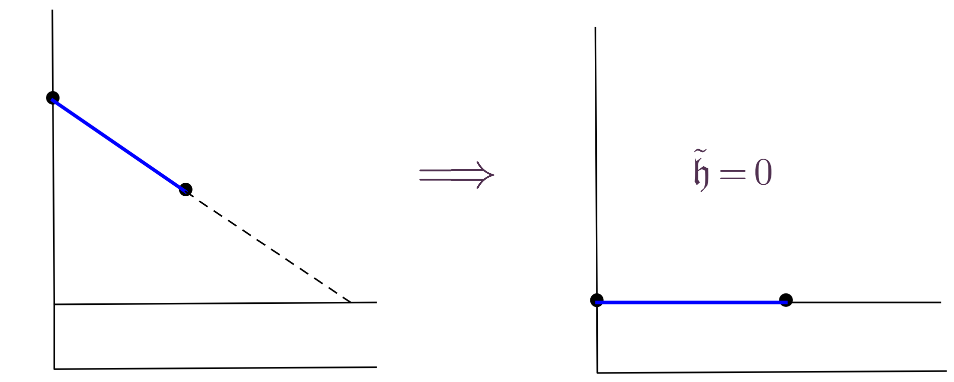
be the blowing-up of $p \in \mathrm{Nilp}(M, \mathcal{F})$ with weight $\mathrm{wt}(\mathfrak{e})$. Then,

$$\forall \tilde{p} \in \Phi^{-1}(p)\colon \quad \tilde{\mathfrak{h}} \leqslant \mathfrak{h} - 1.$$

(very simple) Proof: Firstly, we do not have to care about the y-directional chart

$$x \to y^{\alpha} x, \qquad y \to y^{\beta}$$

as $\tilde{p} = (0\colon 1)$ will always be elementary.



We look the x-directional chart

$$x \to x^{\alpha}, \quad y \to x^{\beta} y$$

Suppose that $\tilde{\mathfrak{h}} = \mathfrak{h}$. Then, there should exists a non-zero constant ξ such that the translation (in blowed-up coordinates)

$$y \to y + \xi$$

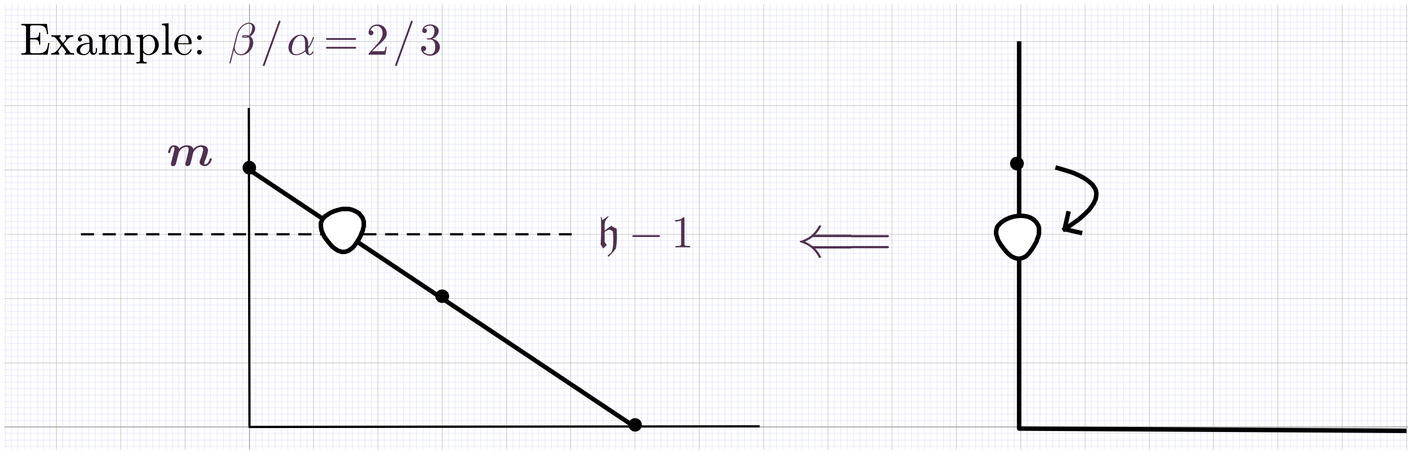
gives a Newton polyhedron with main vertex $\tilde{\boldsymbol{m}} = \boldsymbol{m}$. We split into two cases:

- $\beta / \alpha \in \mathbb{Q}_{>0} \setminus \mathbb{Z}_{>0}$.
- $\beta / \alpha \in \mathbb{Z}_{>0}$

In the latter case, the above map corresponds (in the original coordinates), to the polynomial map $y \to y + \xi x^{\beta/\alpha}$ (just write $y \to x^{-\beta} y, x \to x^{1/\alpha}$).

The assumption $\tilde{\mathfrak{h}} = \mathfrak{h}$ is equivalent to say that $\mathrm{New}_{(x,y)}(\partial)$ is edge-unstable, which contradicts the hypothesis of the Theorem.

In the former case ($i.e.\, \beta/\alpha \notin \mathbb{Z}$), the $\mathfrak{e}$-initial form of ∂ has a gap at height $\mathfrak{h}-1$.



After blowing-up, followed by an arbitrary translation $y \to y+\xi$, we have

$$(y^{\mathfrak{h}})\left(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}\right)+\text{terms in } y^{\leqslant \mathfrak{h}-2} \longrightarrow (y+\xi)^{\mathfrak{h}}\left(\alpha x\frac{\partial}{\partial x}+\beta(y+\xi)\frac{\partial}{\partial y}\right)+\cdots$$

which gives a monomial on the support at height $\mathfrak{h}-1$.

(Abhyankar called this argument the "lazy Tschirnhaussen").

What is "behind" this argument?

To simplify, let us look at the case of function germs:

$f\in\mathcal{O}_p$ is "elementary" iff f is a unit (i.e. iff $0\in\mathrm{New}_{(x,y)}(f)$).

Supposing that $\boldsymbol{m}=(0,\mathfrak{h})$, the $\mathfrak{e}$-initial part of f is a (α,β)-homogeneous polynomial

$$f_{\mathfrak{e}}=c y^{\mathfrak{h}}+\sum_{\substack{\alpha i+\beta j=d\\ j\geqslant 1}} c_{ij}x^i y^j$$

(i.e. $f_{\mathfrak{e}}$ is a section a line (orbi)-bundle $\mathcal{L}\to\mathbb{P}^1_{(\alpha,\beta)}$, equal to $\mathcal{O}_{\mathbb{P}^1}(d)$ in the classical homogeneous case).

We can look at the divisor $\mathrm{Div}(f_{\mathfrak{e}})=\sum m_i[\xi_i]$ on $\mathbb{P}^1_{(\alpha,\beta)}$ (write $f_{\mathfrak{e}}(1,y)=\prod(y-\xi_i)^{m_i}$)

The choice of $\mathfrak{e}$ implies that $\mathrm{Div}(f_{\mathfrak{e}})\neq\mathfrak{h}[(1:0)]$. (i.e. the support of the divisor is not concentrated at $[(1:0)]$)

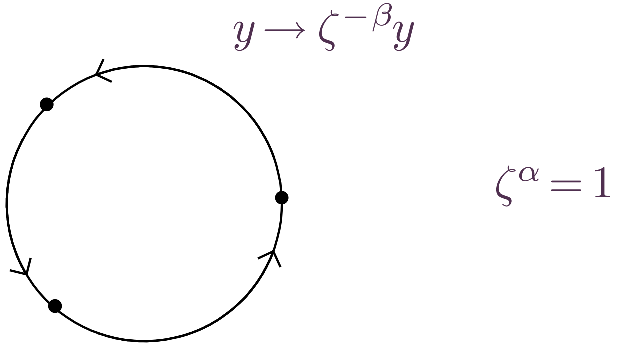
$\mathrm{New}_{(x,y)}(f)$ is **edge-unstable** iff $\mathrm{Div}(f_{\mathfrak{e}})=\mathfrak{h}\,[\xi]$ (i.e. the support of the divisor is a point $\xi\neq(1:0)$. In this case:

(1) This point is <u>necessarily unique</u> and,

(2) $\beta/\alpha\notin\mathbb{Z}_{>0}$

Simply because there is a $\mathbb{Z}/\alpha\mathbb{Z}$-symmetry on the divisor.

Symmetry breaking



It remains to prove that the following

Theorem (on edge stabilization)

(Existence) There exists adapted coordinates (x, y) such that

$$\mathrm{New}_{(x,y)}(\partial)$$

is edge stable.

(Uniqueness of the associated filtration) Let $(x, y), (x', y')$ be coordinates such that $\mathrm{New}_{(x,y)}(\partial)$ and $\mathrm{New}_{(x',y')}(\partial)$ are edge stable. Then the local resolution algorithm (i.e. the local filtration of the local ring) defined throught these coordinates coïncide.

In other words, the filtration is intrinsically determined by ∂ (and the divisor E).

Proof: We start with an arbitrary adapted coordinate system (x, y_0).

1) If $\mathrm{New}_{(x,y_0)}(\partial)$ is edge-stable, we stop

2) If $\mathrm{New}_{(x,y_0)}(\partial)$ is edge-unstable, we choose a polynomial coordinate change $(x, y_0) \to (x, y_1)$, where

$$y_1 = y_0 + \xi_0 \, x^{k_0}, \qquad k_0 = \beta_0 / \alpha_0$$

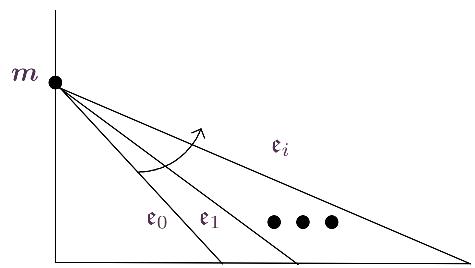
eliminates the main edge $\mathfrak{e}_0$.

We now consider the new coordinates (x, y_1) and apply the same argument. I claim that this procedure eventually stops with an edge stable situation.

Indeed, assume the contrary. Then, we end-up with an infinite sequence of coordinate changes

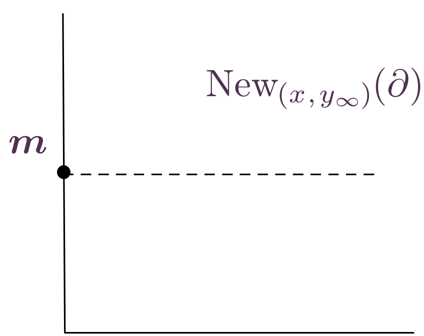
$$y_{i+1} = y_i + \xi_i x^{k_i}, \quad i \geqslant 1$$

where $\{k_i = \beta_i / \alpha_i\}$ forms an strictly increasing sequence of integers, corresponding to the successive slopes of the edges $\mathfrak{e}_i$.



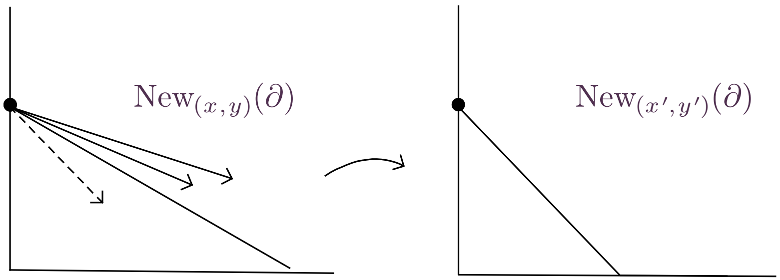
The composition of these maps converges to a formal coordinate change $\widehat{y_\infty} = y_0 + \sum \xi_i x^{ki}$

In these coordinates,



i.e. $(\widehat{y_\infty} = 0) \subset \mathrm{Nilp}(M, \mathcal{F})$. Contradiction.

Uniqueness of the filtration. Suppose that $\mathrm{New}_{(x,y)}(\partial)$, $\mathrm{New}_{(x',y'')}(\partial)$ are edge stable



write $x' = x\,f(x,y)$, $y' = g(x,y)$. We claim that this map preserves the $\mathrm{wt}(\mathfrak{e})$ filtration.

Let us write $g(x,y) = g_0(x) + yG(x,y)$. Then, the change of coordinates preserves the filtration if and only if

$$g_0(x) = O\left(x^{\frac{\beta}{\alpha}}\right)$$

Suppose that this is not the case. Then, looking at the smallest order term of g_0, we find a polynomial change of coordinates $y_1 = y + \xi x^k$ with $\xi \neq 0$ and $\boldsymbol{k} < \frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}}$ such that

$$\mathrm{New}_{(x,y_1)}(\partial)$$

is has a main edge $\mathfrak{e}'$ of slope $k < \beta/\alpha$ (because the action of $y \to y + \xi x^k$ on $\mathrm{New}_{(x,y)}(\partial)$ is effective).

However, $\mathrm{New}_{(x,y_1)}(\partial)$ should also be edge-stable.

(because the $(1,k)$-initial part of ∂ with respect to (x,y_1) equals its $\mathfrak{e}'$-initial part with respect to (x',y'), which is stable by the hypothesis).

But this contradicts the fact that the **inverse transformation** $y = y_1 - \xi x^k$ eliminates the main edge.

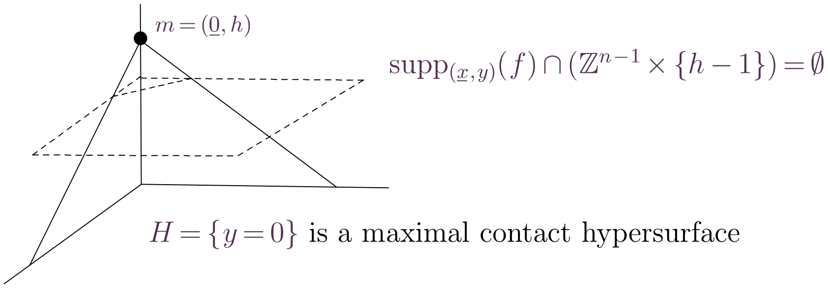
Some general remarks:

1) We cannot expect to obtain a **fully convergent Tchirnhaussen preparation** (or, more generally, a maximal contact hypersurface which would allow to use induction in the dimension)

Recall that, in the classical case of a germ of singular hypersurface S, this corresponds to choose a local equation of the form

$$f(\underline{x},y)=y^h+\sum a_i(\underline{x})y^{h-i}$$

and eliminate the term in y^{h-1} by the local change of coordinates $y\to y-\frac{1}{h}a_1$ (Tschirnhaussen transformation)



As a consequence, simply because $\boxed{(\partial/\partial y)^{h-1}f=y}$, the *multiplicity h-locus* $\mathrm{Sing}^h(f)$ is contained in the hypersurface $H=\{y=0\}$

and this remains true for all blowings-up with center on $\mathrm{Sing}^h(f)$.

Analogous question for vector fields, say in dim. 2:

$$\partial = y^h\left(ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y}\right) + \sum y^{h-i}a(x)$$

The differential operator $\left(\frac{\partial}{\partial y}\right)$ acts on $\mathrm{Der}(\mathcal{O})$ by Lie brackets.

$$\delta = (\mathrm{ad}_{\partial/\partial y})^h \partial = \left(\left[\frac{\partial}{\partial y}, \cdot\right]\right)^h \partial = \boxed{(h+1)!by\frac{\partial}{\partial y} + h!ax\frac{\partial}{\partial x}} + (\text{terms of higher order})$$

In this situation, the analogous of a maximal contact surface should be

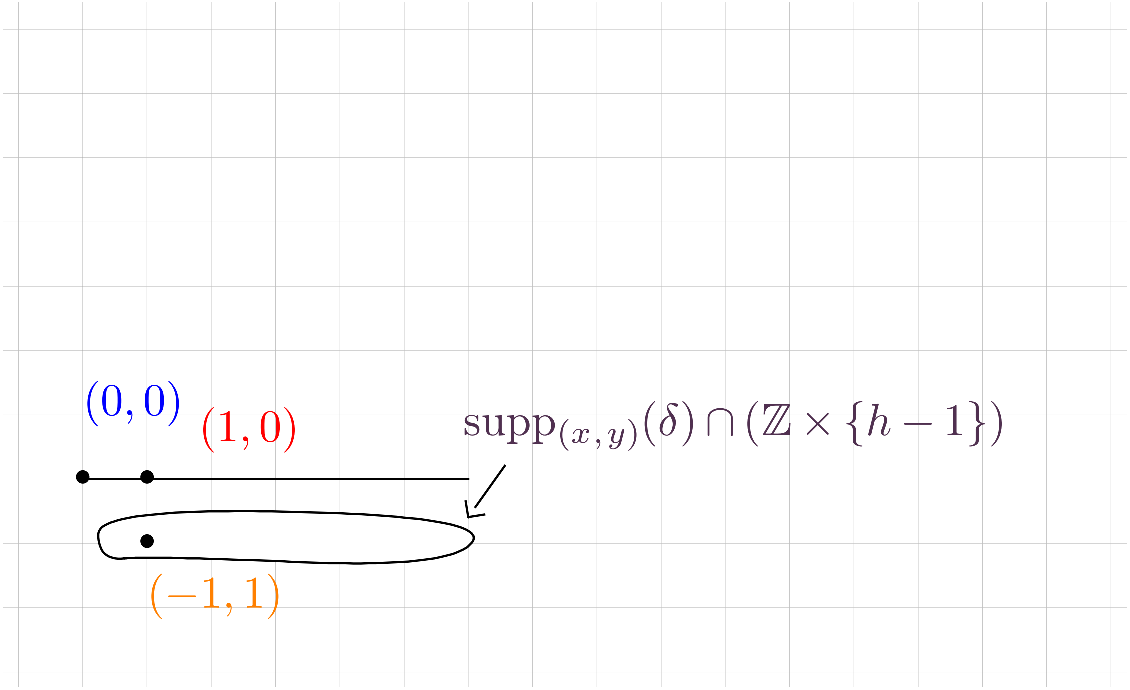
an *invariant curve* for δ of the form $H = \{y = f(x)\}$.

i.e. satisfying

$$\delta(y-f) \subset \langle y-f\rangle$$

Example (Euler's equation): Assume that $\delta = \mathrm{ad}_{(\partial/\partial y)^r}(\partial)$ has the form

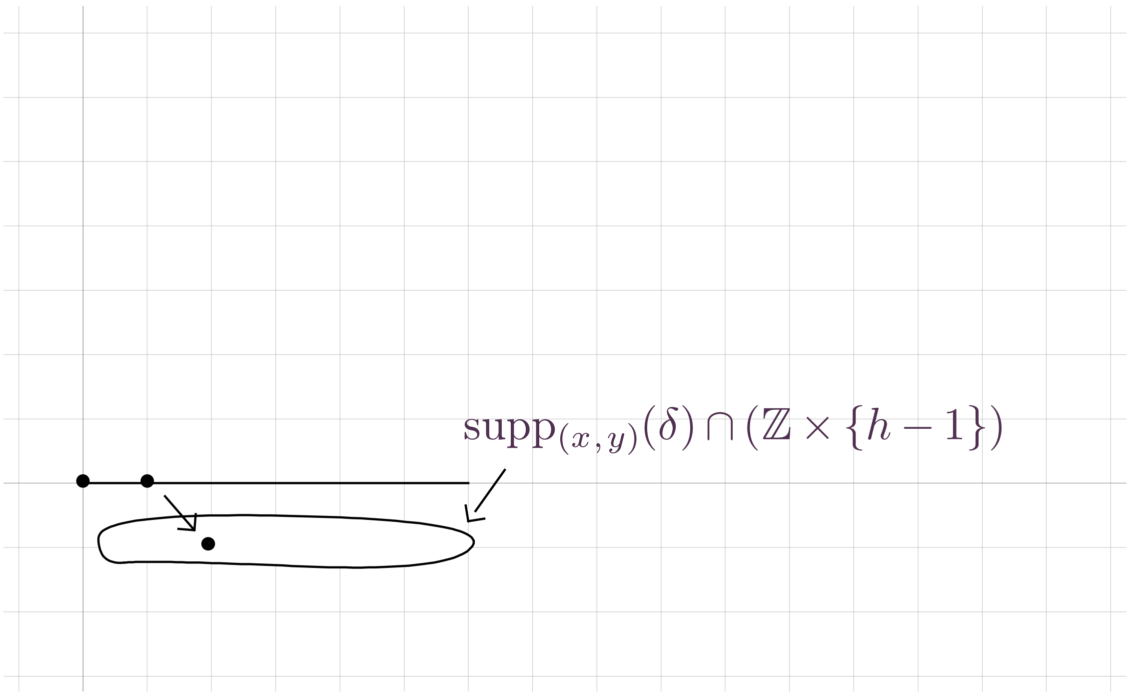
$$\delta = x^2 \frac{\partial}{\partial x} + (y - x)\frac{\partial}{\partial y}$$



$$y \to y - x$$

Example (Euler's equation): Assume that $\delta = \mathrm{ad}_{(\partial/\partial y)^r}(\partial)$ has the form

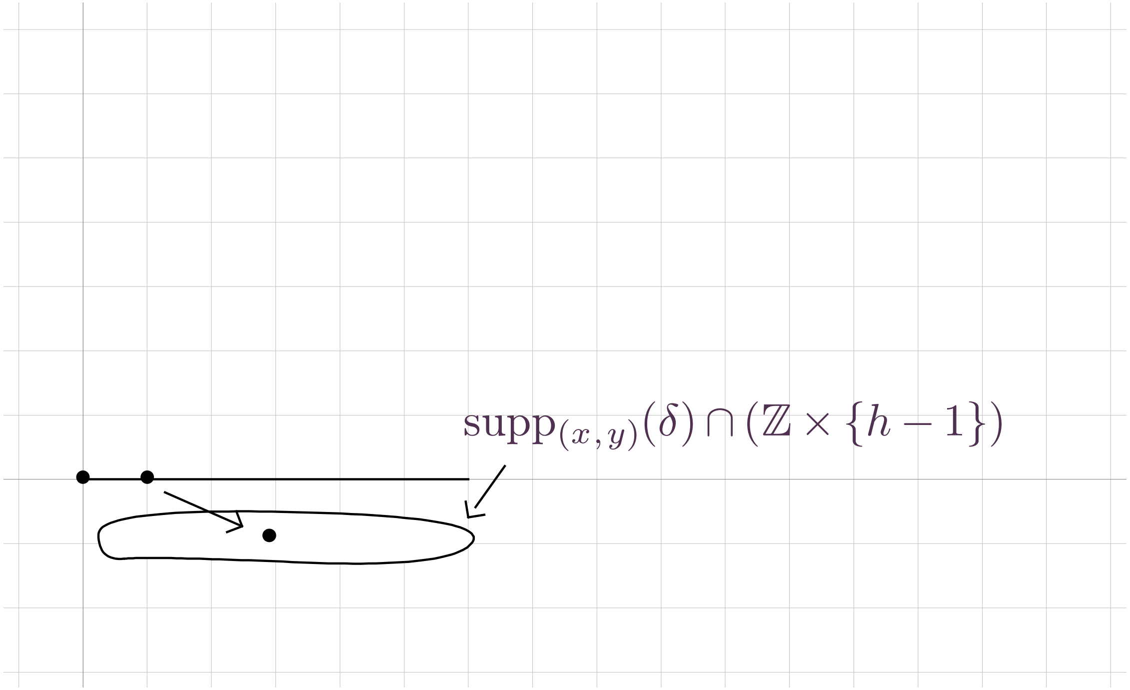
$$\delta = x^2 \frac{\partial}{\partial x} + (y - x^2)\frac{\partial}{\partial y}$$



$$y \to y - x^2$$

Example (Euler's equation): Assume that $\delta = \mathrm{ad}_{(\partial/\partial y)^r}(\partial)$ has the form

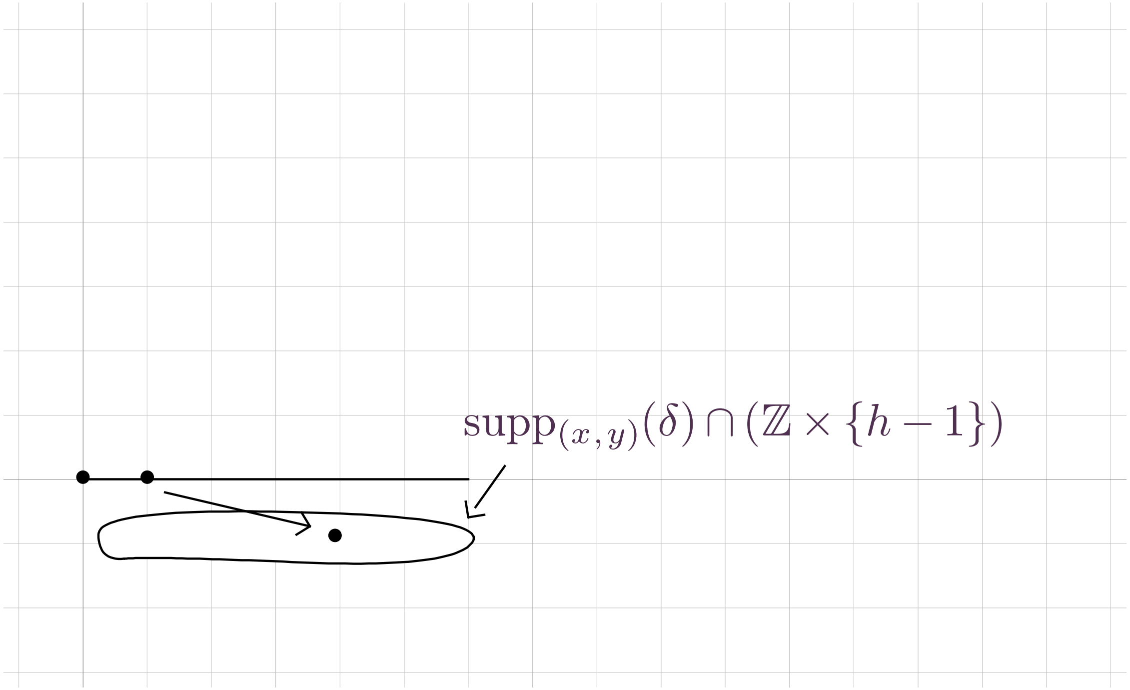
$$\delta = x^2 \frac{\partial}{\partial x} + (y - 2x^3)\frac{\partial}{\partial y}$$



$$y \to y - 2x^3$$

Example (Euler's equation): Assume that $\delta = \mathrm{ad}_{(\partial/\partial y)^r}(\partial)$ has the form

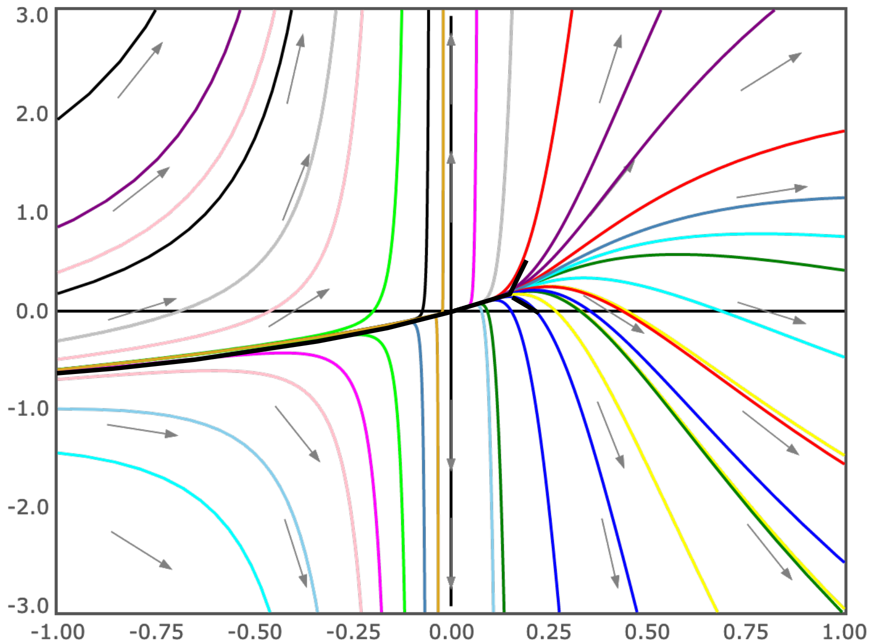
$$\delta = x^2 \frac{\partial}{\partial x} + (y - 3x^4)\frac{\partial}{\partial y}$$



At the “Krull”-limit, we obtain

$$H = \left\{ y = \sum_{n \geqslant 1} (n-1)! x^n \right\}$$

which is the so-called “center manifold” of the Euler’s equation.



In this case, the maximal contact surface is a formal, non-convergent curve.

But which is a C^∞-curve, lying on the pfaffian extension of $\mathbb{R}_{\text{an}}$.

What comes next:

1) How to generalize these ideas to eliminate the nilpotent locus for foliations in dimension three?

2) What to do with the final models in dimension three? (There is no such well developped theory)

3) Interesting particular case for the Hilbert's 16^th problem: The case "2+1".

Attainable goal: stuty of **one-parameter** families of planar analytic foliations.

- Full catalog of final cases
- Study of normal forms
- Finite cyclicity conjecture for one-parameter families of planar analytic foliations.

4) New ideas for dimension greater or equal than four (The Kempf's unstability approach)

Some new phenomena in for final models in dimension three . . .

1) Center manifolds are not necessarily C^∞.

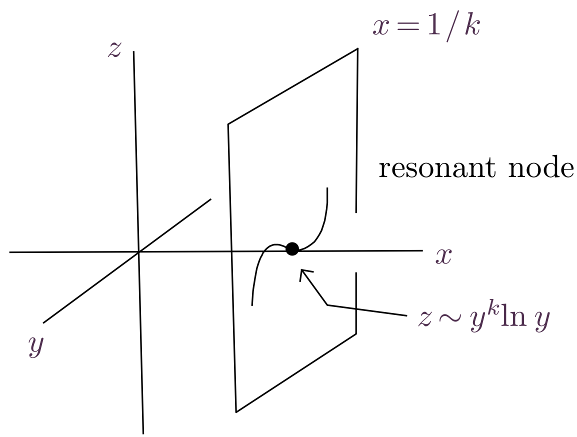
Example (van Strien 1979 - further simplyfied by M. Mcquillan)

a.k.a. “THE MONSTER”

$$\partial = xy\frac{\partial}{\partial y} + \left(z - \frac{y}{1-y}\right)\frac{\partial}{\partial z}$$

$$C = \{z = f(x,y)\}, \qquad \left(1 - xy\frac{\partial}{\partial y}\right)f = \frac{y}{1-y}$$

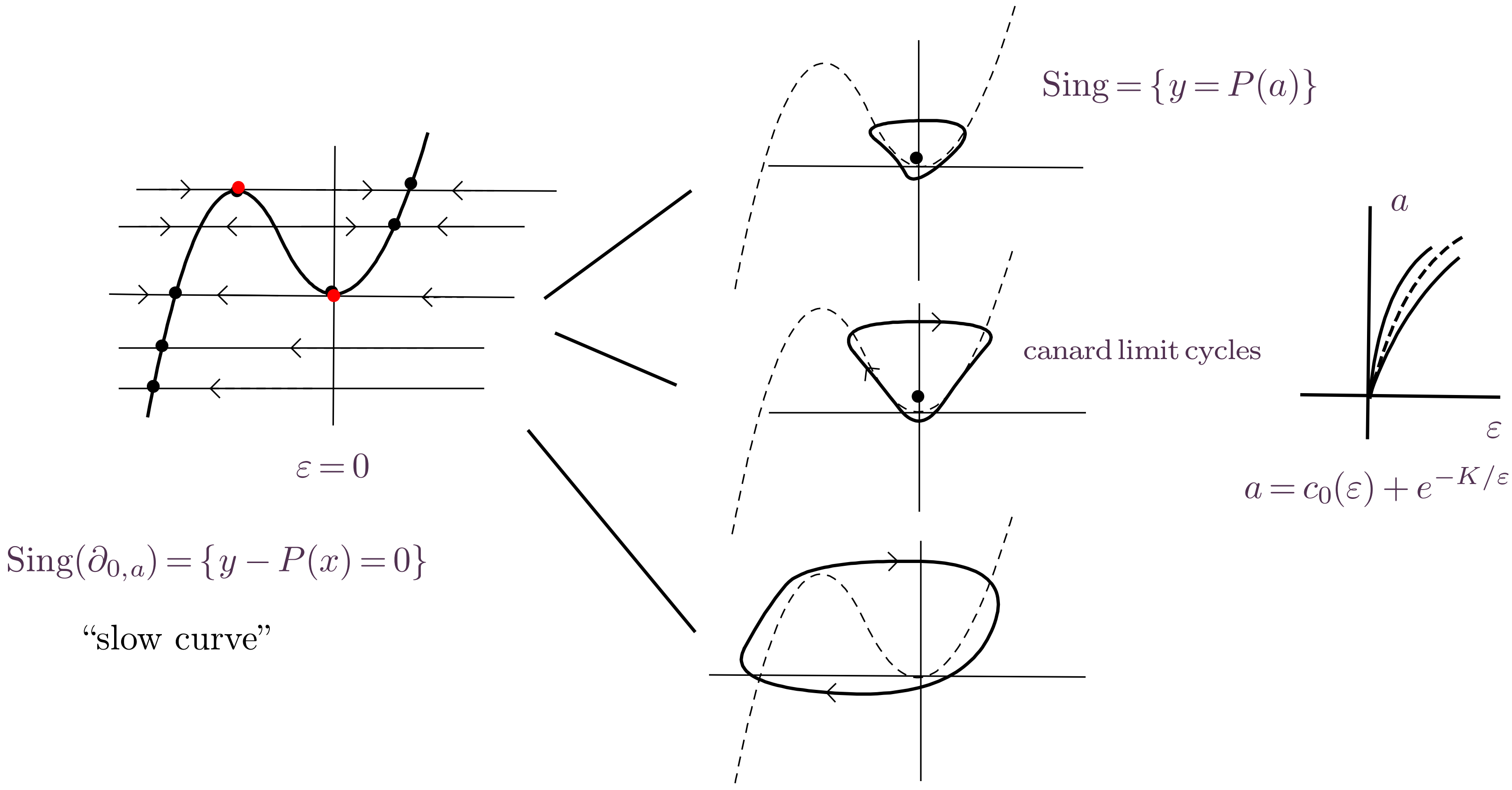
$$f = \sum a_k(x)y^k \Longrightarrow a_k = \frac{1}{1-kx}$$

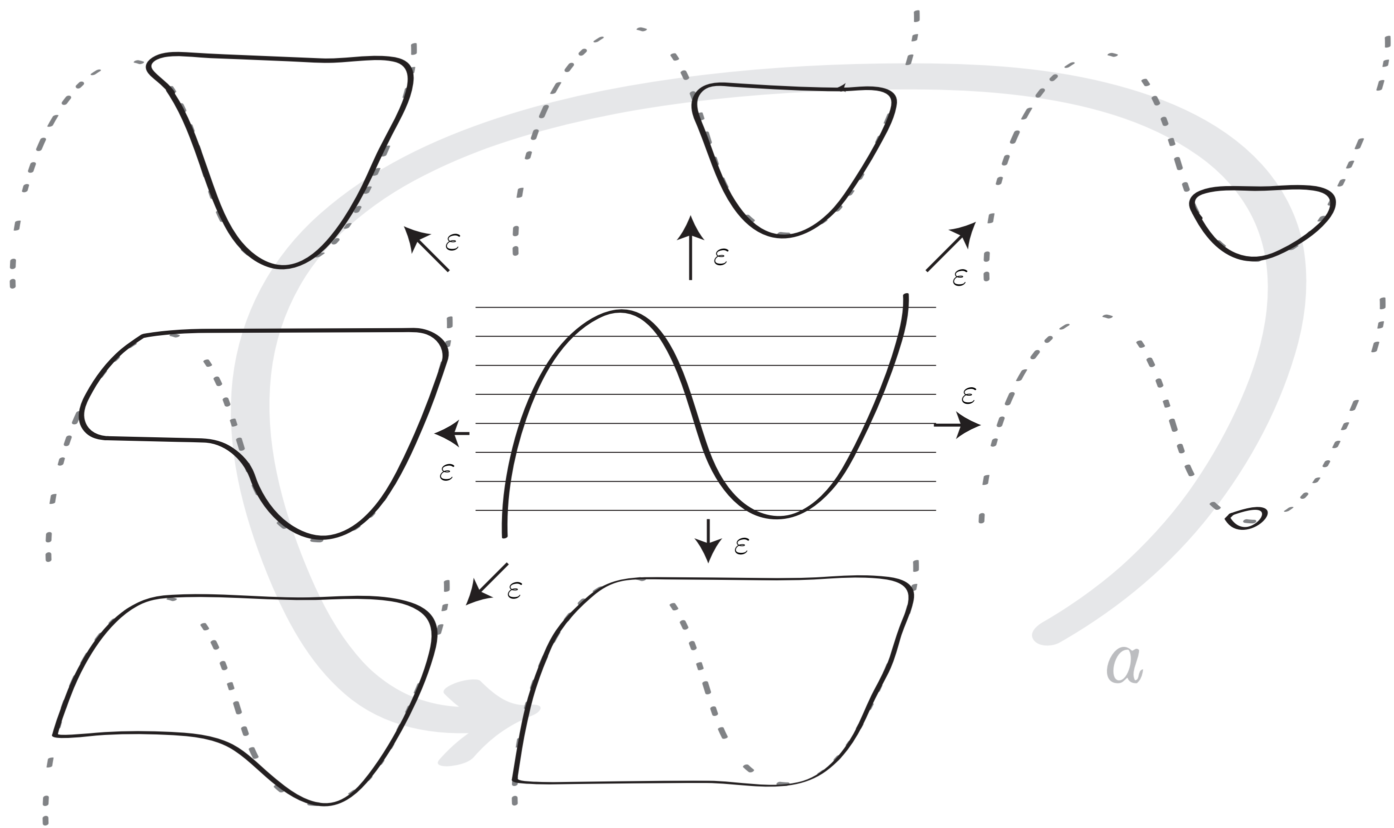


2) Geometric Theory of Singular perturbations (Dumortier-Roussarie)

Example of $(2+1)$ foliations: Singularly perturbed van der Pol's equation

$$\partial_{\varepsilon,a} = \left(y - \frac{x^2}{2} - \frac{x^3}{3}\right)\frac{\partial}{\partial x} + \varepsilon\,(a-x)\frac{\partial}{\partial y}, \qquad (x,y)\in\mathbb{R}^2, \varepsilon\in\mathbb{R}_{\geqslant 0}$$




ε
ε
ε
ε
ε
ε
ε
a

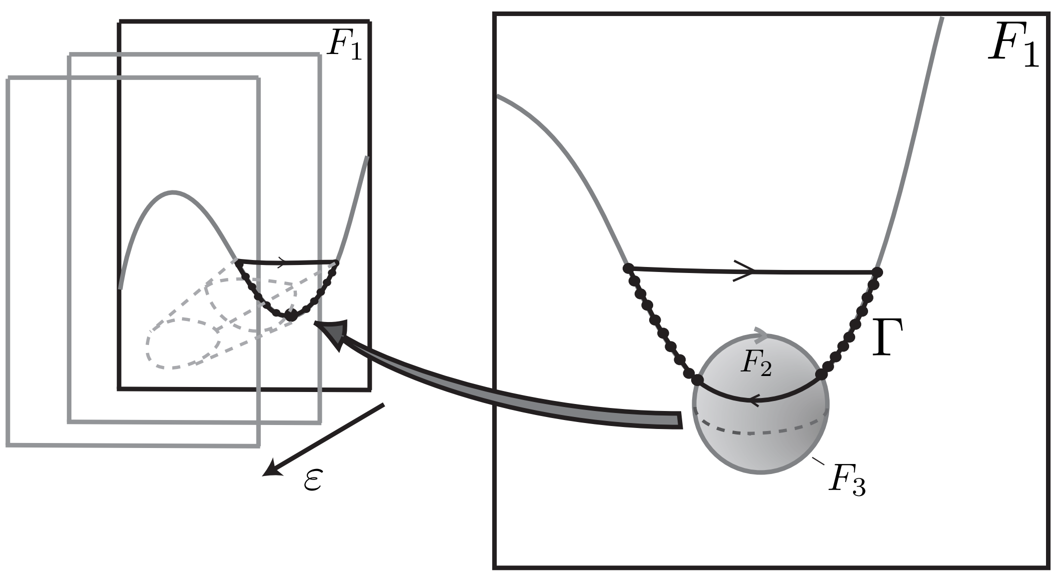
Resolution (in families). We assume $a=0$ to simplify

$$\left(y-\frac{x^2}{2}-\frac{x^3}{3}\right)\frac{\partial}{\partial x}-\varepsilon x\frac{\partial}{\partial y}=\left(x^{-1}y-\frac{x}{2}-\frac{x^2}{3}\right)\left(x\frac{\partial}{\partial x}\right)-\varepsilon x y^{-1}\left(y\frac{\partial}{\partial y}\right)$$

Is a three dimensional foliation **Tangent to the fibration**: $F=\{d\varepsilon=0\}$

Choice of weights: $-\mathrm{wt}(x)+\mathrm{wt}(y)=\mathrm{wt}(x), \quad \mathrm{wt}(\varepsilon)+\mathrm{wt}(x)-\mathrm{wt}(y)=\mathrm{wt}(x)$

$$\mathrm{wt}(x)=1, \mathrm{wt}(y)=2, \mathrm{wt}(\varepsilon)=2$$



Some computations...

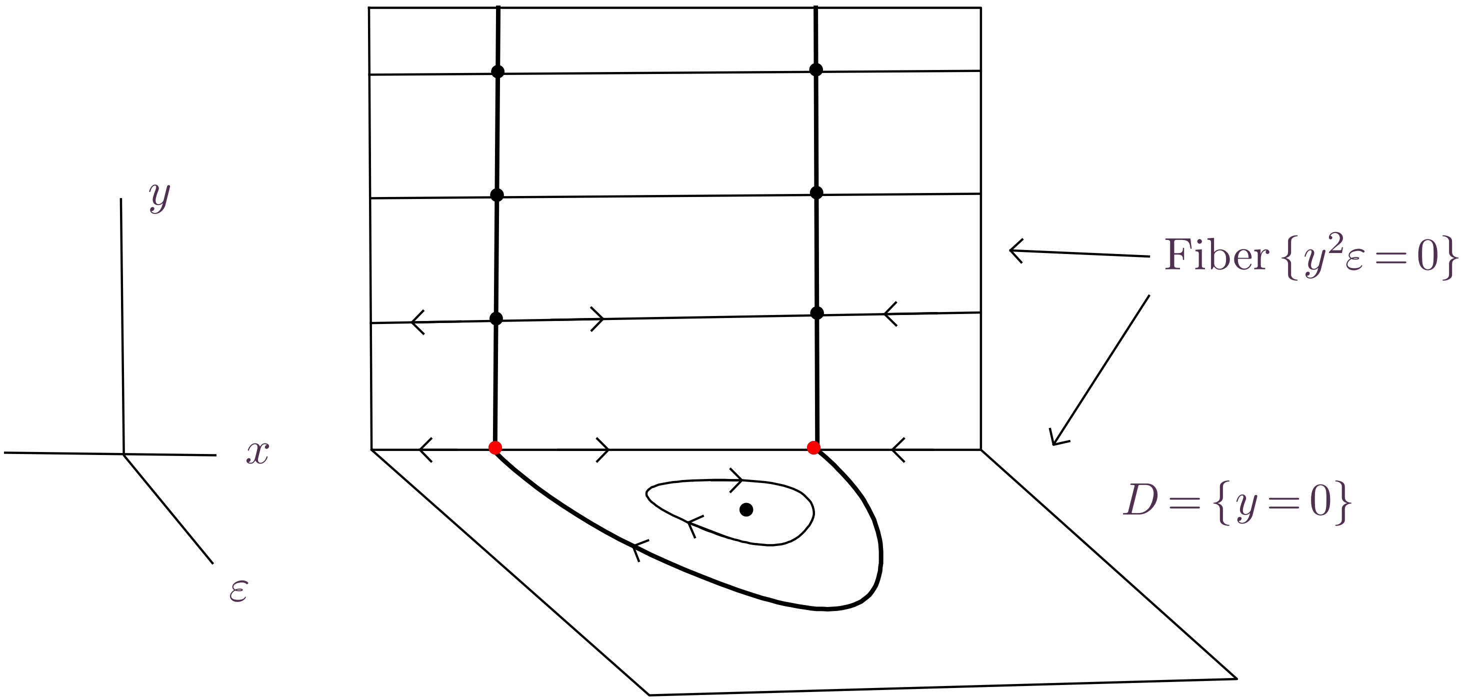
ε − directional blowing up: $x \to \varepsilon x, \quad y \to \varepsilon^2 y, \varepsilon \to \varepsilon^2$

$$\left(y - \frac{x^2}{2} - \frac{\varepsilon x^3}{3}\right)\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}, \quad F = \{d\varepsilon = 0\}$$

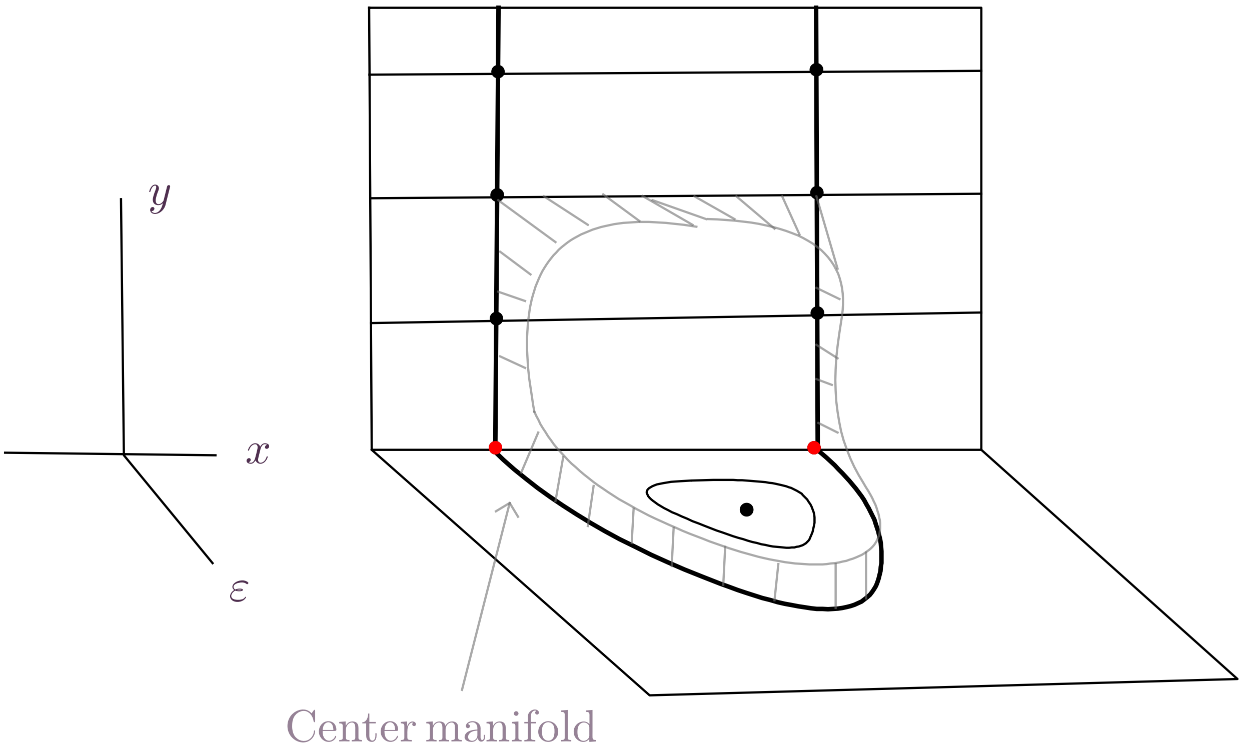
In the classical singular perturbation theory, this is the so-called a *rescaling*.

y − directional blowing up: $x \to yx, \quad y \to y^2 \quad , \varepsilon \to y^2\varepsilon$

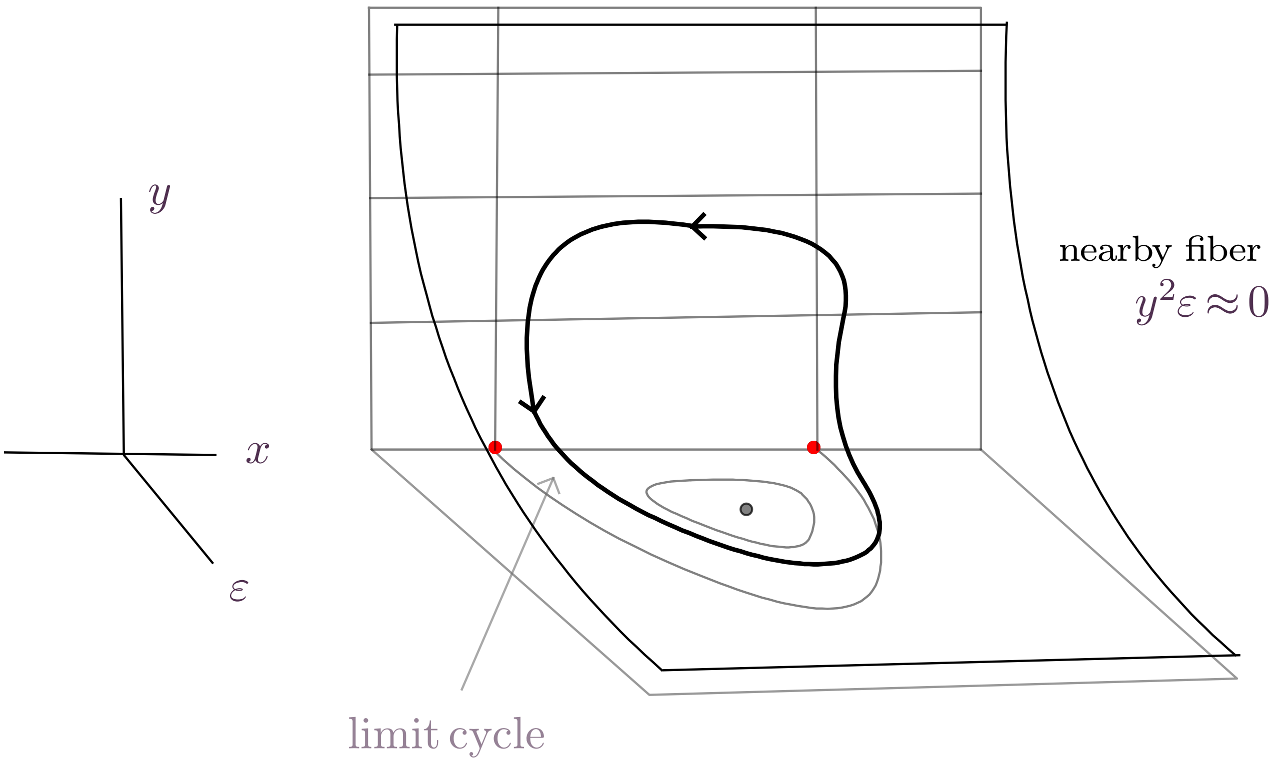
$$\left(1 - \frac{x^2}{2} - \frac{x^3 y}{3}\right)\frac{\partial}{\partial x} - \frac{\varepsilon x}{2}\left(y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} - 2\varepsilon\frac{\partial}{\partial \varepsilon}\right), \quad F = \{d(y^2\varepsilon) = 0\}$$



y
x
ε
Fiber $\{y^2\varepsilon = 0\}$
$D = \{y = 0\}$



(matching of asymptotic expansions)



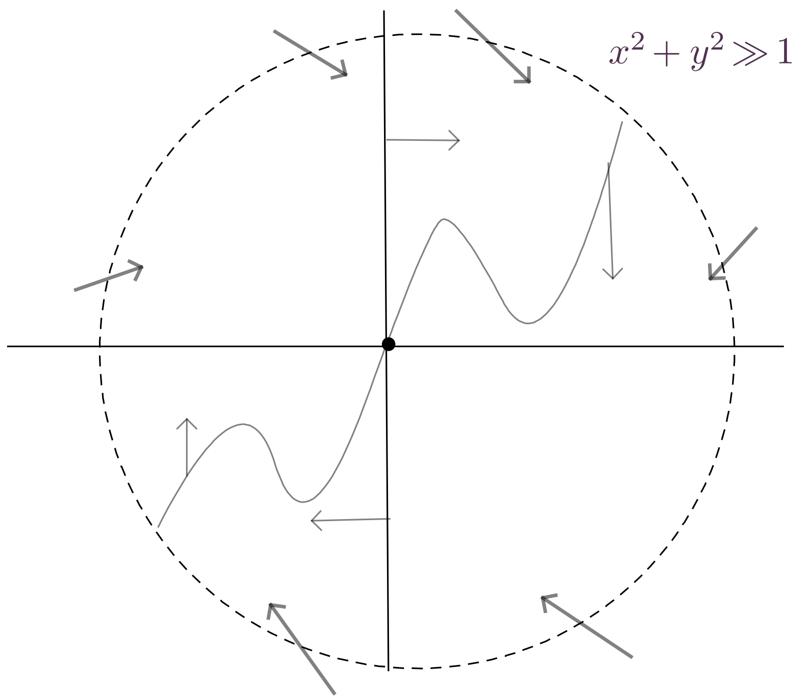
y
x
ε
nearby fiber
$y^2\varepsilon \approx 0$
limit cycle

Smale's 13$^{\text{th}}$ Problem

Prove the finite cyclicity for the Liénard family $x'' + p(x)x' + x = 0$, or equivalently

$$(y - P(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \qquad (\mathrm{Lie}_n)$$

where $P = \int p$ is a **real polynomial of degree $2n+1$** with $P(0) = 0$.



Smale's 13$^{\text{th}}$ Problem (particular case of Hilbert's 16^{th} Problem)

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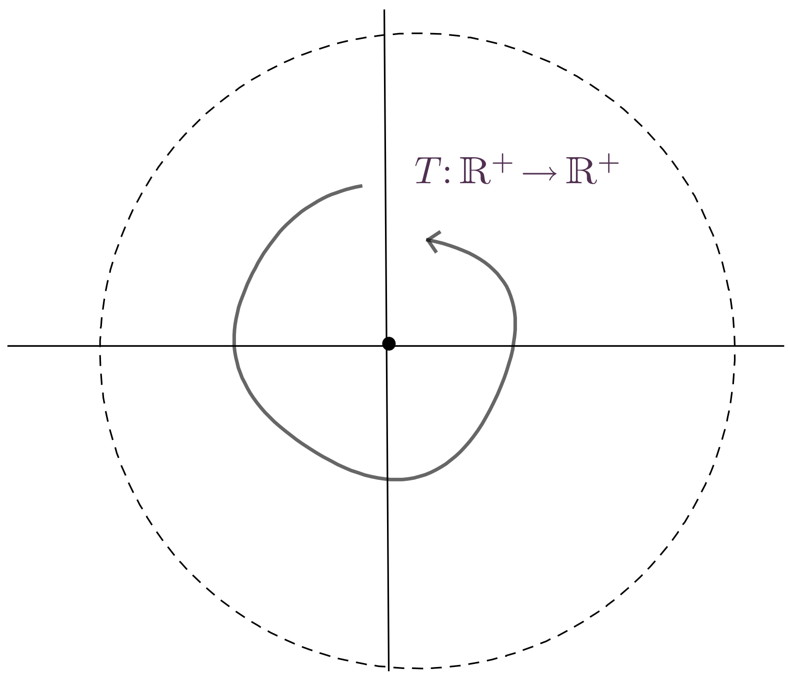
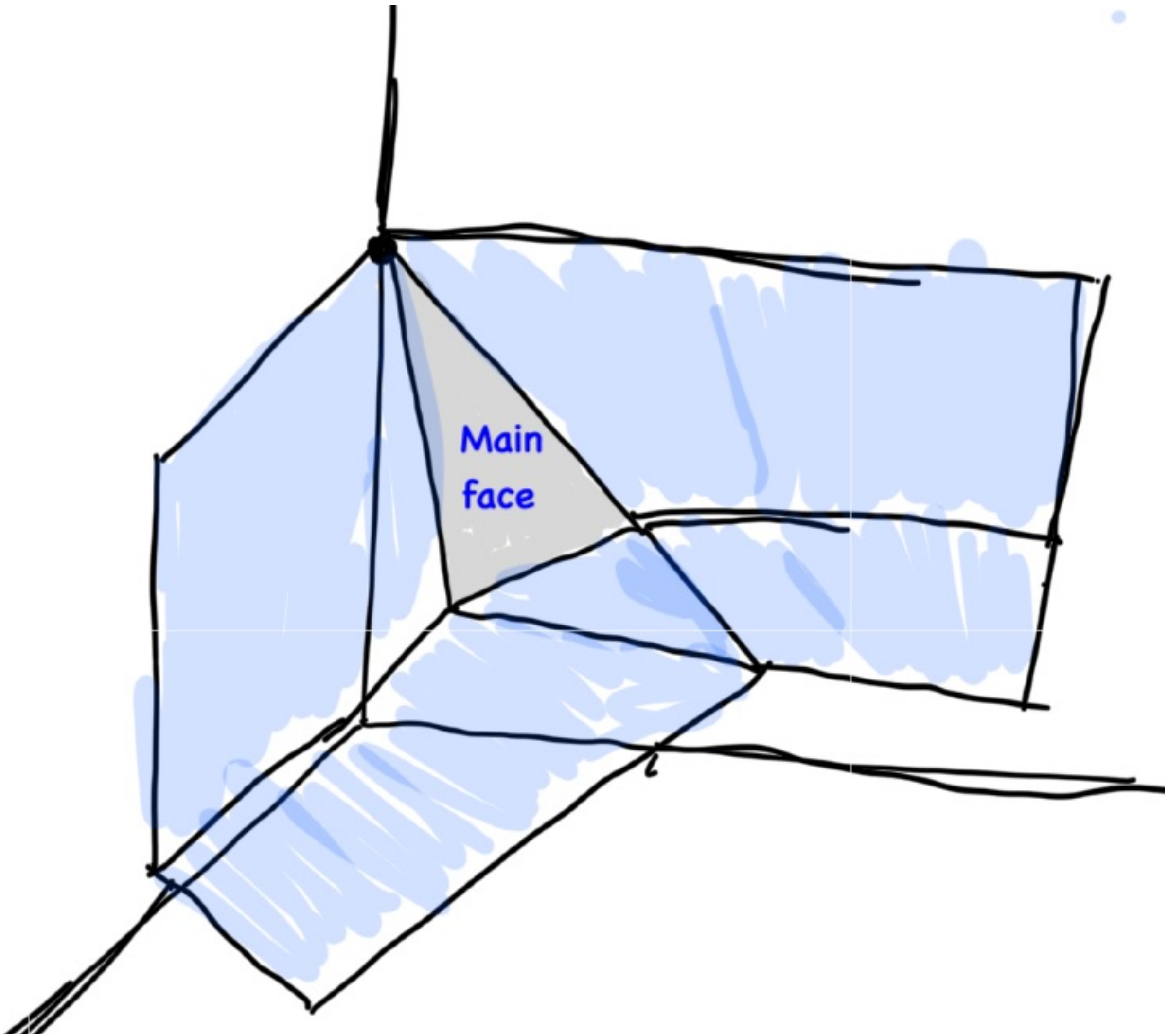


For fixed P, the Poincaré first return map T is **analytic**, up to the origin (for $P(x) = o(x)$)


Main
face

Elimination of nilpotent points in dimension three

$(M, E, \mathcal{F})$

M a three dimensional real analytic manifold with corners

E is the boundary of M

$\mathcal{F}$ is a singular foliation by curves, tangent to E and such that

$$\operatorname{codim} \operatorname{Nilp}(M, \mathcal{F}) \geqslant 2$$

To explain the invariant, let us consider the following **typical** situation

- $M = (\mathbb{R}^3_{(x,y,z)}, 0)$, and that $0 \in \operatorname{Nilp}(M, \mathcal{F})$
- The divisor E is given either by $\{x = 0\}$ or by $\{xy = 0\}$
- The vertical axis $\{x = y = 0\}$ is not entirely contained in $\operatorname{Nilp}(M, \mathcal{F})$

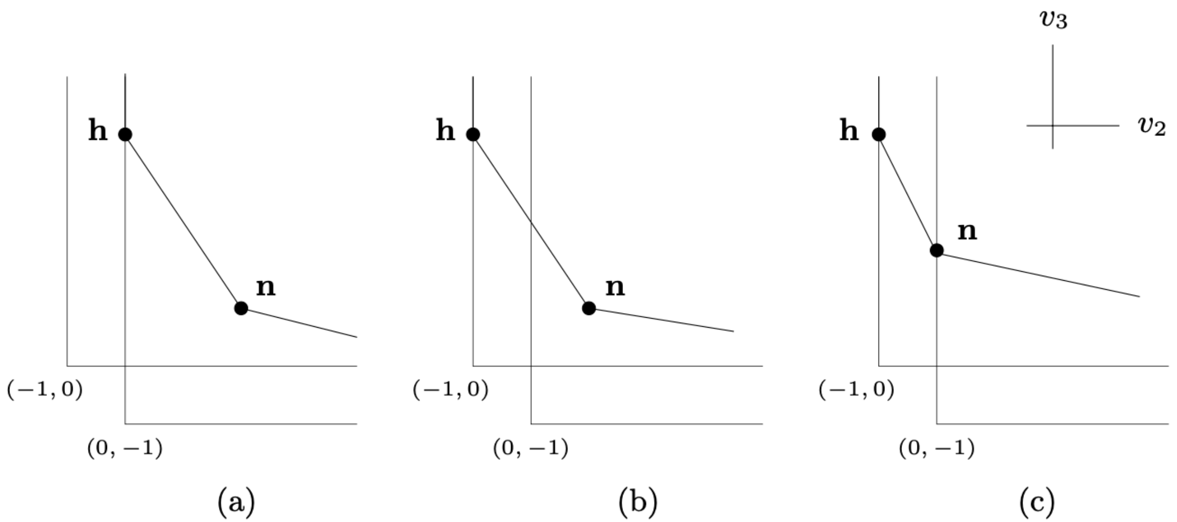
Let $\mathcal{N} = \operatorname{New}_{(x,y,z)}(\partial)$ be the Newton polyhedron of $\mathcal{F}$ with respect to these coordinates.

Definition: The **higher vertex** is the vertex $\boldsymbol{h} \in \mathcal{N}$ which is minimal with respect to the lexicographical ordering in $\mathbb{Z}^3$.

By the above assumptions, we have $\boldsymbol{h} = (0, h_2, h_3)$, with $h_2, h_3 \in \mathbb{Z}_{\geqslant -1}$

(because $h_1 > 0 \Longrightarrow \{x = 0\} \in \operatorname{Nilp}(M, \mathcal{F})$).

Moreover, the intersection of $\mathcal{N}$ with the plane $\{\boldsymbol{v} \in \mathbb{R}^3 \colon v_1 = 0\}$ is in one of the situations illustrated below



Regular and nilpotent configurations.

(because otherwise $\{x = y = 0\} \subset \mathrm{Nilp}(M, \mathcal{F})$)

Definition: • Cases (a) and (b) are called *regular configurations* and case (c) is called *nilpotent configuration*.

• The *main vertex* of $\mathcal{N}$ is given by $\boldsymbol{m} = \boldsymbol{h}$ in the regular configurations and by $\boldsymbol{m} = \boldsymbol{n}$ in the nilpotent configuration.

We now consider the intersection

$$\mathcal{N}' = \mathcal{N} \cap \left\{ \boldsymbol{v} \in \mathbb{R}^3 \colon v_3 = m_3 - \frac{1}{2} \right\}$$

which we will call the *derived polygon*.

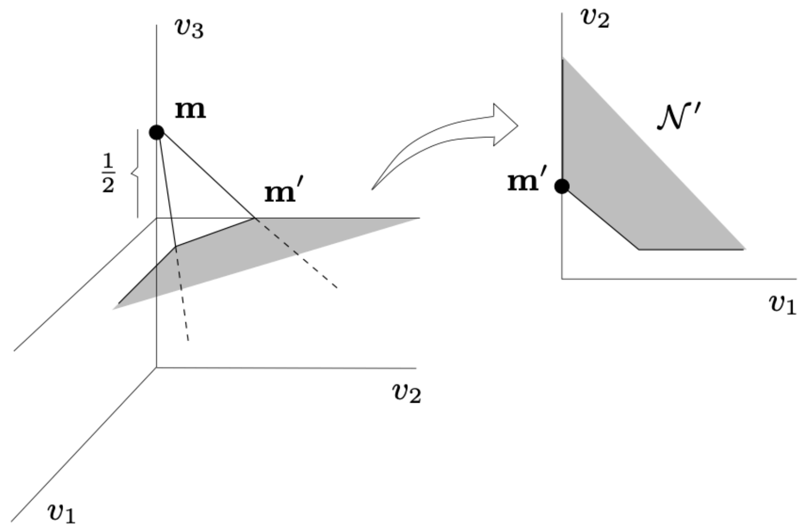


Figure 2. The derived polygon.

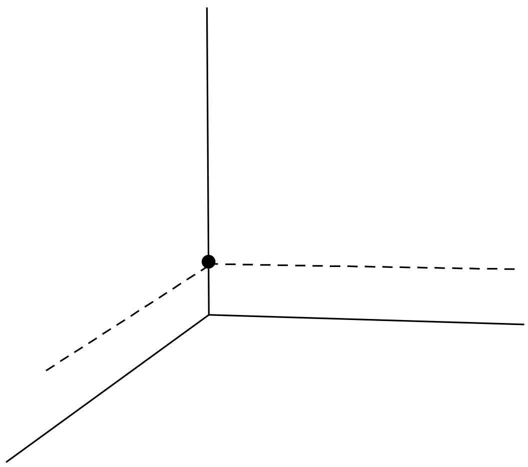
Let $\boldsymbol{m}' = \left(m_1', m_2', m_3 - \frac{1}{2}\right)$ be the minimal vertex of $\mathcal{N}'$ (with respect to the lexicographical ordering), and write the **vertical displacement vector $\boldsymbol{m}' - \boldsymbol{m}$** as $\frac{1}{2}(\Delta_1, \Delta_2, -1)$

$$\Delta = (\Delta_1, \Delta_2) \in \mathbb{Q}^2$$

Remark: We observe that if the main vertex $\boldsymbol{m}$ is such that $m_3 \geqslant 1$ then the derived polgon is non-empty.

Indeed, if this where not the case, the Newton polyhedron should be contained in the region

$$\{\boldsymbol{v} \in \mathbb{R}^3 \quad | \quad v_3 \geqslant 1\}$$

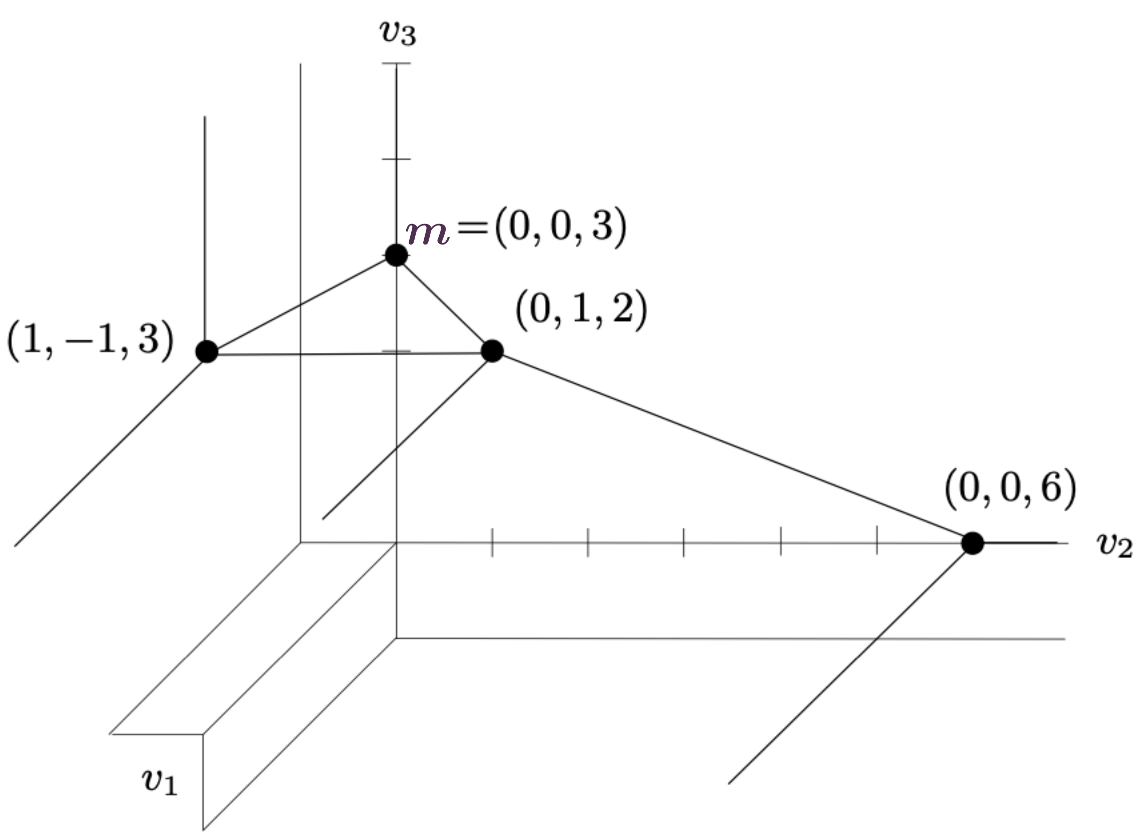


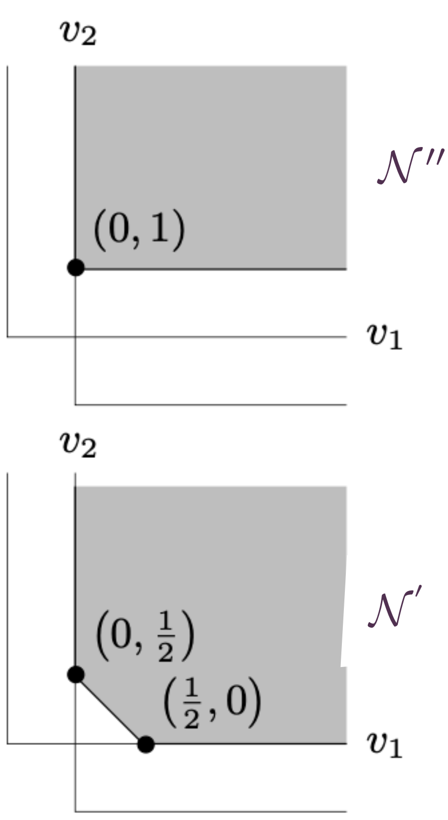
But this would imply that $\{z=0\} \subset \mathrm{Nilp}(M, \mathcal{F})$, contradicting the hypothesis that the nilpotent locus has codimension greater or equal than two.

Comparison of the derived polygon $\mathcal{N}'$ with Hironaka's characteristic polygon

Consider the vector field

$$\partial = (z^3x + xyz^2)\frac{\partial}{\partial x} + xz^3\frac{\partial}{\partial y} + y^7\frac{\partial}{\partial z}$$





The invariant

The **main invariant** of $\mathcal{F}$ (with respect to the coordinates (x, y, z)) is the 6-uple of natural numbers

$$\boxed{\text{inv} = (\mathfrak{h}, m_2 + 1, m_3, e - 1, \lambda\Delta_1, \lambda \max\{0, \Delta_2\})}$$

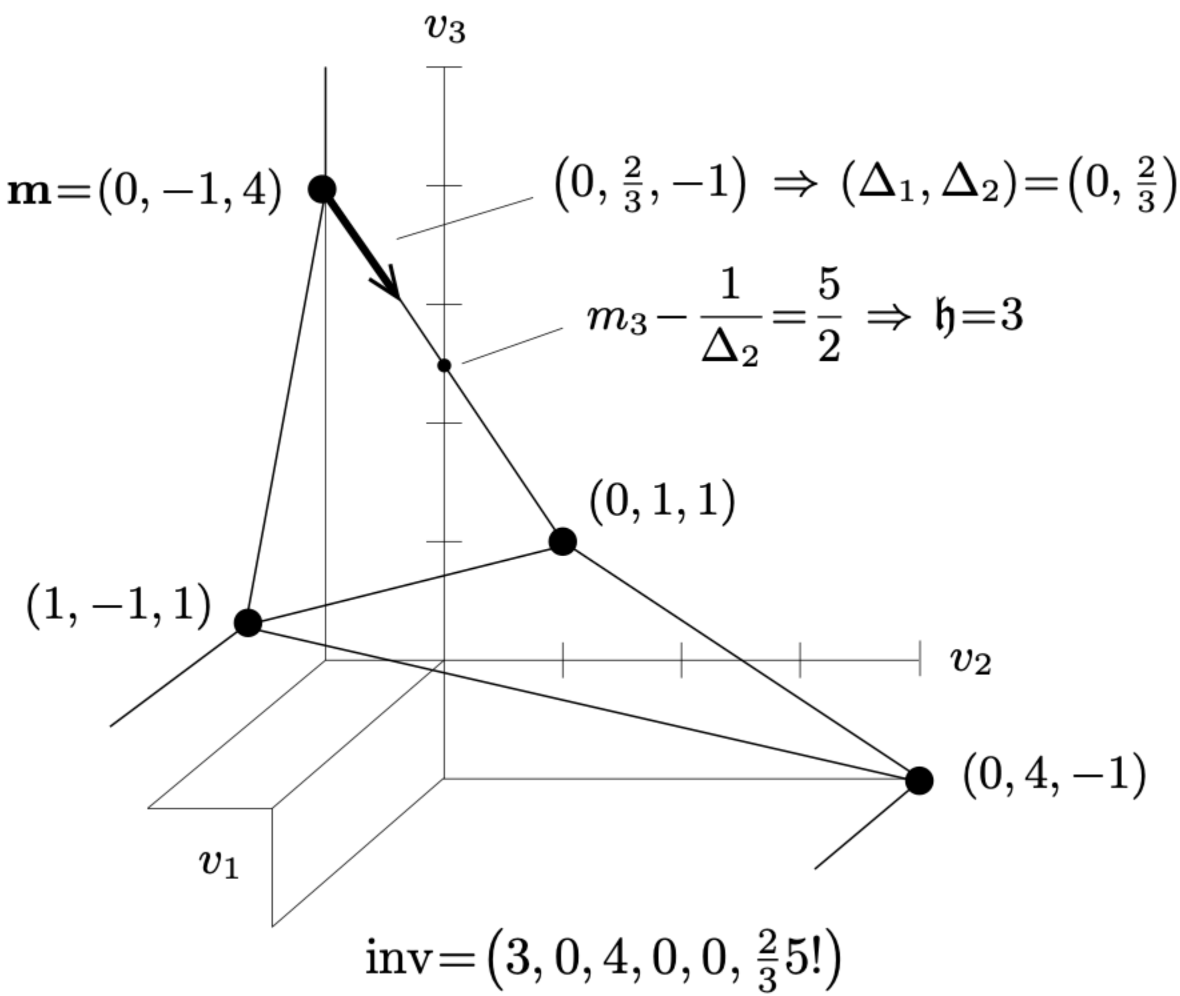
where

- $\lambda = (m_3 + 1)!$
- $e \in \{1, 2\}$ is the number of local irreducible components of E at the origin
- The **virtual height** $\mathfrak{h}$ is the natural number defined by

$$\mathfrak{h} = \begin{cases} \lfloor m_3 + 1 - 1/\Delta_2 \rfloor & \text{, if } m_2 = -1 \text{ and } \Delta_1 = 0 \\ \\ m_3 & \text{, if } m_2 = 0 \text{ or } \Delta_1 > 0 \end{cases}$$

Example:

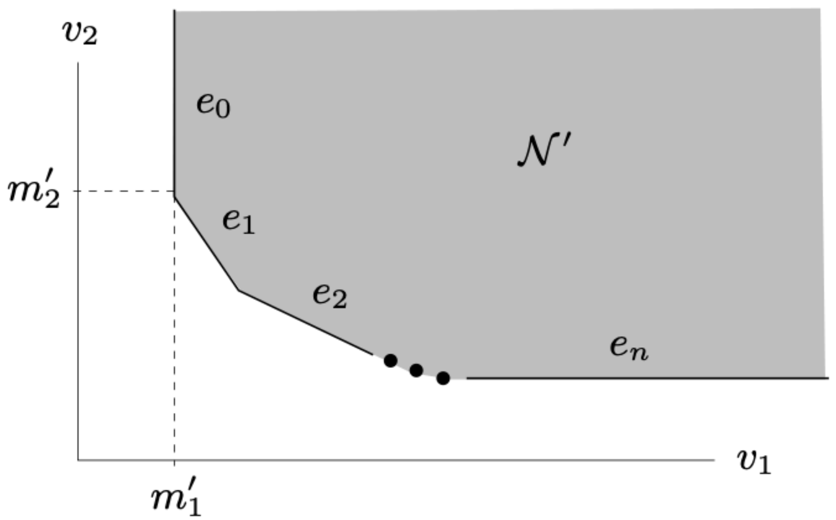
$$\partial = x^2 y \frac{\partial}{\partial x} + (z^4 + x z)\frac{\partial}{\partial y} + y^4 \frac{\partial}{\partial z}$$



The main face and the local desingularization strategy

Let $\boldsymbol{m}' = (m_1', m_2', m_3 - 1/2)$ be the main vertex of the derived polygon $\mathcal{N}'$. The **main side** of $\mathcal{N}'$ is defined (according to the figure below) by

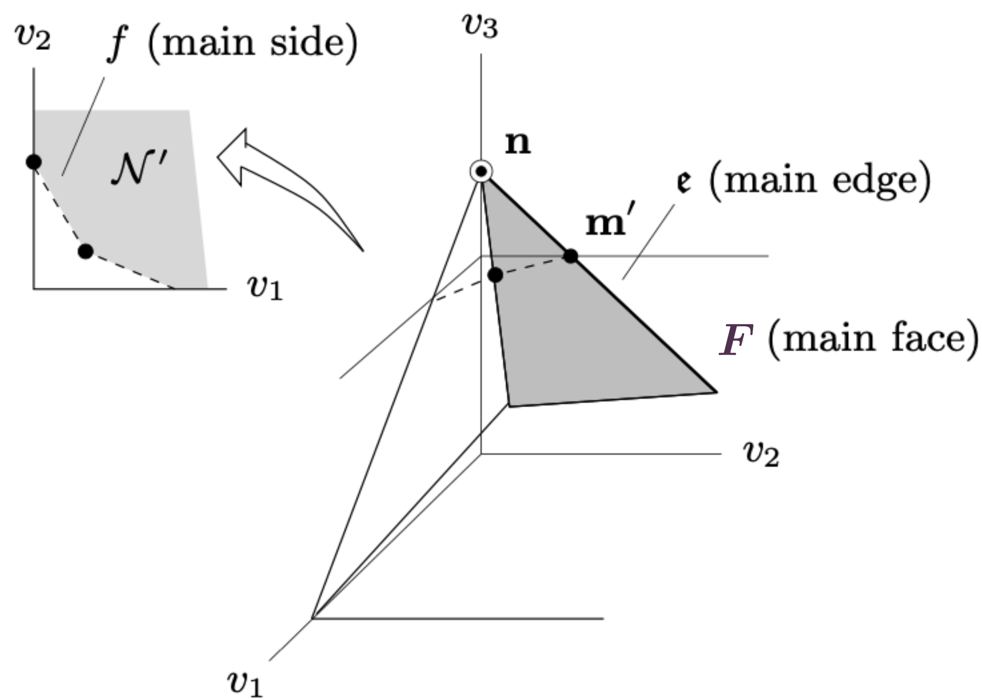
$$\boldsymbol{f}(\mathcal{N}') = \begin{cases} e_0 & \text{if } \ m_1' > 0 \\ \\ e_1 & \text{if } \ m_1' = 0 \end{cases}$$



The derived polygon.

The **main edge** of $\mathcal{N}$ is the edge $\mathfrak{e}$ containing the segment $[\boldsymbol{m}, \boldsymbol{m}']$.

The **main face** of $\mathcal{N}$ is the unique face $\boldsymbol{F} \subset \mathcal{N}$ such that $\boldsymbol{F} \cap \mathcal{N}' = \boldsymbol{f}(\mathcal{N}')$



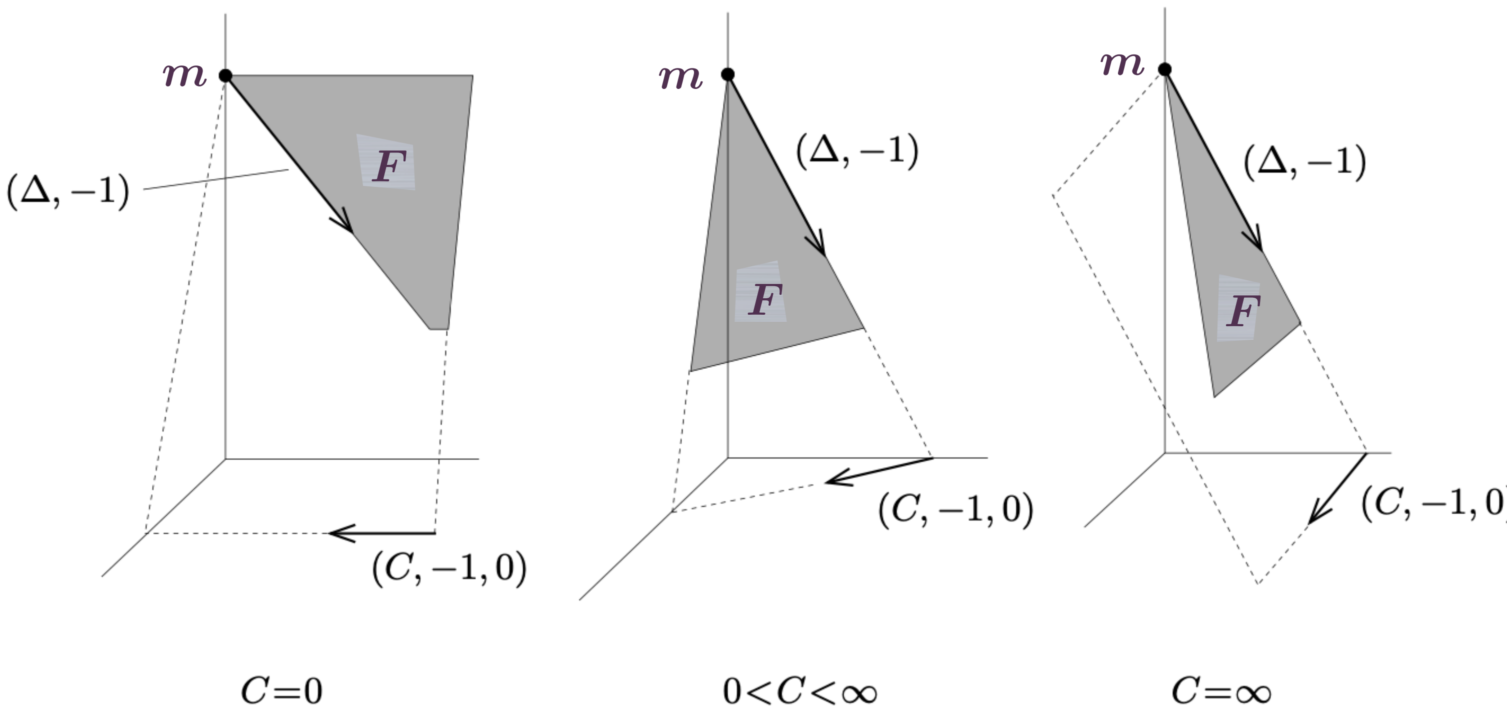
We recall that the vertical displacement vector is given by $\Delta = \boldsymbol{m}' - \boldsymbol{m}$

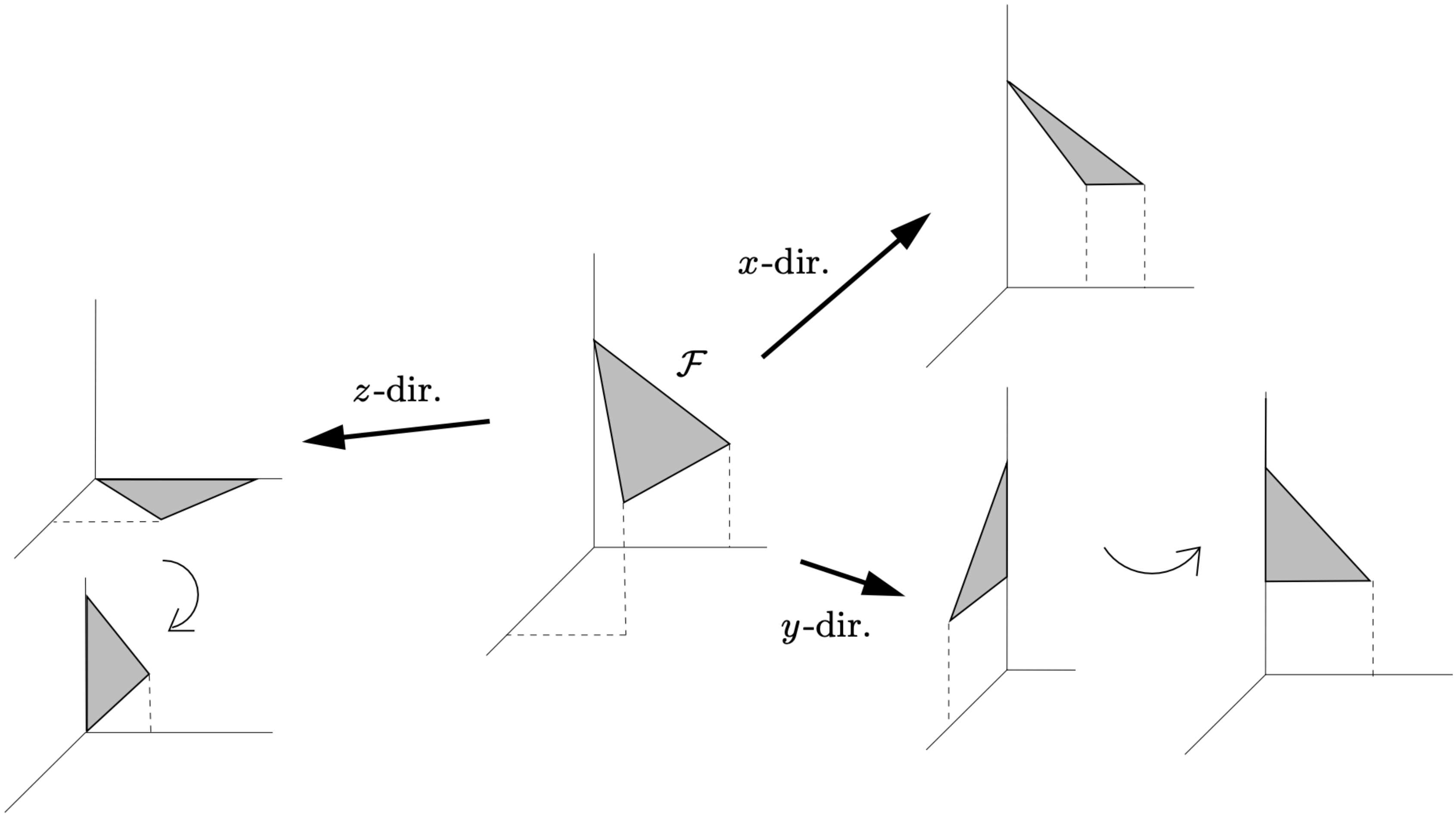
The main side can be uniquely written

$$f(\mathcal{N}') = \{\boldsymbol{m}' + t(C, -1, 0) : \quad t \in I\}$$

for some $C \in \overline{\mathbb{Q}}_{\geqslant 0} = \mathbb{Q}_{\geqslant 0} \cup \{\infty\}$.

We say that $(C, -1, 0)$ is the **horizontal displacement vector** of $\mathcal{N}$.

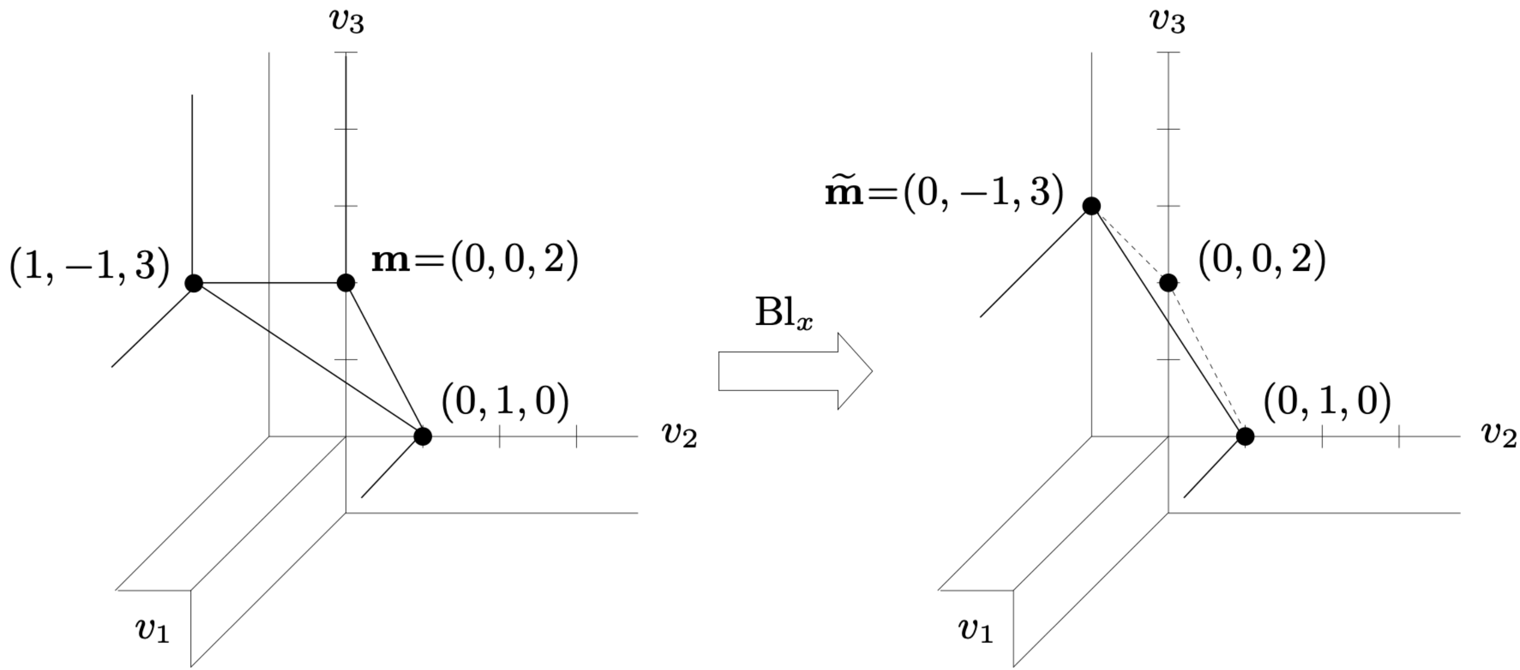




The directional blowing-ups.

Why the height of the main vertex is not the first entry in the invariant?

Example: $\partial = (y^2 + x\,z^3)\partial/\partial y + z^3\partial/\partial z$, and $\mathrm{inv} = (\mathfrak{h}, m_2 + 1, m_3, \dots) = (2, 1, 2, \dots)$



The height of the main vertex increases after an x-directional blowing-up.

After a x-directional blowing-up, we get $\tilde{\partial} = (y^2 + z^3)\partial/\partial y + z^3\partial/\partial z$

and $\widetilde{\mathrm{inv}} = (\tilde{\mathfrak{h}}, \widetilde{m}_2 + 1, \widetilde{m}_3, \dots) = (2, 0, 3, \dots)$.

Here, $\widetilde{\mathrm{inv}} <_{\mathrm{lex}} \mathrm{inv}$ because $\widetilde{m}_2 = -1 < 0 = m_2$.

As in the case of dimension two, we need to compute the invariant with respect to *stable coordinates*.

As we shall see, for (x, y, z) given as above, a stable coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ will be obtained by an analytic change of coordinates in the triangular form

$$\tilde{x} = x, \quad \tilde{y} = y + G(x), \qquad \tilde{z} = z + F(x, y) \qquad (\star)$$

The invariant, when computed with respect to a stable coordinate system, will be intrinsically attached to the germ of $\mathcal{F}$, up to an additional geometric structure on the ambient space, called an **axis**.

The local strategy of blowing-up will be read out from the Newton polyhedron and the main invariant...

Provided that these objects are computed with respect to a stable coordinate system.

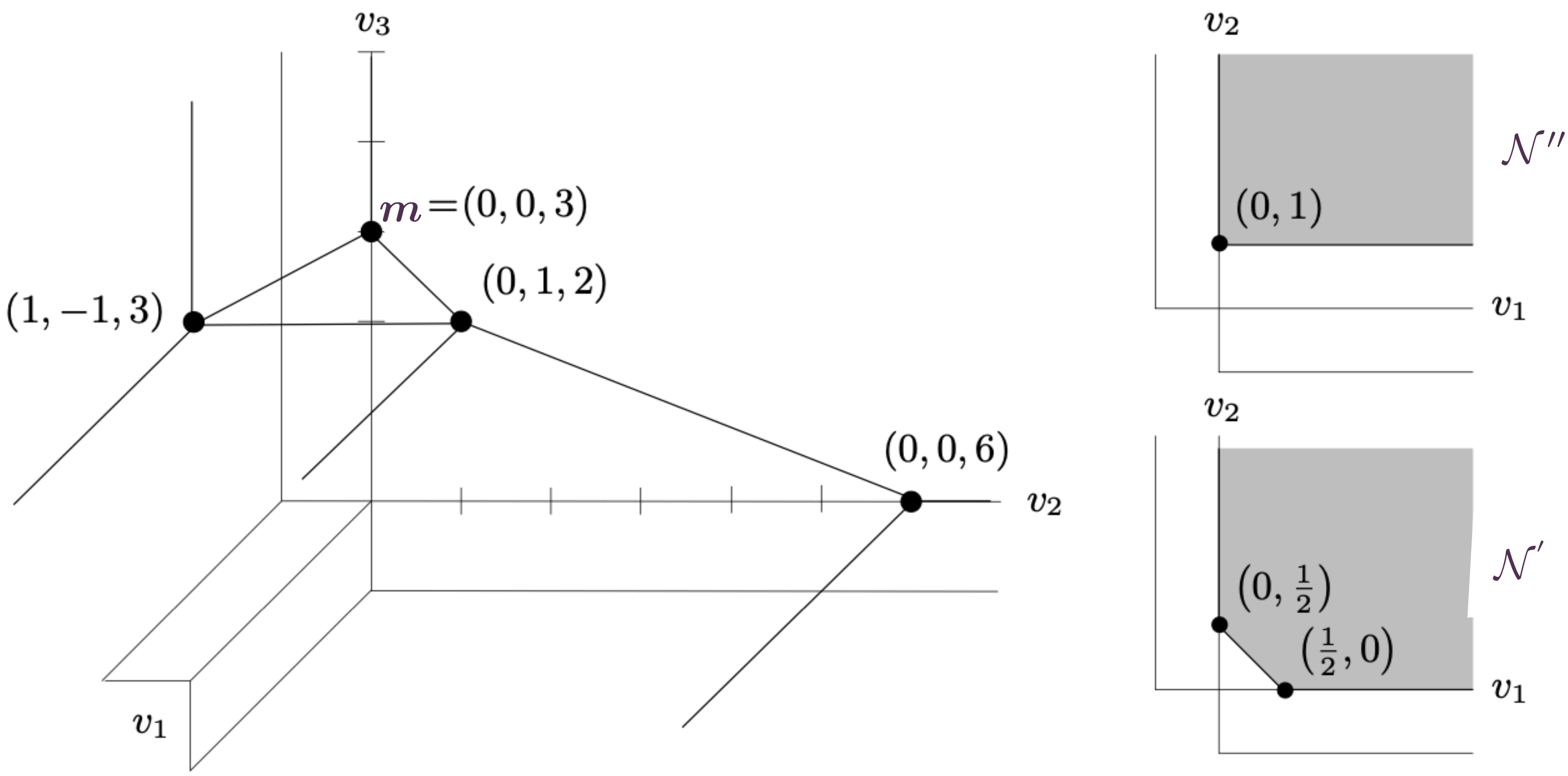
Remark 1. The notion of stable coordinates is similar to the notions of *well-prepared* and *very well-prepared coordinates* in Hironaka's paper *Desingularization of excellent surfaces.*

..But new diffulties appear in the context of vector fields because the action of the Lie group defined by $(\star)$ is much harder to study.

Comparison of the derived polygon $\mathcal{N}'$ with Hironaka's characteristic polygon

Consider the vector field

$$\partial = (z^3x + xyz^2)\frac{\partial}{\partial x} + xz^3\frac{\partial}{\partial y} + z^7\frac{\partial}{\partial z}$$



The Axis

The main goal is to *rigidify* the choice of local coordinates.

Definition: An **axis** for $(M, E, \mathcal{F})$ is a pair $\mathrm{Ax} = (U, \mathcal{A})$, where $U \subset M$ is an open neighborhood of $\mathrm{Nilp}(M, \mathcal{F})$ and $\mathcal{A}$ is an analytic foliation by curves defined on U such that:

- $\mathcal{A}$ is tangent to the divisor E
- $\mathrm{Sing}(\mathcal{A}) = \emptyset$ (i.e. $\mathcal{A}$ is everywhere non-singular)
- For each point $p \in E \cap U$, if (x, y, z) are local coordinates such that $\mathcal{A} = \left\langle \frac{\partial}{\partial z} \right\rangle$ then

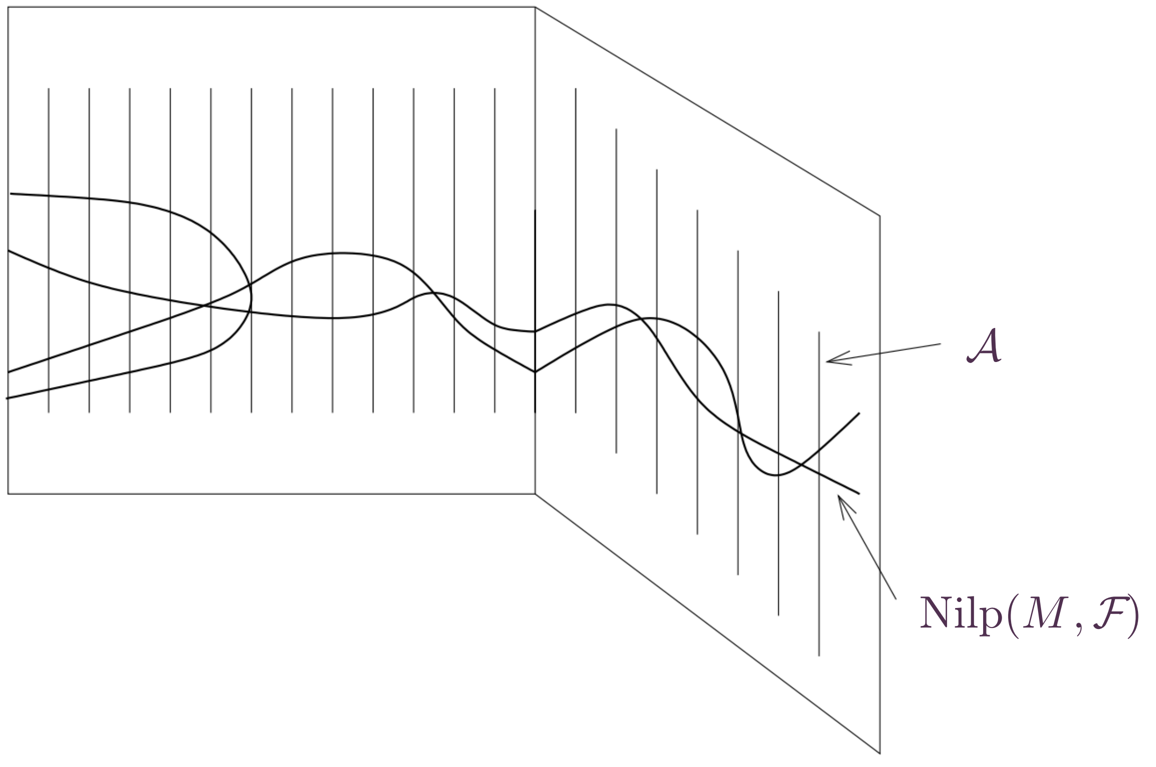
$$I(\mathrm{Nilp}(M, \mathcal{F})) \not\subset \langle x, y \rangle$$

(i.e. the nilpotent locus of $\mathcal{F}$ does not contains the axis through p)

- For each point $p \in U \setminus E$, if (x, y, z) are local coordinates such that $\mathcal{A} = \left\langle \frac{\partial}{\partial z} \right\rangle$ then

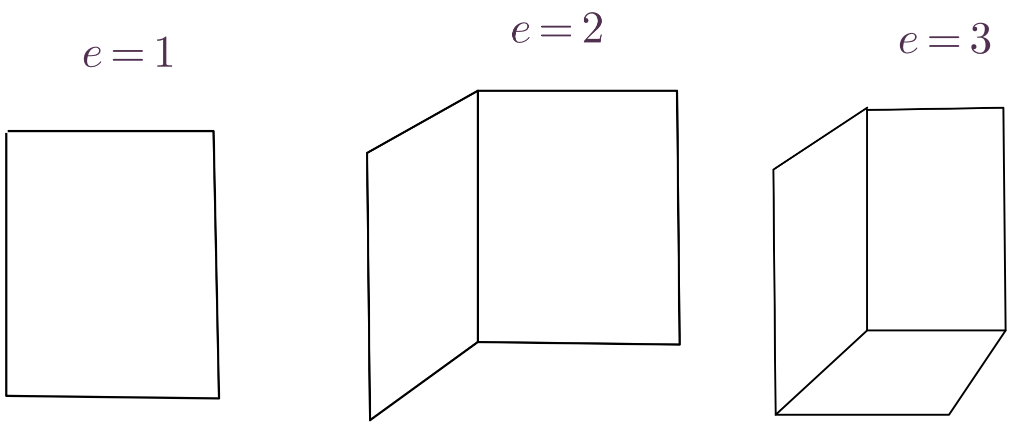
$$\partial(\langle x, y \rangle) \not\subset \langle x, y \rangle$$

where ∂ is a local generator of $\mathcal{F}$ (i.e.the axis through p is not an invariant curve for $\mathcal{F}$)


$\mathcal{A}$
$\mathrm{Nilp}(M, \mathcal{F})$

Remark: Notice that an axis cannot exist if there exists a point $p \in \mathrm{Nilp}(M, \mathcal{F})$ such that

$$e(p) = 3$$



(because the tangency to E would force $p \in \mathrm{Sing}(\mathcal{A})$).

We say that $(M, E, \mathcal{F})$ is **controllable** if there exist an axis Ax as above. The 4-uple

$$(M, E, \mathcal{F}, \mathrm{Ax})$$

will be called a **controlled singularly foliated manifold**.

Proposition: *Let $(M, E, \mathcal{F})$ be a singularly foliated manifold such that $E = \emptyset$. Then, there exists an axis for $(M, E, \mathcal{F})$.*

Sketch of the proof: Since $E = \emptyset$, this amounts to choose a regural one-dimensional foliation in the vicinity of $\mathrm{Nilp}(M, \mathcal{F})$ which contains no invariant curve of $\mathcal{F}$.

By an easy perturbation argument this can be easily done locally at each point $p \in \mathrm{Nilp}(M, \mathcal{F})$.

Using partitions of unity, we can glue together and define a C^∞ foliation $\tilde{\mathcal{A}}$ satisfying all the requirements.

Then, we use Grauert's embedding theorem to approach $\tilde{\mathcal{A}}$ by an *analytic foliation* satisfying all the requirements.

Remark: This last statement does not hold in the complex setting because not every complex manifold is Stein. Thus, in the resolution of singularities for vector fields over $\mathbb{C}$ (joint work with M. Mcquillan), we need to introduce the weaker notion of "quasi axis".

Adapted local charts

Let $(M, E, \mathcal{F}, \mathrm{Ax})$ be a controlled singular foliated manifold, where $\mathrm{Ax} = (U, \mathcal{A})$ is the axis.

We shall also fix a **tagging** of E, namely a bijection

$$\tau\colon \{1, \ldots, n\} \to \{\text{irreducible compotents of } E\}$$

which defines an enumeration of the irreducible components. (The tag will record the year of creation of the divisor component in the resolution process).

A local chart (x, y, z) centered at a point $p \in U$ is **adapted** if

- $\mathcal{A}$ is locally generated by $\partial / \partial z$
- If $e(p) = 1$ then $E = \{x = 0\}$
- If $e(p) = 2$ and $E = D_i \cup D_j$ with $i > j$ then $D_i = \{x = 0\}$ and $D_j = \{y = 0\}$

In other words, the divisor $\{x = 0\}$ is always *younger* than the divisor $\{y = 0\}$.

Let us see how the concept of adapted local charts rigidifies the choice of local coordinates.

Proposition: Let (x, y, z) and (x', y', z') be local adapted charts at a point $p \in U$. Then, the transition map has the form

$$x' = F(x, y), \quad y' = G(x, y), \quad z' = f(x, y) + z w(x, y, z)$$

where $\partial(F, G)/\partial(x, y)(0, 0) \neq 0$ and w is a unit.

Moreover,

- if $e(p) = 1$ then $F(x, y) = x u(x, y)$ and $G(x, y) = g(x) + y v(x, y)$
- if $e(p) = 2$ then $F(x, y) = x u(x, y)$ and $G(x, y) = y v(x, y)$

where u, v are units.

Proof: The coordinate change $(x, y, z) \to (x', y', z')$ should map the vector field $\partial/\partial z$ to

$$U \frac{\partial}{\partial z'}$$

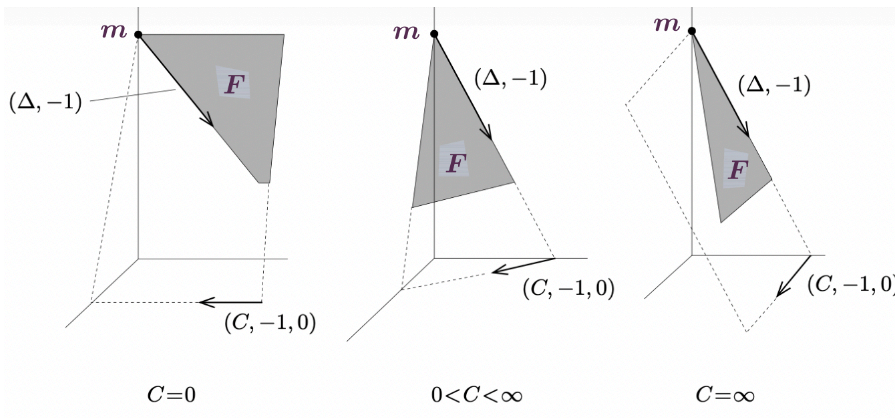
where U is a unit. This implies that x', y' cannot depend upon z.

The other assertions are easily deduced from the fact that the components of the divisor (and their tagging) should be preserved.

From now on, we will only consider adapted coordinate sytems.

Let $p\in\mathrm{Nilp}(M,\mathcal{F})\cap E$ and $\mathrm{New}_{(x,y,z)}(\partial)$ be the Newton polyhedron at p with respect to (x,y,z).

We recall the definition of the main vertex $\boldsymbol{m}$, the displacement vectors $\Delta\in\mathbb{Q}^2$, $C\in\overline{\mathbb{Q}}_{\geqslant 0}$ and the main face $\boldsymbol{F}$.



We denote by $\mathbb{N}^{\boldsymbol{m}}_{\Delta,C}$ the set of all polyhedra having a same main vertex $\boldsymbol{m}$ and displacement vectors Δ, C. (but possibly with different main faces). We denote by $G_{\Delta,c}$ the group of polynomial changes of coordinates

$$\tilde{x}=x,\quad \tilde{y}=y+g(x),\qquad \tilde{z}=z+f(x,y)$$

which *respects* the quasi-homogeneous graduation determined by $\boldsymbol{F}$.

$$\tilde{x} = x, \quad \tilde{y} = y + g(x), \qquad \tilde{z} = z + f(x, y)$$

i.e. $\mathrm{wt}(z) = \mathrm{wt}(f)$ and $\mathrm{wt}(y) = \mathrm{wt}(g)$. In other words, such that

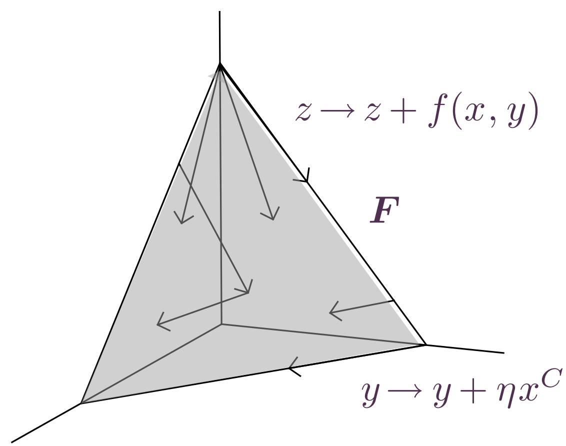
$$\mathrm{supp}(f) \subset \{(a, b) \in \Delta + s(C, -1) \mid s \in \overline{\mathbb{Q}}_{\geqslant 0}\} \cap \mathbb{N}^2, \qquad \text{if } \Delta_1 = 0$$

$$\mathrm{supp}(f) \subset \{\Delta\} \cap \mathbb{N}^2, \qquad \text{if } \Delta_1 > 0$$

and

$$\mathrm{supp}(g) \subset \{C\} \cap \mathbb{N}, \quad \text{if } \Delta_1 = 0$$

$$\mathrm{supp}(g) = \emptyset, \qquad \text{if } \Delta_1 > 0$$



Denote by $(f,g)\in G_{\Delta,C}$ the element corresponding to the map

$$\tilde{x}=x, \quad \tilde{y}=y+g(x), \qquad \tilde{z}=z+f(x,y)$$

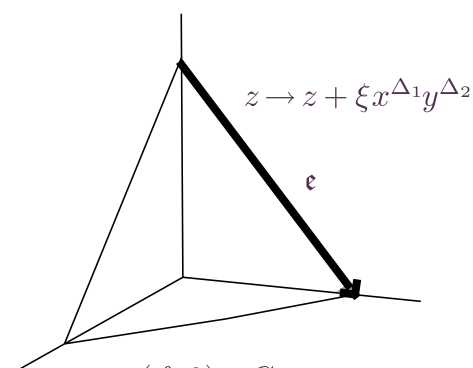
We split $G_{\Delta,C}$ as a semi-direct sum

$$G_{\Delta,C}=G^{+}_{\Delta,C}\rtimes G_{\Delta}$$

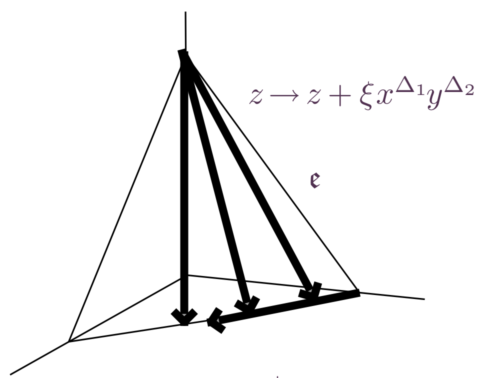
where $G_{\Delta}=\{(f,g)\in G_{\Delta,C}\mid\ g=0,\, f=\xi x^{\Delta_1}y^{\Delta_2},\, \xi\in\mathbb{R}\}$ is and

$$G^{+}_{\Delta,C}=\{(f,g)\in G_{\Delta,C}|\ \Delta\notin \operatorname{supp}(f)\}$$

is a the subgroup of *edge-preserving* maps.



$(f,0)\in G_{\Delta}$



$(f,g)\in G^{+}_{\Delta,C}$

Let (x, y, z) be an (adapted) system of coordinates at $p \in \mathrm{Nilp}(M, \mathcal{F}) \cap E$, and suppose that

$$\mathrm{New}_{(x,y,z)}(\partial) \in \mathbb{N}^{\boldsymbol{m}}_{\Delta,C}$$

Definiton: We say that (x, y, z) is a **stable** system of coordinates (for $(M, \mathcal{F}, E, \mathrm{Ax})$) at p if **for all $(f, g) \in G_{\Delta,C}$,**

$$\mathrm{New}_{(x,y+g,z+f)}(\partial) \in \mathbb{N}^{\boldsymbol{m}}_{\Delta,C}$$

In other words, the action of the group $G_{\Delta,C}$ cannot modify supporting plane of the main face.

Using stable coordinates, we can now identify the **final situations**

Proposition. Suppose that (x,y,z) is a stable coordinate system at $p \in \mathrm{Nilp}(M,\mathcal{F}) \cap E$. Then, none of the following configurations can occur for $\mathrm{New}_{(x,y,z)}(\partial)$.

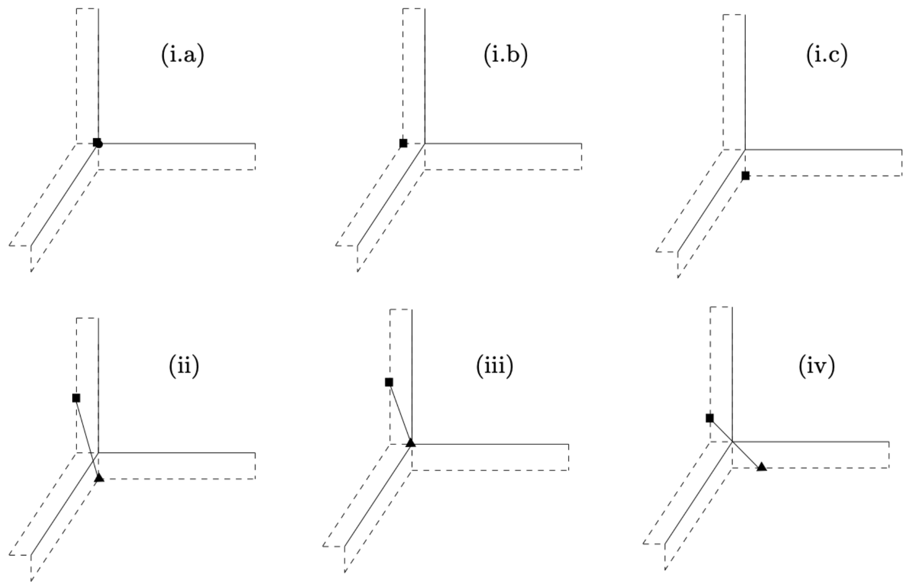


Figure 14. The final situations.

because $0 \in \mathrm{New}(\partial)$ "irremovably" in each one of these cases (i.e. p is *elementary*).

Intrinsic definition of the invariant and local strategy

We recall that the invariant is given by

$$\boxed{\mathrm{inv}_{(x,y,z)} = (\mathfrak{h}, m_2+1, m_3, e-1, \lambda\Delta_1, \lambda \max\{0, \Delta_2\})}$$

Theorem 1: Suppose that (x, y, z) and (x', y', z') are stable coordinates at a point $p \in \mathrm{Nilp}(M, \mathcal{F}) \cap \mathrm{E}$. Then,

- The invariants $\mathrm{inv}_{(x,y,z)}$ and $\mathrm{inv}_{(x',y',z')}$ coïncide.
- The change of coordinates $(x, y, z) \rightarrow (x', y', z')$ preserves the quasi-homogeneous filtration $\mathrm{Gr}_{\geqslant}$ determined by the main face $\boldsymbol{F}$.

Definition. Let (x, y, z) be an arbitrary stable coordinate system.

1) The **invariant** at p is the 6-uple $\mathrm{inv}_p(M, E, \mathcal{F}, \mathrm{Ax}) = \mathrm{inv}_{(x,y,z)}$

2) the **local resolution strategy** at p is the weighted blowing-up defined by the main face $\boldsymbol{F}$ of $\mathrm{New}_{(x,y,z)}(\partial)$.

The local resolution theorem

Theorem 2: Let $(M, E, \mathcal{F}, \mathrm{Ax})$ be a controlled singularly foliated manifold. Consider the local blowing-up at $p \in \mathrm{Nilp}(M, \mathcal{F}) \cap E$

$$\Phi\colon \tilde{M} \to M$$

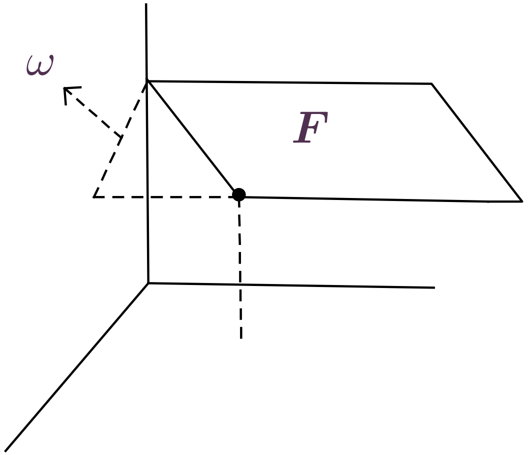
which is determined by the local strategy, and let $\tilde{E}, \tilde{\mathcal{F}}, \widetilde{\mathrm{Ax}}$ denote the strict transforms of $E, \mathcal{F}, \mathrm{Ax}$ by this map.

Then, for each point $p \in \Phi^{-1}(p) \cap \mathrm{Nilp}(\tilde{M}, \tilde{\mathcal{F}})$,

$$\mathrm{inv}_{\tilde{p}}(\tilde{M}, \tilde{E}, \tilde{\mathcal{F}}, \widetilde{\mathrm{Ax}}) <_{\mathrm{lex}} \mathrm{inv}_p(M, E, \mathcal{F}, \mathrm{Ax})$$

Remark: The local center is always contained in Nilp$(M,\mathcal{F})$.

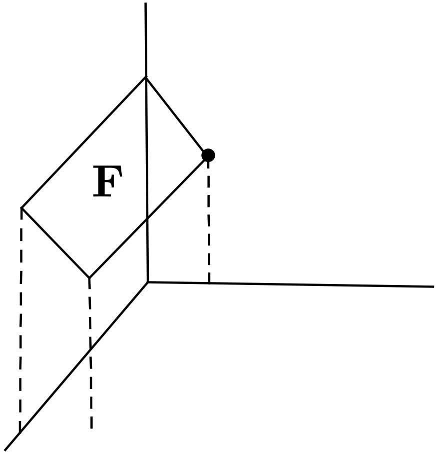
1) $\Delta_1 > 0$.



$$\omega = (\omega_1, 0, \omega_3) = (1, 0, \Delta_1)$$

$$C = \{x = z = 0\} \subset \mathrm{Nilp}(M,\mathcal{F})$$

2) $\Delta_1 = 0, C = \infty$



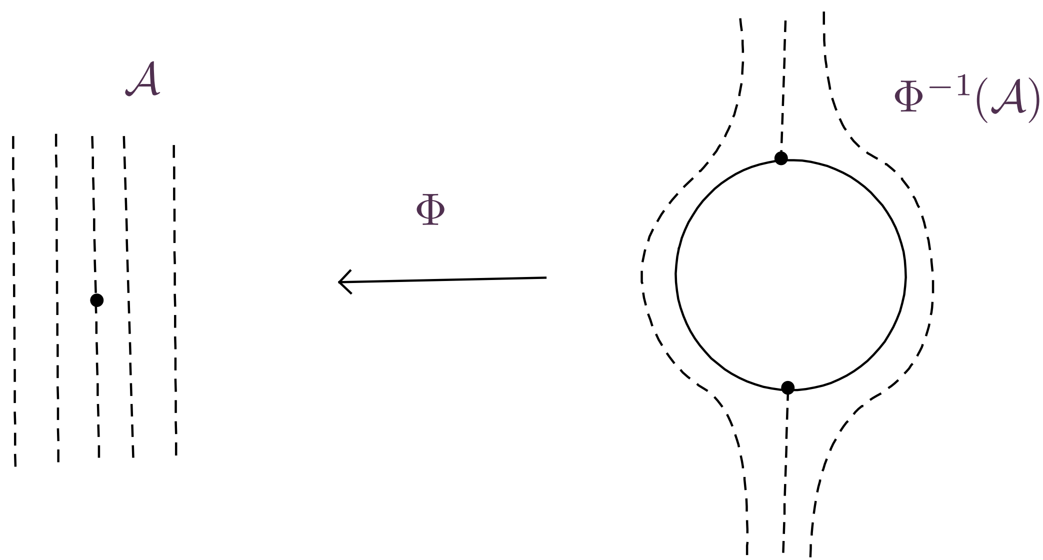
$$\omega = (0, \omega_2, \omega_3) = (0, 1, \Delta_2)$$

$$C = \{y = z = 0\} \subset \mathrm{Nilp}(M,\mathcal{F})$$

Remark. The strict transform of the axis Ax by the local blowing-up determined by the local strategy

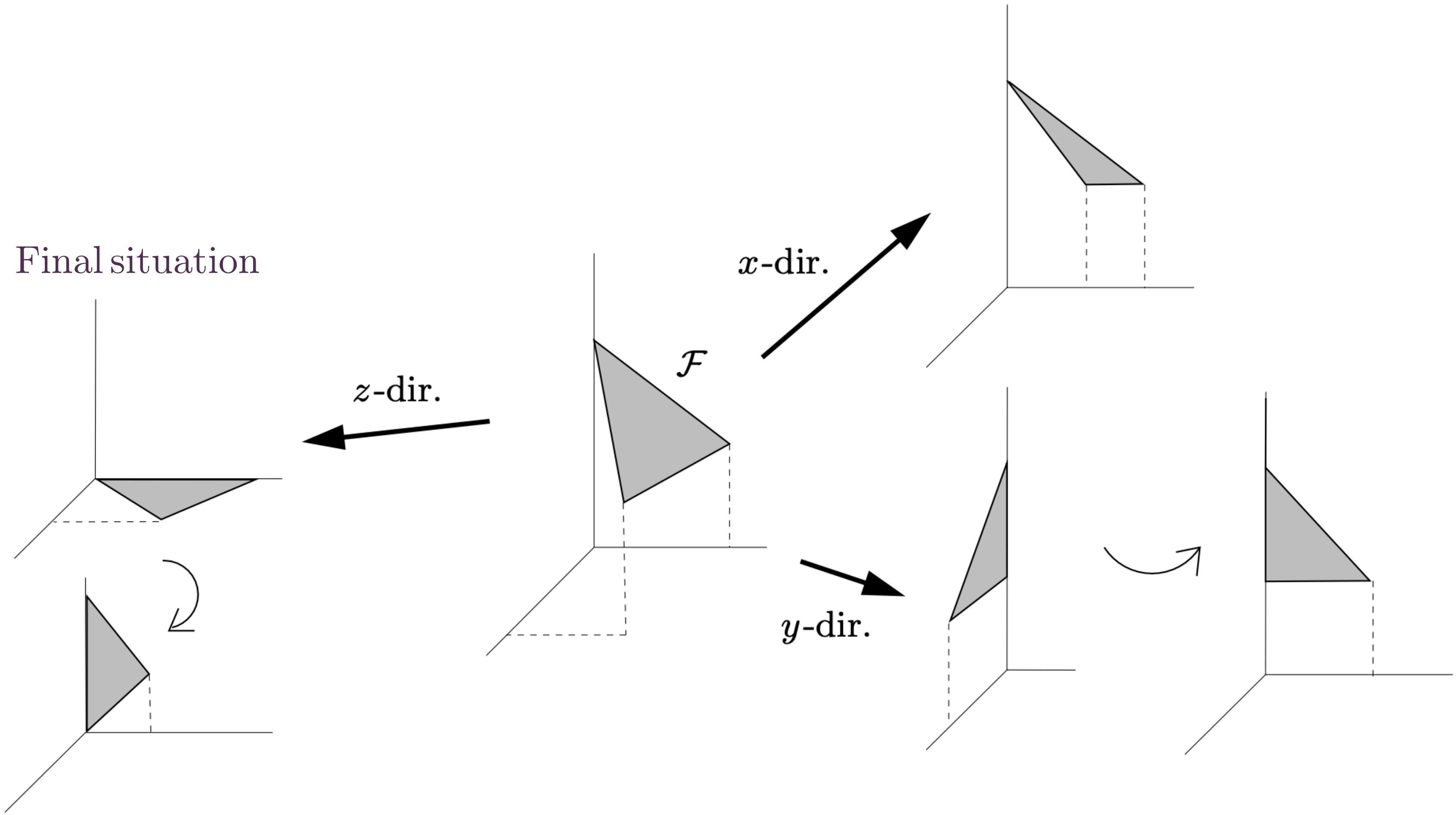
$$\Phi\colon \tilde{M} \to M$$

defines an axis $\widetilde{\mathrm{Ax}}$ for $(\tilde{M}, \tilde{E}, \tilde{\mathcal{F}})$.



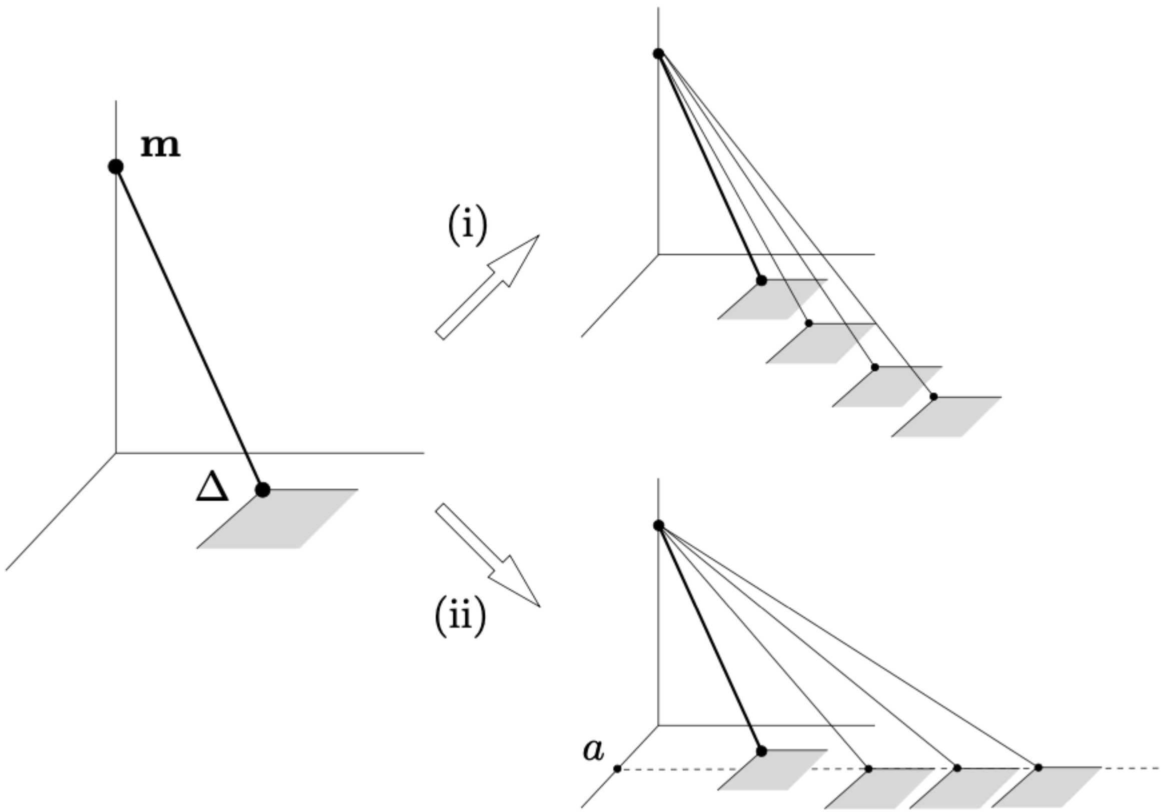
The unique two singular points of $\Phi^{-1}(\mathcal{A})$ occurs at the points $p_{\pm} = (0\colon 0\colon \pm 1) \in \Phi^{-1}(p)$.

But, by construction, $p_{\pm} \notin \mathrm{Nilp}(\tilde{M}, \tilde{\mathcal{F}})$.

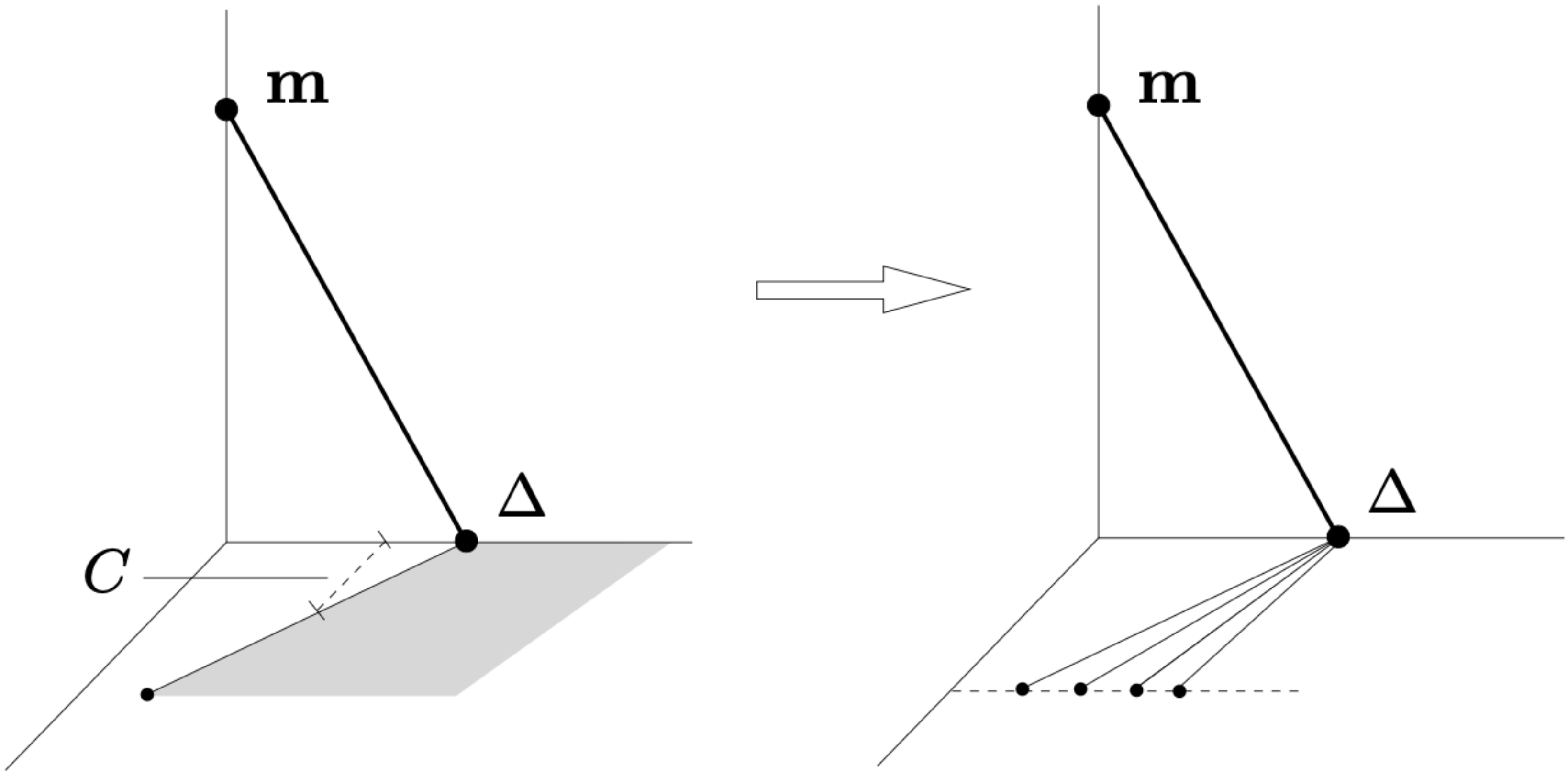


The directional blowing-ups.

Edge stabilization by the action of G_Δ



Face stabilisation by the action of $G^{+}_{\Delta,C}$



Generically, points in $\mathrm{Nilp}(M, \mathcal{F})$ will be **equireducible.**

Namely, there is a discrete set of points $N \subset \mathrm{Nilp}(M, \mathcal{F})$ such that,

$$\forall \ p \in \mathrm{Nilp}(M, \mathcal{F}) \backslash \mathrm{N}$$

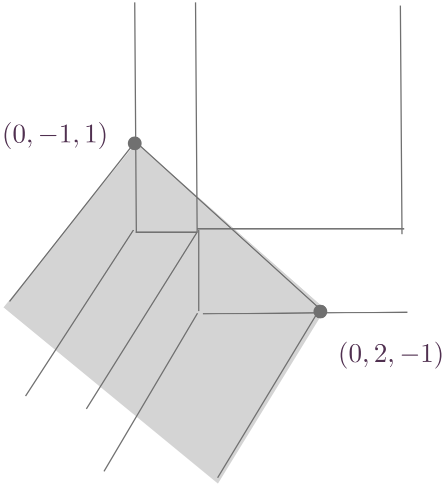
• The germ $\mathrm{Nilp}(M, \mathcal{F})_p$ is a locally smooth curve.

• A weighted blowing-up with center $C = \mathrm{Nilp}(M, \mathcal{F})_p$ (and appropriate weights) reduces the invariant.

• Each nilpotent point which is infinitely near p satisfies the same conditions.

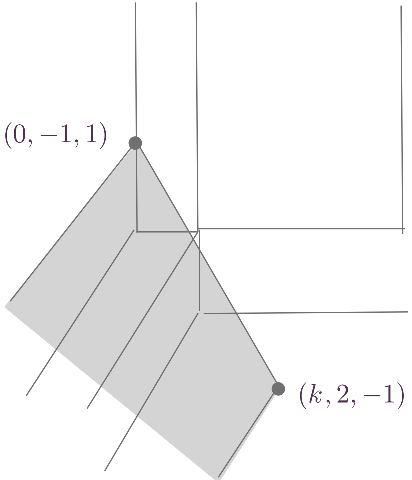
The initial step of the algorithm, so-called *distinguished vertex blowing-up* consists in including all non-equireducible points into the divisor by taking them as blowing-up centers.

Example: $\partial = \boxed{z\frac{\partial}{\partial y} - y^2\frac{\partial}{\partial z}}$, with $\Delta \in \mathrm{Gr}_{\geqslant 2}\left(\cdot, 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right)$



The curve $C = \{y = z = 0\} \in \mathrm{Nilp}$ is equireducible

Example: $\partial = \boxed{z\frac{\partial}{\partial y} - f(x)y^2\frac{\partial}{\partial z}}$, $f(x) = x^k$



The curve $C = \{y = z = 0\} \in \mathrm{Nilp}$ is equireducible for $x \neq 0$

A new strategy via GIT (work in progress...)

The previous strategy cannot be easily adapted to higher-dimensions

• The axis does not behave so-well under blowings-up. (We remark in passing that Haüser defined a notion of "local flag" which generalizes this concept for higher dimensions)

• The presence of negative vertices makes it very hard to capture a good filtration of the local ring and define a good invariant (intrinsic, upper semicontinuous, etc.).

Basic goal:

we have to look for an **invariant** and a **filtration** which are intrinsically attached to the local object, such that

• (Local resolution) The local blowing-up with the center determined by the filtration strictly reduces the invariant.

• (Global resolution) The invariant is upper semi-continuous with respect to the analytic (or Zariski) topology.

Guiding principle: To treat "on an equal footing" germs of vector fields than germs of function?

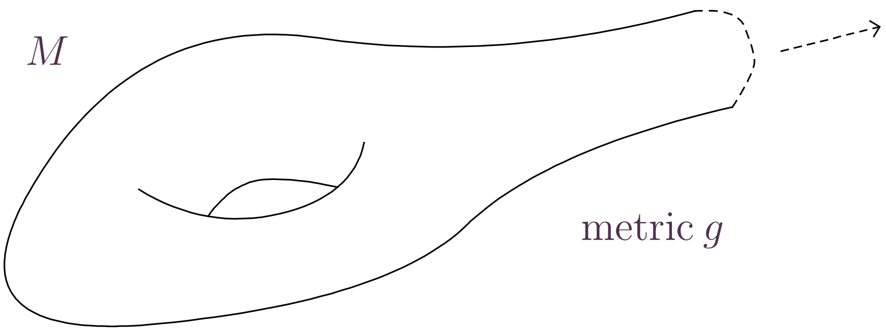
They are both **differential operators**.

By observing things from this more general perspective, we will see a broader panorama...

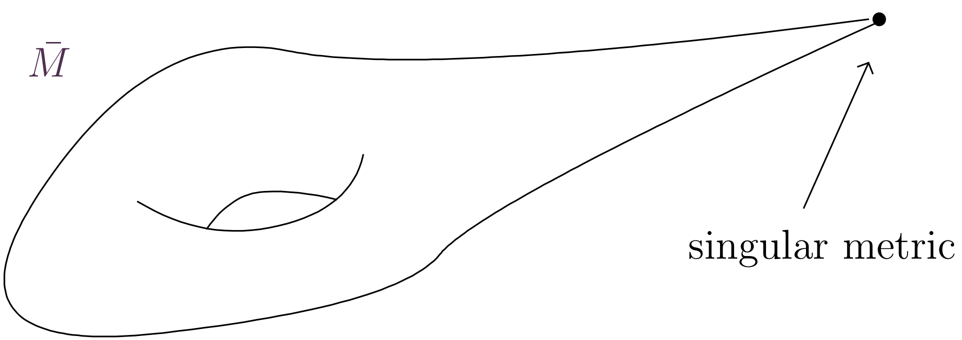
Example of singular differential operator: Laplace equation on open manifolds.

(M, g) a Riemannian manifold and $\Delta = \Delta_g$ the Laplace-Beltrami operator

$$\Delta f = 0$$



Compactification



The associated Laplace-Beltramy operator becomes singular at the new boundary.

Differential operators on manifolds (or orbifolds)

Consider a manifold (real analytic or holomorphic) M and two vector bundles

$$\begin{array}{ccc} E & & F \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

A (E,F)-**differential operator** is a $\mathbb{C}-$linear map $\Phi\colon \mathcal{E}\rightarrow \mathcal{F}$ between (local) sections $\mathcal{E}=\Gamma(E), \mathcal{F}=\Gamma(F)$ of these bundles.

Example: For a global holomorphic function $f\in \mathcal{O}(M)$, the **multiplication** operator

$$\mu_f\colon \mathcal{O}\rightarrow \mathcal{O}$$

defined by $\mu_f(g)=fg$ is a $(\mathcal{O},\mathcal{O})$-differential operator.

More generally, for any bundle E, as $\mathcal{E}$ is a sheaf of $\mathcal{O}$-modules, the multiplication by f defines a differential operator $\mu_f\colon \mathcal{E}\rightarrow \mathcal{E}$.

The order of a differential operator

We say that Φ has **order 0** if it commutes with the (local) multiplication operator, namely

$$\mu_f \Phi = \Phi \mu_f, \qquad \forall f \in \mathcal{O}$$

More generally, we say that Φ is of order d if

$$[\mu_{f_{d+1}}, \cdots [\mu_{f_2}, [\mu_{f_1}, \Phi]]] = 0, \qquad \forall f_1, \ldots, f_d \in \mathcal{O}$$

Examples: 1) A global holomorphic function $h \in \mathcal{O}(M)$ defines a differential operator

$$\mu_h : \mathcal{O} \to \mathcal{O}$$

of order 0. Since $[\mu_f, \mu_h] = fh - hf = 0$ for all $f \in \mathcal{O}$.

2) A global vector field ∂ defines a differential operator of order 1

$$\partial : \mathcal{O} \to \mathcal{O}$$

Since

$$[\mu_f, \partial](g) = f\partial g - \partial fg = -(\partial f)g = \mu_{-\partial f}(g)$$

By fixing local coordinates $(x_1, \ldots, x_n)$, a differential operator of order d can be written

$$\Phi = \sum_{|k| \leqslant d} \varphi_k(x) \left(\frac{\partial}{\partial x} \right)^k$$

where φ_k are $\operatorname{rk} F \times \operatorname{rk} E$ matrices of holomorphic maps.

In what follows, I will only consider the case where $\operatorname{rk} E = \operatorname{rk} F = 1$ (i.e. E, F are line-bundles), and therefore φ_k are germs of holomorphic functions.

Some problems in the theory:

Local resolubility problem: Given g, find f such that

$$\Phi(f) = g$$

Index problems: Find $\operatorname{rank}(\Phi)$ and $\operatorname{corank}(\Phi)$

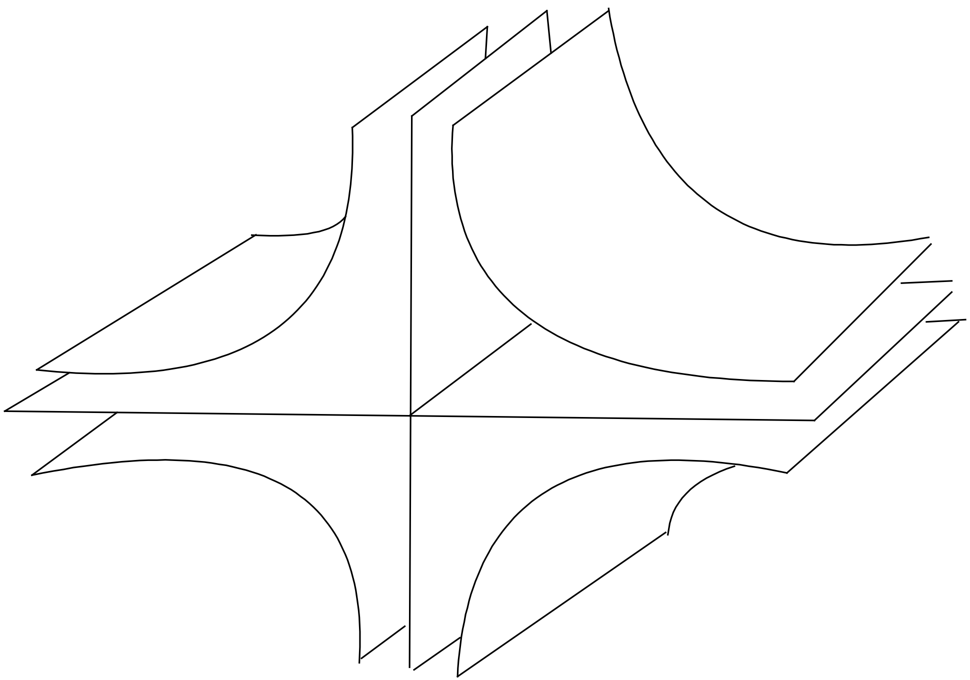
Pseudo differential calculus: Write the inverse operator in a convenient function class.

Basic dichotomies

Global vs **local**

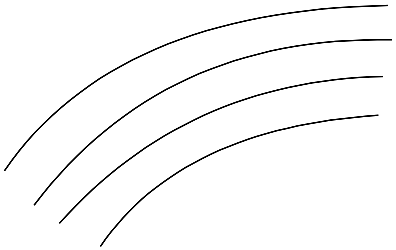
Generic vs **exceptional** phenomena (exceptional = situated on a closed analytic subset of high codimension=**singular set**).

Examples: 1) Level sets of a reduced holomorphic function f are smooth outside a closed subset $\mathrm{Sing}(f) \subset M$ of codimension $\geqslant 2$.



$\mathrm{Sing}(f) = \{df = 0\}$

2) A non-zero vector field ∂ is locally rectifiable, outside a subset $\mathrm{Sing}(\partial) \subset M$ of codimension $\geqslant 1$.



(i.e. we can find local coordinates such that $\partial = \frac{\partial}{\partial x_1}$).

If we write $\partial = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ then

$$\mathrm{Sing}(\partial) = \{ f_1 = \ldots = f_n = 0 \}$$

3) The **Cauchy-Kowalevski theorem** applies locally near all points where a differential operator is not totally characteristic.

What about the behaviour near these **singular sets**?

Resolution/Reduction of singularities approach for diff. operators.

First step: Define $\mathrm{Sing}(\Phi)$, generalizing both the function and vector field case.

The local behaviour should be **simple** outside $\mathrm{Sing}(\Phi)$.

Second step: Prove the existence of a modification

$$(M, \Phi) \xleftarrow{\varphi} (M', \Phi')$$

that is, a morphism φ such that:

1) $\varphi \colon M' \to M$ is proper and restricts to a biholomorphism outside $\mathrm{Sing}(\Phi)$.

2) The operator Φ' is the strict transform of Φ under this morphism

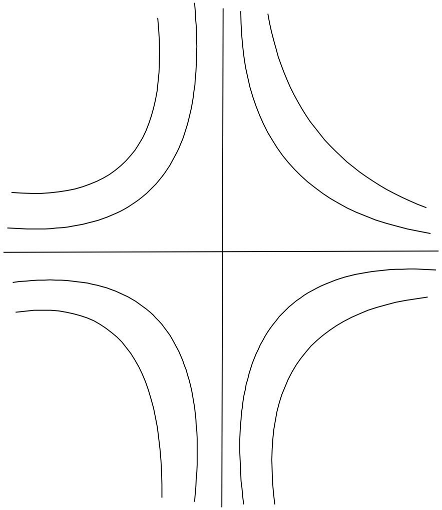
3) All singularities in $\mathrm{Sing}(\Phi')$ should be amenable to a normal form theory (so-called **final models**)

Confession: I don't know (for the moment) how these final models can be useful for the general theory of linear PDE, but there exists a whole theory of PDE and pseud-differential calculus on manifolds with boundary and corners

(see e.g. The *b-calculus* proposed by Melrose's paper on its ICM'90 paper).

Known cases:

Functions (0-order differential operators): This is a consequence of Hironaka's Theorem on resolution of singularities

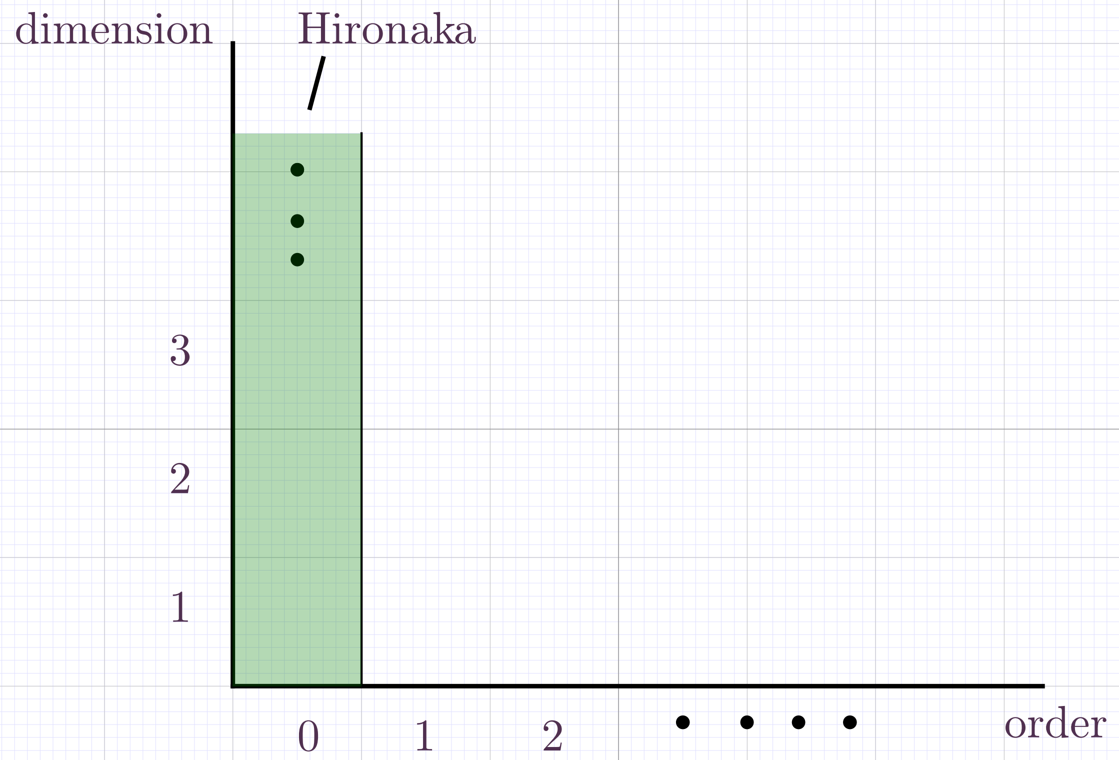
The final models are **monomials**, i.e. $f = x_1^{k_1} \ldots x_n^{k_n}$ 

Vector fields (1-order differential operators): The reduction of singularities in known to hold when $\dim M \leqslant 3$.

The final models are **elementary singularities** (also called **canonical**) of a vector field

$$\partial = f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n}$$

Current status



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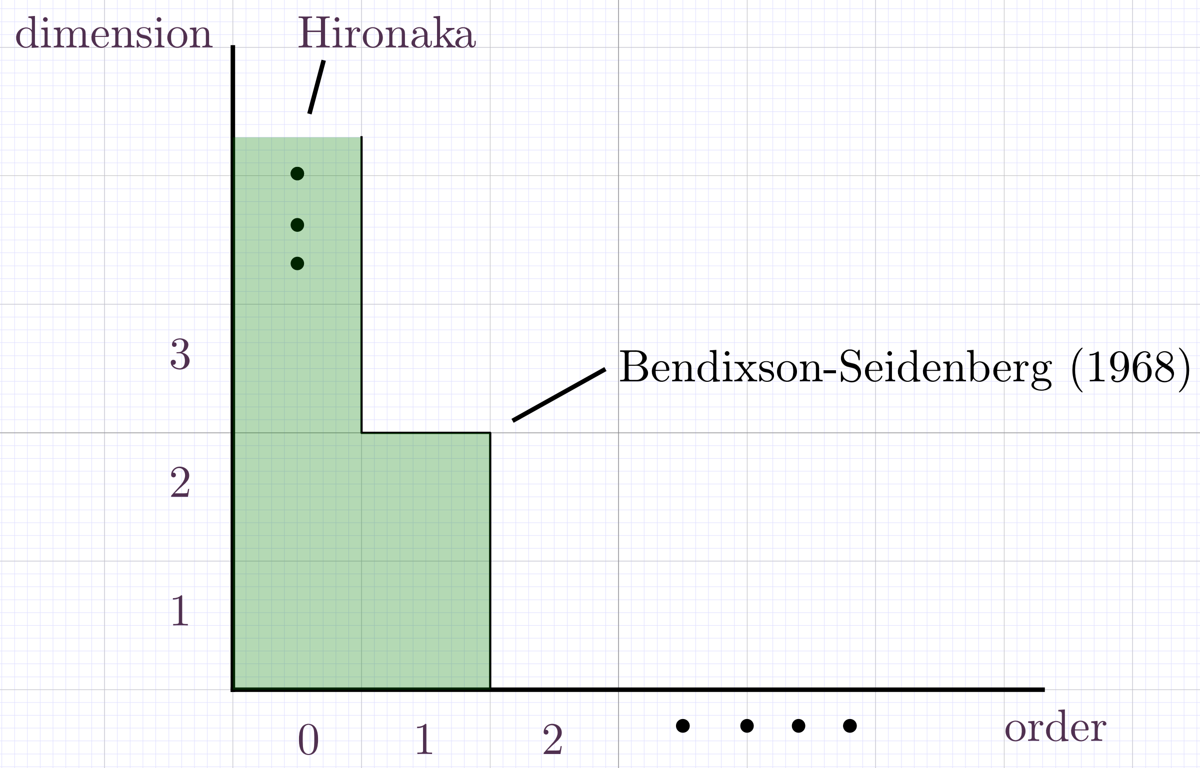

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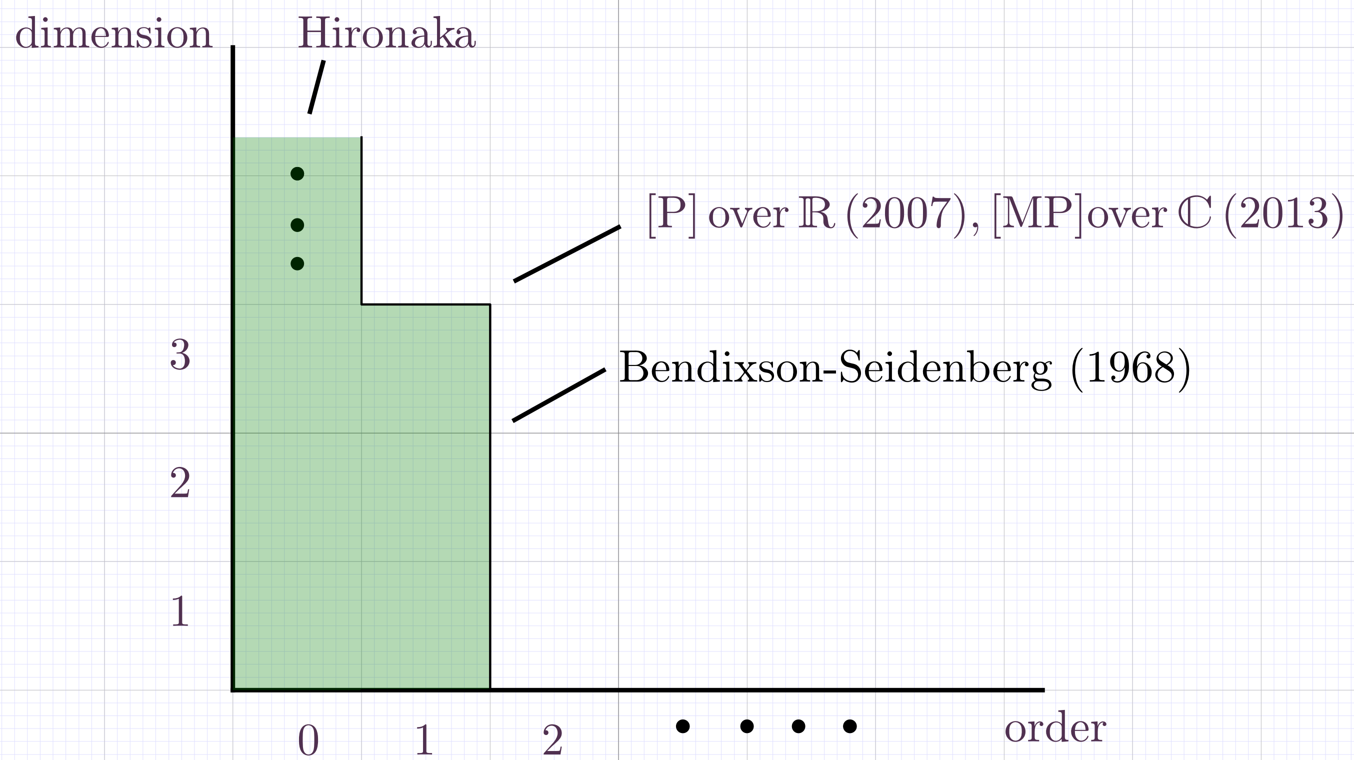

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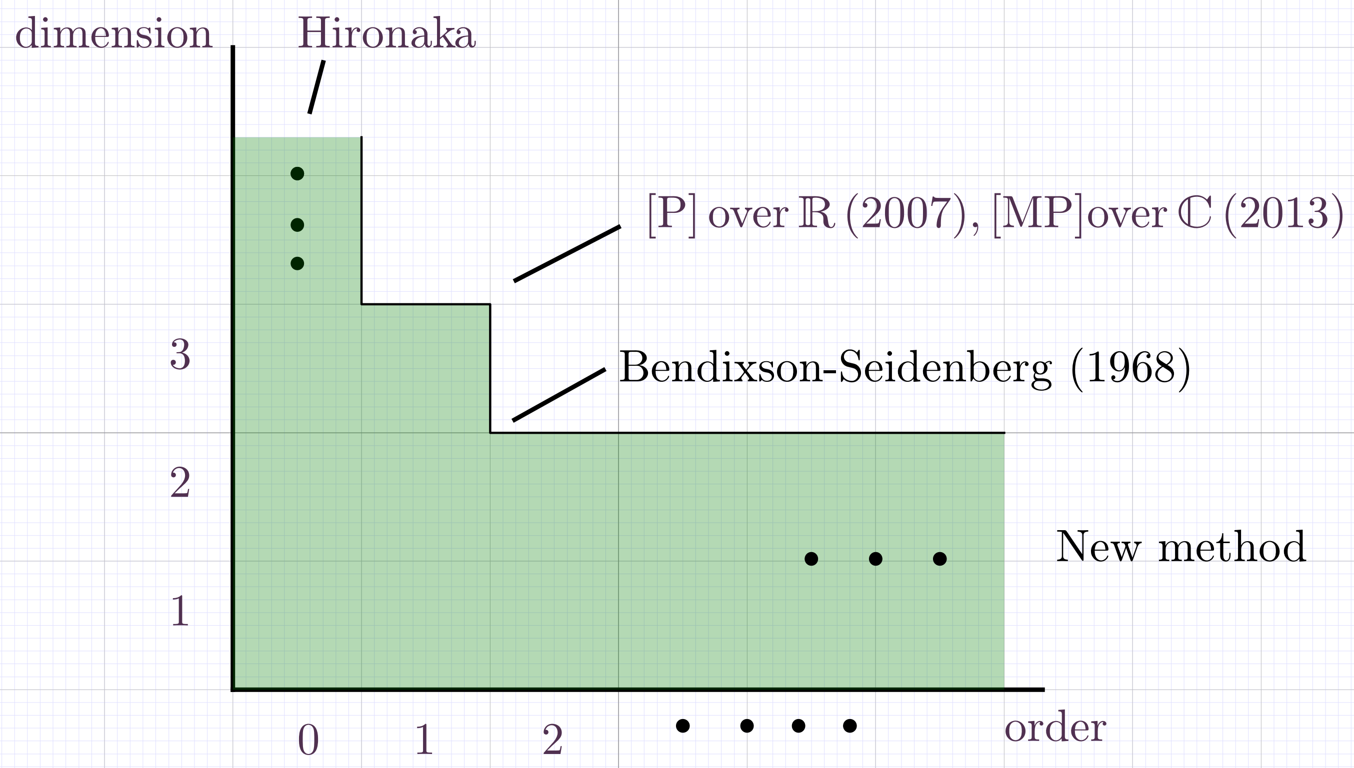


The main combinatorial object linked to a germ of singular operator is its **Newton polyhedron**. We now adopt a more abstract language, better suited to the GIT analogy.

Let $(\mathcal{O}, \mathfrak{m})$ be the local ring at a point $p \in M$.

Notation:

- $\mathrm{End}(\mathcal{O})$ is the module of continuos $\mathbb{C}$-endomorphisms $\Phi\colon \mathcal{O} \to \mathcal{O}$ for which there exists a $l \in \mathbb{Z}$ such that $\Phi(\mathfrak{m}^k) \subset \mathfrak{m}^{k+l}\ \forall k \in \mathbb{Z}_{\geqslant 0}$. We denote by $\mathrm{End}(\Phi, \mathfrak{m})$ those for which $l = 0$.
- $\mathrm{Der}(\mathcal{O}, \mathfrak{m}) \subset \mathrm{End}(\mathcal{O}, \mathfrak{m})$ is the Lie algebra of derivations (satisfying Leibniz rule)

$$\forall f, g \in \mathcal{O}, \quad \partial(fg) = (\partial f)g + f(\partial g)$$

- $\mathrm{Aut}(\mathcal{O}, \mathfrak{m}) \subset \mathrm{End}(\mathcal{O}, \mathfrak{m})$ is the group of automorphisms, satisfying

$$\forall f, g \in \mathcal{O}, \quad \varphi(fg) = \varphi(f)\varphi(g)$$

We denote by $\mathrm{End}(\hat{\mathcal{O}}, \hat{\mathfrak{m}}), \ldots$ their formal counterparts.

The action of the group $\mathrm{End}(\mathcal{O}, \mathfrak{m})$ on $\mathcal{O}$ induces an action $\mathrm{End}(\mathcal{O}, \mathfrak{m})$ into itself by conjugation - i.e. for $\psi, \varphi \in \mathrm{End}(\mathcal{O}, \mathfrak{m})$, $f \in \mathcal{O}$, the condition $\psi \cdot (\varphi f) = (\psi \cdot \varphi) f$ gives

$$\psi \cdot \varphi = \psi\, \varphi\, \psi^{-1}$$

Definition: A **maximal torus** in $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ is a subgroup $\mathbb{T} \subset \mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ which is isomorphic to a multiplicative torus $(\mathbb{C}^\star)^n$.

We denote by $\mathfrak{t} \subset \mathrm{Der}(\mathcal{O}, \mathfrak{m})$ the Lie algebra of $\mathbb{T}$.

Example: We fix local coordinates $(x_1, \ldots, x_n)$. Then, the $(\mathbb{C}^\star)^n$-action on $\mathcal{O}$ defined by

$$(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1 x_1, \ldots, t_n x_n)$$

defines an embedding $(\mathbb{C}^\star)^n \hookrightarrow \mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ whose image is a maximal torus.

The associated Lie algebra is the $\mathbb{C}$-submodule $\mathfrak{t}$ of derivations generated by

$$x_1 \frac{\partial}{\partial x_1}, \ldots, x_n \frac{\partial}{\partial x_n}$$

We say that $\mathbb{T}$ is the **standard torus** associated to these coordinates, notes $\mathbb{T}_{x,\mathrm{st}}$

Proposition. Let $\mathbb{T} \subset \mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ be a maximal torus. Then, there exists an unique (up to permutation of indices) system of coordinates $(x_1, \ldots, x_n)$ such that $\mathbb{T}_{x,\mathrm{st}}$.

Proof: Each vector field $\partial \in \mathfrak{t}$ is semi-simple. If we use Poincaré's-Dulac normal form, we can (formally) diagonalize simultaneously all $\mathfrak{t}$.

Now, if we take a $\partial \in \mathfrak{t}$ with a generic spectra (it suffices to require that $\mathrm{spec}(\partial) = \{\lambda_1, \ldots, \lambda_n\}$ is $\mathbb{Q}$-independent), we see that the formal diagonalization is unique, up to permutation of indices.

Moreover, taking $\mathrm{spec}(\partial) = \{\lambda_1, \ldots, \lambda_n\}$ of Bryuno type

(i.e. such that the numbers $\{\langle \lambda, k \rangle \mid \quad k \in \mathbb{Z}^n\}$ are not **abnormally small**)

we guarantee that such diagonalization is indeed convergent.

Corollary. Let $\mathrm{Center}(\mathbb{T})$ and $\mathrm{Norm}(\mathbb{T})$ denote respectively the subgroup of automorphisms whose action (under conjugation) centralizes (i.e. fixes pointwise each element of $\mathbb{T}$) and normalizes $\mathbb{T}$ (i.e. maps $\mathbb{T}$ into itself). Then the so-called Weyl group

$$\mathrm{Norm}(\mathbb{T}) / \mathrm{Center}(\mathbb{T}) \approx \mathrm{Sym}_n$$

where Sym_n the group of permutations in n-elements.

Proof: Just consider the group of automorphisms which map $\mathbb{T}_{\mathrm{st},x}$ into itself.

General property of Torus actions

Let $\mathbb{T}$ be a torus acting (regularly) on a finite dimensional vector space V. Then, there exists a direct sum decomposition

$$V = \bigoplus_{\alpha \in X(\mathbb{T})} \mathrm{Gr}_\alpha(V, \mathbb{T})$$

where $X(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^\star)$ is the group of characters of $\mathbb{T}$ and

$$\mathrm{Gr}_\alpha(V, \mathbb{T}) = \{v \in V : \forall t \in \mathbb{T}, t \cdot v = \alpha(t)\, v\}$$

(i.e. for each (α, t), $\mathrm{Gr}_\alpha(V, \mathbb{T})$ is the **eigenspace** for t associated to the eigenvector $\alpha(t)$).

In our setting, since the action of $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ is local (i.e. compatible with truncations), each maximal torus defines a direct sum decomposition

$$\mathcal{O} = \bigoplus \mathrm{Gr}_\alpha(\mathcal{O}, \mathbb{T})$$

and also, writing $\mathrm{End} = \mathrm{End}(\mathcal{O}, \mathfrak{m})$,.. for shortness, we have

$$\mathrm{End} = \bigoplus_\alpha \mathrm{Gr}_\alpha(\mathrm{End}, \mathbb{T}), \quad \mathrm{Der} = \bigoplus_\alpha \mathrm{Gr}_\alpha(\mathrm{Der}, \mathbb{T}), \quad \mathrm{Aut} = \bigoplus_\alpha \mathrm{Gr}_\alpha(\mathrm{Aut}, \mathbb{T})$$

Example: $\mathbb{T} = T_{\mathrm{st},x}$. Then the diagonal action on the variables $x_1, \ldots, x_n$ induces an action on the monomials $x^k = x_1^{k_1} \ldots x_n^{k_n}$,

$$(t_1, \ldots, t_n) \cdot x^k = t^k x^k$$

Therefore, identifying each element of $X(\mathbb{T})$ to $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ via the map

$$k(t) = t_1^{k_1} \ldots t_n^{k_n}$$

we have

$$\mathrm{Gr}_k(\mathcal{O}, \mathbb{T}) = \mathbb{C} x^k$$

$$\mathrm{Gr}_k(\mathrm{Der}, \mathbb{T}) = x^k \left(\mathbb{C} x_1 \frac{\partial}{\partial x_1} + \ldots + \mathbb{C} x_n \frac{\partial}{\partial x_n} \right)$$

$$\mathrm{Gr}_k(\mathrm{End}, \mathbb{T}) = \{ \varphi \in \mathrm{End} \colon \forall n \colon \mathbb{C} x^n \rightarrow \mathbb{C} x^{n+k} \}$$

It is also possible to define such graduation with respect to the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$

$$\mathrm{Gr}_k(\mathcal{O}, \mathbb{T}) = \mathrm{Gr}_k(\mathcal{O}, \mathfrak{t}) = \left\{ f : x_1 \frac{\partial}{\partial x_1} f = k_1 f, \ldots, x_n \frac{\partial}{\partial x_n} f = k_n f \right\}$$

Given an endomorphism Φ and a maximal torus $\mathbb{T}$, we consider the direct sum decomposition

$$\Phi = \sum_{\alpha \in X(\mathbb{T})} \Phi_\alpha$$

and define

$$\mathrm{supp}(\Phi, \mathbb{T}) = \{\alpha \in X(\mathbb{T}) \mid \Phi_\alpha \neq 0\}$$

and, upon identification of $X(\mathbb{T})$ to $\mathbb{Z}^n \subset \mathbb{R}^n$,

$$\mathrm{New}(\Phi, \mathbb{T}) = \mathrm{conv}(\mathrm{supp}(\Phi, \mathbb{T})) + (\mathbb{R}_{\geqslant 0})^n$$

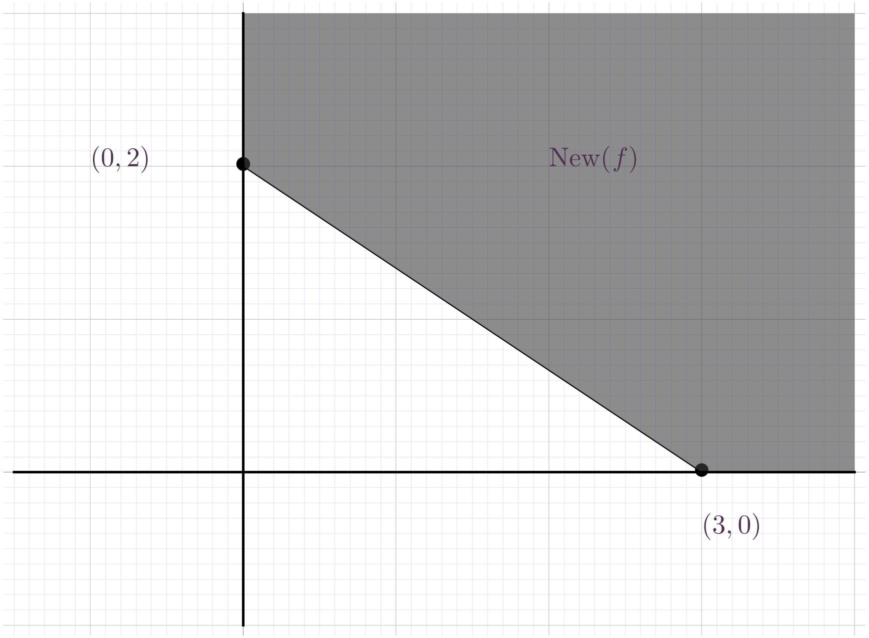
For instance, suppose that Φ is a **differential operator of order *d*.** Then, and that t $\mathbb{T} = \mathbb{T}_{\mathrm{st},x}$, for some local coordinates $(x_1, \ldots, x_n)$. Then, we write

$$\Phi = \sum_{s \in \mathbb{Z}^n} x^s \underbrace{P_s\left(x\frac{\partial}{\partial x}\right)}_{\text{polynomial of total deg} \leqslant d}, \qquad \text{where } x\frac{\partial}{\partial x} = \left(x_1\frac{\partial}{\partial x_1}, \cdots, x_n\frac{\partial}{\partial x_n}\right) \text{ is the } \textbf{logarithmic basis}$$

and $\mathrm{supp}(\Phi, \mathbb{T}) = \{s \in \mathbb{Z}^n \mid P_s \neq 0\}$.

Example (order 0 case): 1) $f = y^2 - x^3$ (diff. operator of order 0)

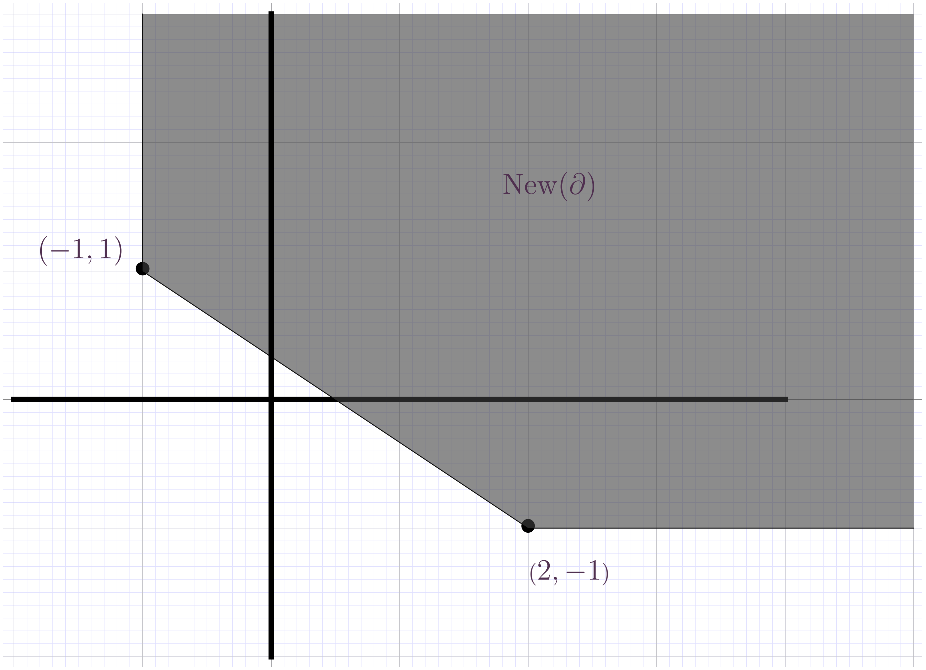
$$\mathrm{supp}(\Phi, \mathbb{T}) = \{(3,0), (0,2)\}$$



Example (order 1 case): Vector field (diff. operator of order one)

$$\partial = y\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$$

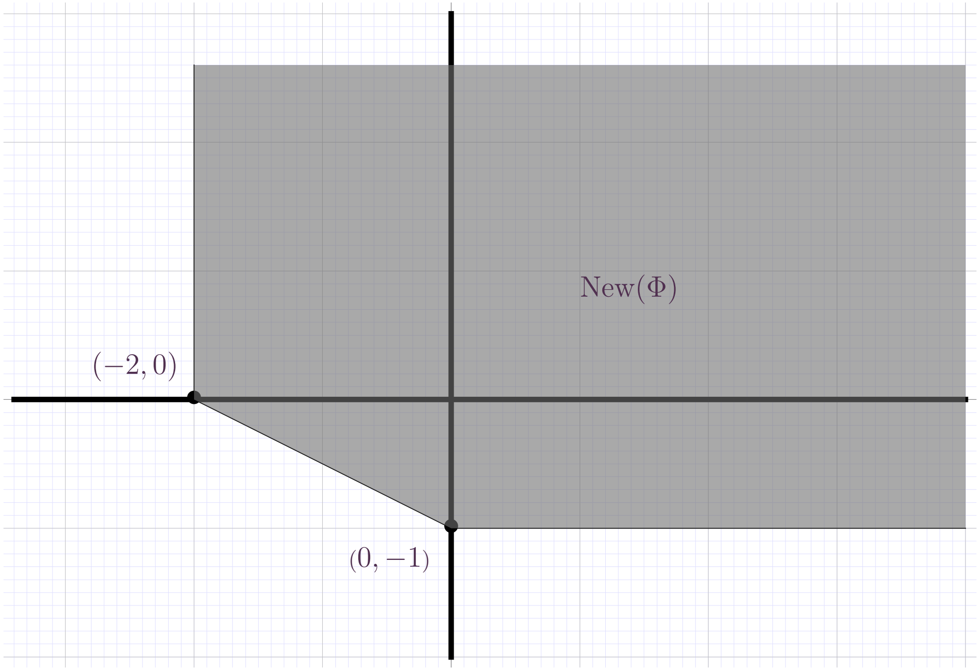
$$\partial = \underbrace{x^{-1}y}_{(-1,1)}\left(x\frac{\partial}{\partial x}\right) + \underbrace{x^2y^{-1}}_{(2,-1)}\left(y\frac{\partial}{\partial y}\right)$$



Example (order 2 case): Heat equation (diff. operator of order 2)

$$\Phi = \left(\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial t}\right)$$

$$\Phi = 2x^{-2}\binom{x\frac{\partial}{\partial x}}{2} - t^{-1}\binom{t\frac{\partial}{\partial t}}{1}, \qquad \text{where } \binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$$



Based on the fundamental dichotomy of GIT (Hilbert-Mumford criteria)

Definition. We say that a germ of endomorphism Φ at p is

- **unstable** if there exists a maximal torus $\mathbb{T} \subset \mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ such that

$$0 \notin \mathrm{New}(\Phi, \mathbb{T})$$

- **semi-stable** if for all maximal torus $\mathbb{T} \subset \mathrm{Aut}(\mathcal{O}, \mathfrak{m})$,

$$0 \in \mathrm{New}(\Phi, \mathbb{T})$$

The **unstable locus** $\mathrm{Unst}(\Phi)$ is the set of points p for which the germ Φ_p is unstable.

Examples: For $\Phi = \mu_f$ the scalar multiplication operator,

$$\mathrm{Unst}(\Phi) = V(f) \quad (i.e.\, p \in \mathrm{Unst}(\Phi) \Longleftrightarrow f \in \mathfrak{m}_p)$$

For $\Phi = \mu_f + \partial$ (general differential operator of order 1),

$$p \in \mathrm{Unst}(\Phi) \Longleftrightarrow f \in \mathfrak{m} \text{ and } \partial \text{ is nilpotent}$$

where, we recall, ∂ is called nilpotent if $\partial(\mathfrak{m}) \subset \mathfrak{m}$ and $\partial_S = 0$.

Alternative caracterization of unstability via one-parameter subgroups

Definition. A one-parameter subgroup of $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ is defined by an embedding λ of the multiplicative group $(\mathbb{C}^\star)$ into $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$. We will denote by $\mathrm{Lie}(\lambda) \subset \mathrm{Der}(\mathcal{O}, \mathfrak{m})$ the associated one-dimensional Lie-subalgebra.

Example: Fixing local coordinates, $(x_1, \ldots, x_n)$, we consider the of action $\mathbb{C}^\star$ on $\mathcal{O}$ by

$$t \cdot x = (t^{\omega_1} x_1, \ldots, t^{\omega_n} x_n)$$

for some $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n \setminus \{0\}$. We say that λ is *positive* is we can choose $\omega_1, \ldots, \omega_n$ of the same-sign. The associated lie algebra is generated (over $\mathbb{C}$) by the diagonal derivation

$$\omega_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \omega_n x_n \frac{\partial}{\partial x_1}$$

Remarks: 1) By Poincaré-Duac's theorem, each one-parameter group can be (formally) diagonalized (i.e. expressed as above in appropriate local coordinates). By Bruno's theorem (condition B), such coordinates can be chosen analytic.

2) Each one-parameter group is contained in a *maximal torus* of $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ (but this torus is far from being unique!).

(analogy: A maximal torus of $\mathrm{GL}(V)$ is defined by a basis of V, but a non-zero vector can belong to infinitely many distinct basis)

Important fact for the future...

Let us denote by $\Gamma(G)$ the set of 1-parameter subgroups of a group G, and by

$$\Gamma(G)/G$$

the cosets for the action of action of G in $\Gamma(G)$ by conjugation $(i.e. \quad g \cdot \lambda = g\lambda g^{-1})$.

Proposition: For each maximal torus $\mathbb{T} \subset \mathrm{Aut}(\mathcal{O}, \mathfrak{m})$

$$\Gamma(\mathrm{Aut}(\mathcal{O}, \mathfrak{m}))/\mathrm{Aut}(\mathcal{O}, \mathfrak{m}) = \Gamma(\mathbb{T})/(\mathrm{Norm}(\mathbb{T})/\mathrm{Cent}(\mathbb{T})) \approx \Gamma(\mathbb{T})/\mathrm{Sym}_n$$

(this is simply the fact that each one-parameter subgroup lies in a maximal torus and that each two maximal tori are $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$-conjugated)

As previously, for each one-parameter subgroup λ, we have a direct sum decomposition

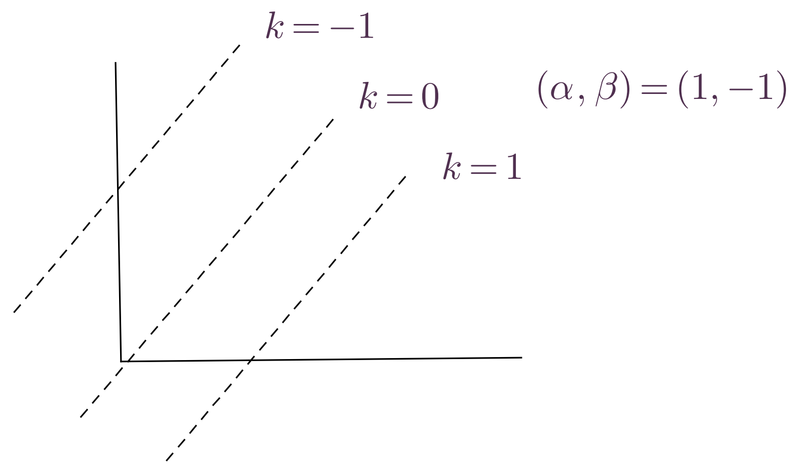
$$\mathcal{O} = \bigoplus_{\alpha \in X(\lambda)} \mathrm{Gr}_\alpha(\mathcal{O}, \lambda)$$

where the group of characters $X(\lambda)$ is now isomorphic to $\mathbb{Z}$.

Example: For $\lambda(t)$ defined by $t\cdot(x,y)=(t^\alpha x, t^\beta y)$, $(\alpha,\beta)\in\mathbb{Z}^2\setminus\{0\}$,

$$\mathrm{Gr}_k(\mathcal{O},\lambda)=\left\{f=\sum_{\alpha u+\beta v=k} a_{uv}\,x^u y^v\right\}$$

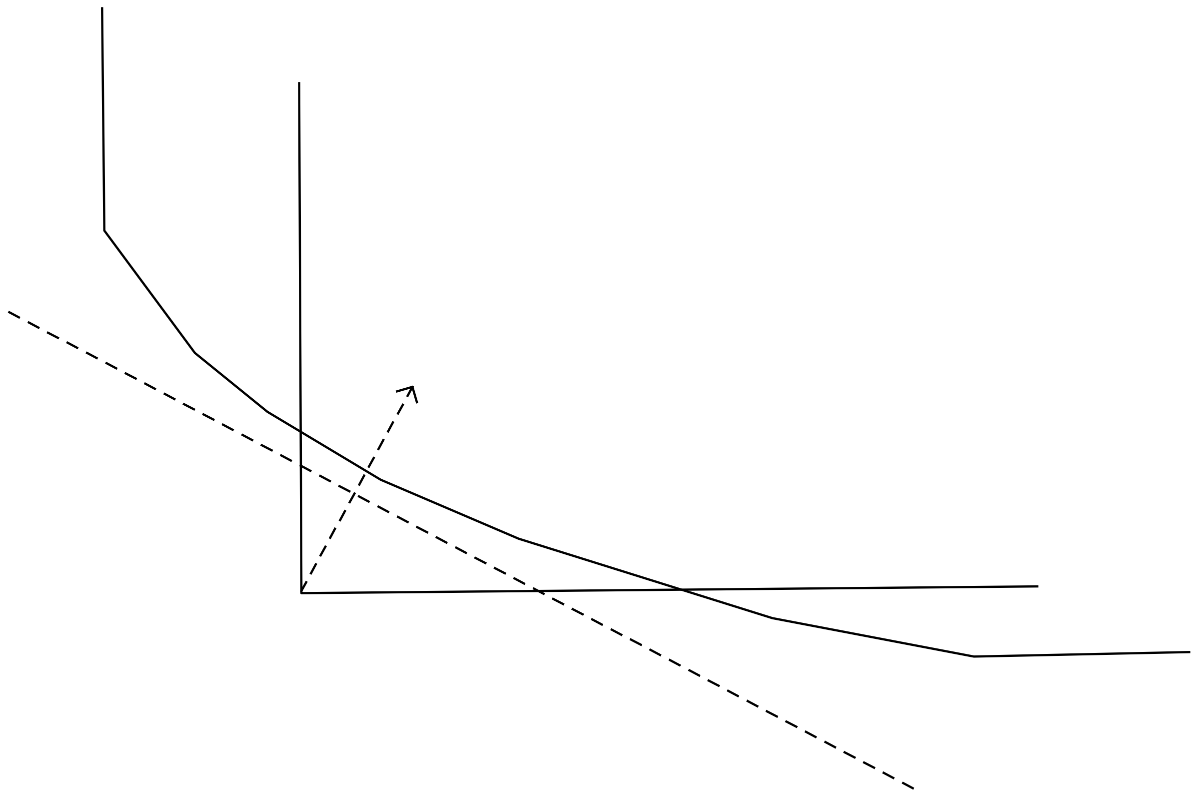
is the vector space of $(\alpha,\beta)-$quasi-homogeneous germs of degree k.



For $\Phi\in\mathrm{End}(\mathcal{O})$, we can define exactly as above its direct sum decomposition with respect to the graduation defined by a 1-psg $\lambda\in\mathrm{Aut}(\mathcal{O},\mathfrak{m})$, and let

Proposition: The germ of Φ is **unstable** if and only if there exists a **positive** 1-psg λ such that $\mathrm{supp}(\Phi,\lambda)\subset\mathrm{Gr}_{>0}(\mathrm{End},\lambda)$.

“Visual” proof:



The above discussion implies that

$$\operatorname{order}(\Phi) \leqslant 1 \Longrightarrow \operatorname{Unst}(\Phi) \text{ is } \textbf{closed}$$

Remark: In the case where $\Phi = \partial$, we recall the condition $\partial_S = 0$ is equivalent to say that the linearization

$$L_\partial\colon \mathfrak{m} / \mathfrak{m}^2 \longrightarrow \mathfrak{m} / \mathfrak{m}^2$$

is a nilpotent endomorphism. In its turn, this corresponds to the fact that the characteristic polynomial χ_{L_∂} is **trivial**.

Each local automorphism φ acts on $\mathfrak{m} / \mathfrak{m}^2$ as a linear coordinate change (isomorphic to $\mathrm{GL}(n, \mathbb{C})$),

And of course the coefficients of χ_{L_∂} are invariant with respect to such action.

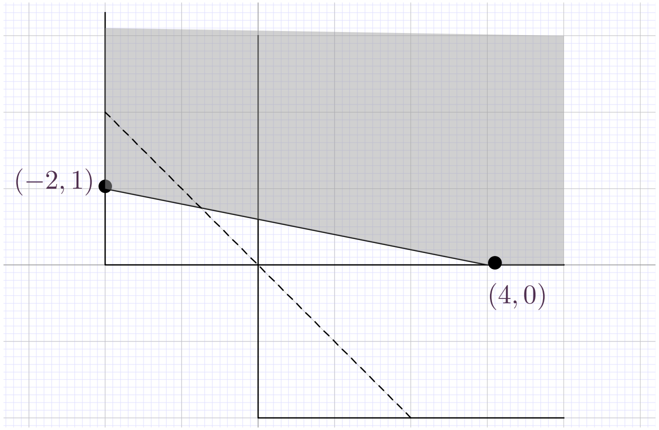
Open Problem: Prove that $\operatorname{Unst}(\Phi)$ is **closed** (wrt the analytic/Zariski topology) for Φ a differential operator of arbitrary order.

Definition: We say that the germ of Φ at p is **strongly unstable** if, for $\mathfrak{m} = \mathfrak{m}_p$,

- $\Phi(\mathfrak{m}^k) \subset \mathfrak{m}^k$ (i.e. Φ is local at p)
- Φ is unstable in the preceeding sense (i.e. $0 \notin \mathrm{New}(\Phi, \mathbb{T})$ for some maximal torus $\mathbb{T}$)

Example:

$$\Phi = x^{-2}y\binom{x\frac{\partial}{\partial x}}{2} + x^4$$



$$\Phi(x^m y^n) = \binom{m}{2} x^{m-2}y^{n+1} + x^{m+4}y \Longrightarrow \Phi(\mathfrak{m}^k) \subset \mathfrak{m}^{k-1} \text{ but } \Phi(\mathfrak{m}^2) \not\subset \mathfrak{m}^2$$

The germ is unstable but not strongly unstable (we note that $\Phi(\mathfrak{m}) \subset \mathfrak{m}$).

We denote by $S.\mathrm{Unst}(\Phi)$ the strongly unstable locus.

Proposition. 1) $S.\mathrm{Unst}(\Phi)=\mathrm{Unst}(\Phi)$ if Φ has order $\leqslant 1$.

2) $S.\mathrm{Unst}(\Phi)$ is closed.

Proof :

1) For $\Phi=\mu_f$ of order 0, we obviously have $\Phi(\mathfrak{m}^k)\subset\mathfrak{m}^k$. Hence, $S.\mathrm{Unst}(\Phi)=\mathrm{Unst}(\Phi)$.

For $\Phi=\partial+\mu_f$ of order one, the condition $\exists k\colon\Phi(\mathfrak{m}^k)\not\subset\mathfrak{m}^k$ is equivalent to the fact that

$$\partial(\mathfrak{m})\not\subset\mathfrak{m}$$

but, from the above characterisation, this implies that p is not an unstable point.

2) We will prove that locally at each point, there exits a finite collection of analytic functions $a_1, \ldots, a_m \in \mathcal{O}$ such that

$$S.\mathrm{Unst}(\Phi) = Z(a_1, \ldots, a_m)$$

Suppose that $p \in M$ is such that $\Phi(\mathfrak{m}^k) \subset \mathfrak{m}^k$ (which is this is expressed by analytic conditions). We fix local coordinates $(x_1, \ldots, x_n)$ and consider the standard maximal torus $\mathbb{T}_{\mathrm{st}} = \mathbb{T}_{\mathrm{st},x}$.

Since the action of $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ on the set of maximal tori is transitive, we have

$$p \in S.\mathrm{Unst}(\Phi) \Longleftrightarrow \boxed{\exists \varphi \in \mathrm{Aut}(\mathcal{O}, \mathfrak{m}) : \mathrm{New}(\varphi \Phi \varphi^{-1}, \mathbb{T}_{\mathrm{st}}) \not\ni 0}$$

Let us consider the 1-psg h associated to $x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$ (homogeneous graduation). The 0-degree component $G = \mathrm{Gr}_0(\mathrm{Aut}(\mathcal{O}, \mathfrak{m}))$ of $\mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ is isomorphic to $\mathrm{GL}(n, \mathbb{C})$.

It acts by conjugation on the degree 0 component of $\mathrm{End}(\mathcal{O}, \mathfrak{m})$,

$$G \times \mathrm{Gr}_0(\mathrm{End}(\mathcal{O}, \mathfrak{m})) \longrightarrow \mathrm{Gr}_0(\mathrm{End}(\mathcal{O}, \mathfrak{m})), \qquad (\varphi_0, \Phi_0) \to \varphi_0\, \Phi_0\, \varphi_0^{-1}$$

The subset of differential operators of degree $\leqslant d$ forms a finite dimensional vector space

$V = \mathrm{Gr}_0(\mathrm{Diff}^{\leqslant d}(\mathcal{O}, \mathfrak{m})) \subset \mathrm{Gr}_0(\mathrm{End}(\mathcal{O}, \mathfrak{m}))$, which is invariant by the G-action.

Some concepts of GIT Let G be a complex reductive group acting linearly on a finite dimensional $\mathbb{C}$-vector space V.

Let $\mathbb{C}[V]^G$ denote the ring of invariant for group action.

Hilbert's theorem There exists polynomials $a_1, \ldots, a_m$ such that $\mathbb{C}[V]^G = \mathbb{C}[a_1, \ldots a_m]$

The algebraic set

$$\mathcal{N}_G(V) = Z(a_1, \ldots, a_m) \subset V$$

is called the **null-cone** for the G action on V. It is the fiber over 0 for the quotient map

$$\pi: V \to V / G$$

How to characterize the null-cone without computing $\mathbb{C}[V]^G$?

Given a torus $T \subset G$, let $V = \bigoplus_\alpha V_\alpha$ denote the direct sum decomposition associated to the corresponding torus-action. (e.g. for $G = \mathrm{GL}(n, \mathbb{C})$, a maximal torus is simply the subgroup of diagonal matrices with respect to a given basis of $\mathbb{C}^n$)

As previously, we can define the **support** $\mathrm{supp}(v, T)$, for each $v \in V$:

$$v = \sum_\alpha v_\alpha, \qquad \text{with } v_\alpha \in V_\alpha, \qquad \Longrightarrow \qquad \mathrm{supp}(v, T) = \{\alpha \in X(T): v_\alpha \neq 0\}$$

Hilbert-Mumford criteria.

Theorem (Hilbert-Mumford) $v \in \mathcal{N}_G(V)$

$\iff \exists$ maximal torus $T \subset G$ such that $\mathrm{conv}(\mathrm{supp}(v, T)) \not\ni 0$

$\iff \exists$ a 1-psg $\lambda \subset G$ such that $\mathrm{supp}(v, \lambda) \subset \mathrm{Gr}_{>0}(V, \lambda)$.

Remark: The first $\iff$ allows us to "eliminate the $\exists$ quantifier" in this finite dimensional setting, since $N_G(V)$ is defined by the vanishing locus of $a_1, \ldots, a_m$ (of generators of $\mathbb{C}[V]^G$)

Remark: Geometrically. $v \in N_G(V) \iff 0 \in \overline{G \cdot v}$ (i.e. 0 lies in the closure of the G-orbit of v)

On the other hand, the last statement in the equivalence means that

$$\lim_{t \to \infty} \lambda(t) \cdot v = 0$$

Therefore, the HM criteria says that

0 belongs to the closure of the orbit $G \cdot v$ **iff** then there exists a 1-psg which **steers** v to 0.

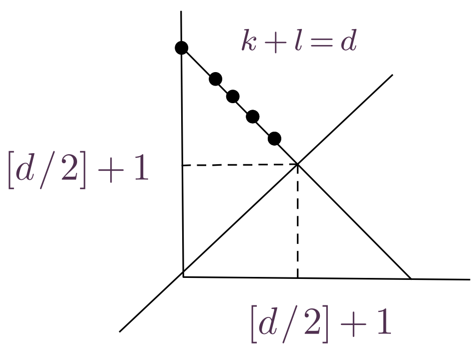
Example (classical) We consider the space of homogeneous d polynomials in two variables, where the reductive group $\mathrm{SL}(2,\mathbb{C})$ acts by linear change of coordinates

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \quad \Longrightarrow \quad (g\cdot p)(x,y) = p(dx-by,-cx+ay),$$

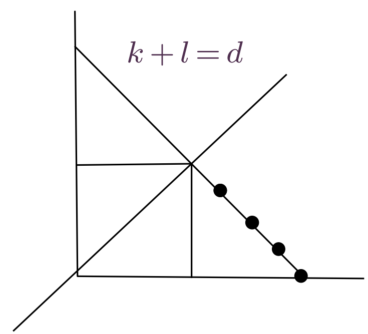
The standard maximal torus in $\mathrm{SL}(2,\mathbb{C})$ is given by $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$,

which acts on a monomial $x^k y^l$ by mapping it to $t^{k-l}x^k y^l$.

Therefore, $\mathrm{supp}(p,\lambda) \subset \mathrm{Gr}_{>0}(V,\lambda)$ if and only if p is divisible by $x^{[d/2]+1}$ or by $y^{[d/2]+1}$.



or



By HM, p is in the null-cone if and only if it has a root of multiplicity **at least $[d/2]+1$.**

Example (classical) We consider action of $\mathrm{GL}(n,\mathbb{C})$ on the matrices $\mathrm{gl}(n,\mathbb{C})$ by conjugation. Then,

the ring of invariants is given by the coefficients of the characteristic polynomial

$$\chi_A(s) = \det(sI - A)$$

and A lies in the null-cone if and only if it is nilpotent.

The standard maximal torus $\mathbb{T}$ is given by the embedding of $(\mathbb{C}^\star)^n$ into the diagonal matrices

$$\mathrm{diag}(t_1, \ldots, t_n)$$

and, for $e_{ij} = (\delta_{ij})$ the basis elements of $\mathrm{gl}(n,\mathbb{C})$,

$$\mathrm{diag}(t) \cdot e_{ij} \cdot \mathrm{diag}(t^{-1}) = (t_i t_j^{-1})\, e_{ij}$$

If A is nilpotent and in jordan normal form then $\mathrm{supp}(A,\mathbb{T})$ can be separated from 0 by a hyperplane.

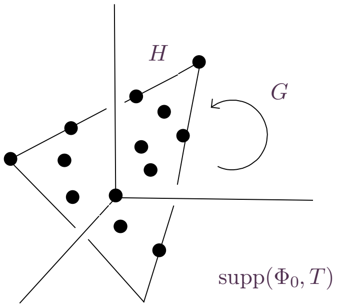
Hence, $A \in \mathcal{N}_G(V) \Longleftrightarrow A \,\text{is nilpotent}$.

Back to the original problem: Prove that $S.\mathrm{Unst}(\Phi)$ is closed.

Let $\Phi_0 \in \mathrm{Gr}_0(\mathrm{End}, h)$ be the degree 0 component of Φ with respect to $\mathrm{Gr}(\cdot, h)$.

For each fixed coordinates $(x_1, \ldots, x_n)$, $\mathrm{supp}(\Phi_0, T_{\mathrm{st},x})$ is a finite subset of

$$H = \{k \in \mathbb{Z}^n_{\geqslant -d} \mid k_1 + \cdots + k_n = 0\}$$



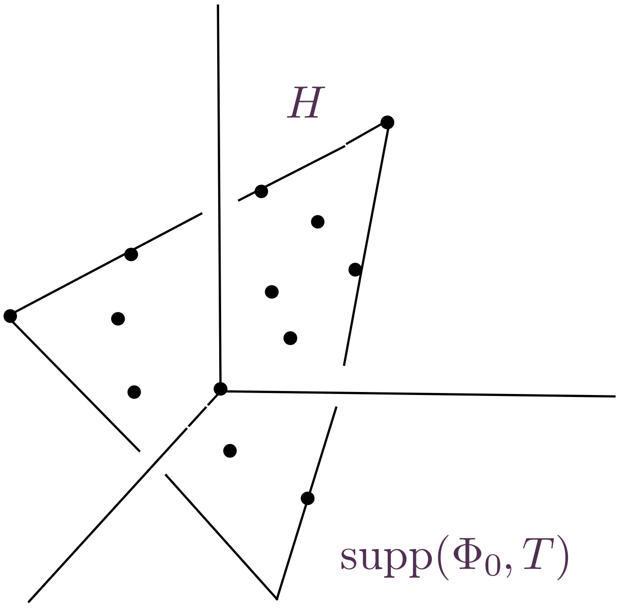
We claim that p belongs to $S.\mathrm{Unst}(\Phi)$ if and only if $\Phi_0 \in N_G(V)$, where $V = \mathrm{Gr}_0(\mathrm{End}, h)$.

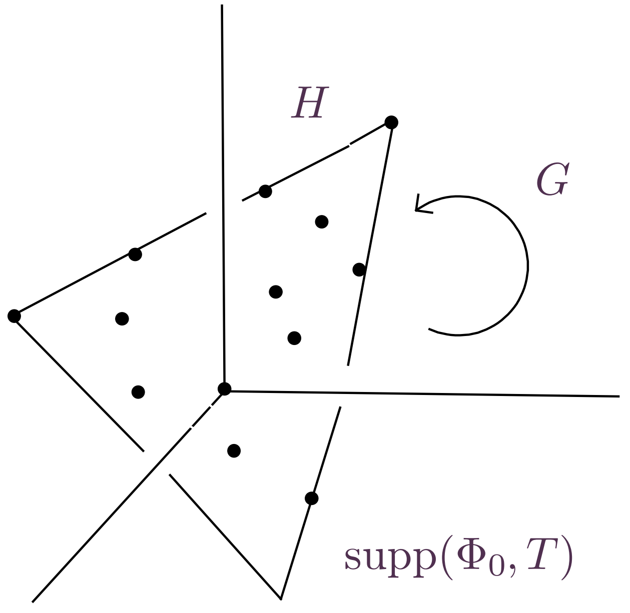
(indeed, $S.\mathrm{Unst}(\Phi)$ means that there exists a maximal torus $\mathbb{T} \in \mathrm{Aut}(\mathcal{O}, \mathfrak{m})$ such that

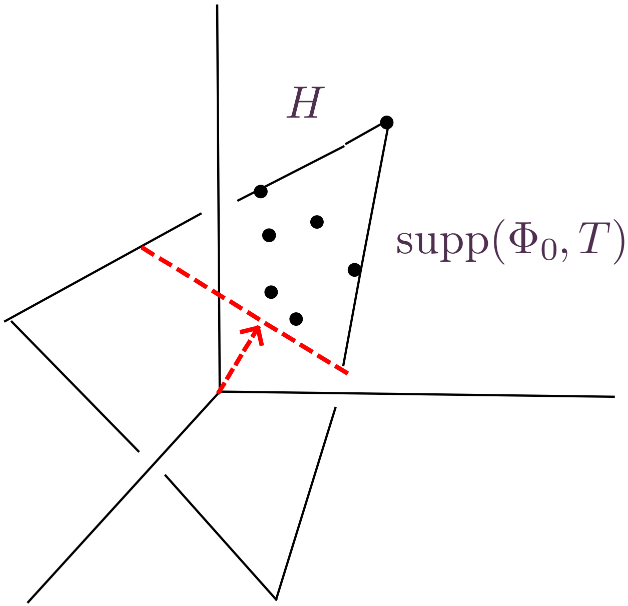
$$0 \notin \mathrm{supp}(\Phi, \mathbb{T})$$

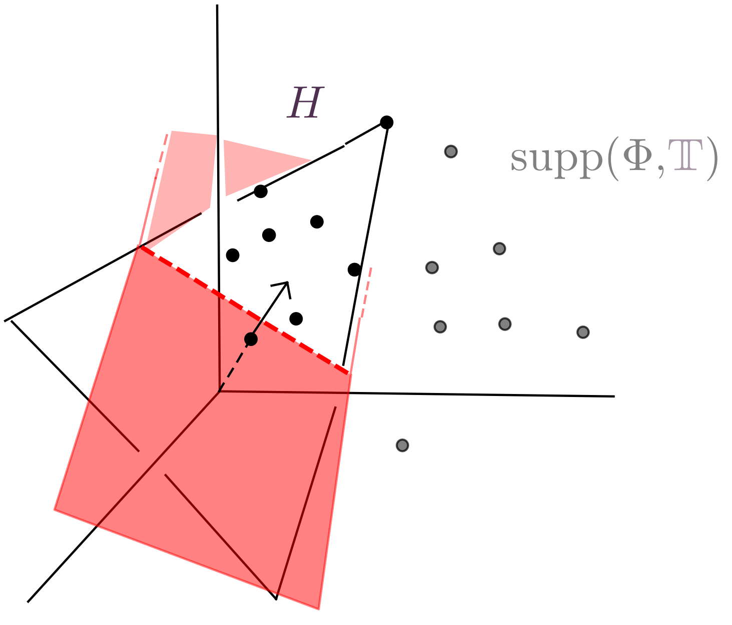
but this holds if and only if we can find a maximal torus $T \subset G = (\text{linear part of } \mathrm{Aut}(\mathcal{O}, \mathfrak{m}))$

such that $0 \notin \mathrm{supp}(\Phi_0, T)$. By the HM criteria, this condition is determined by a finite number of polynomial equations the coefficients of Φ_0.

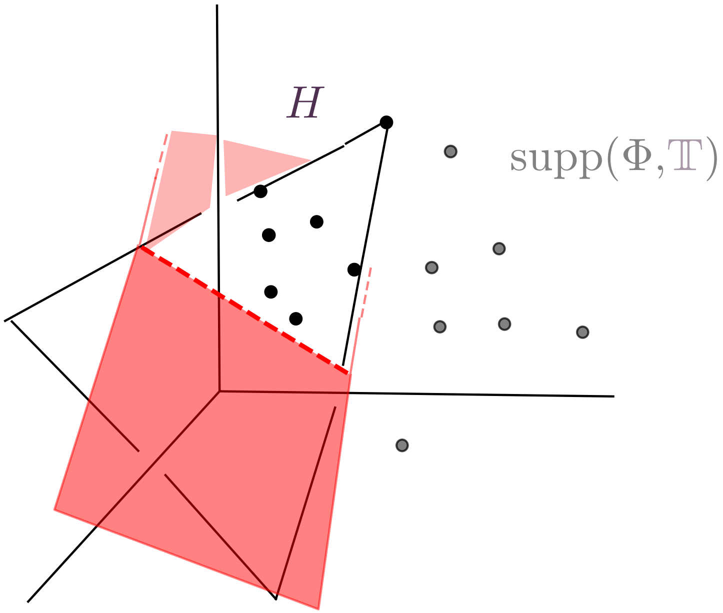

H
supp(Φ0, T)


H
G
$\mathrm{supp}(\Phi_0, T)$


H
$\mathrm{supp}(\Phi_0, T)$



H
supp(Φ,𝕋)



By the  - method

Problem of elimination of the S.Unst locus

Let $\Phi \in \mathrm{Diff}^{\star}(M)$ be a differential operator on a manifold M. Is there a locally finite sequence of blowing-ups

$$(M, \Phi) = (M_0, \Phi_0) \longleftarrow (M_1, \Phi_1) \longleftarrow \cdots \longleftarrow (M_r, \Phi_r) = (M', \Phi')$$

with center on the Strongly unstable locus, and such that

$$S.\mathrm{Unst}(\Phi') = \emptyset$$

Remark: In this case, we are **not** requiring a logarithmic resolution, i.e. that the blowing-up center has normal crossings with the exceptional divisor.

The basic idea of the unstability approach

Question of Mumford-Tits. Let G be a reductive group acting on a vector space V.

Assuming that $v \in \mathcal{N}_G(V)$. Then, by $\mathrm{HM}-\mathrm{criterium}$, there exists a one-parameter group $\lambda \subset G$ such that

$$\lambda(t) \cdot v = O(t^k) v$$

for some $k \geqslant 1$ (i.e. λ *steers* v to 0 at order k) (we note $\mu(v, \lambda) = k$). Can we caracterise the subset of one-parameter groups for which such order is **maximal** ?

(We have to "normalize") because if we define $\lambda_1(t) = \lambda(t^r)$ for some $r \in \mathbb{Z}_{>0}$ then

$$\mu(v, \lambda_1) = r \mu(v, \lambda)$$

G. Kempf, *Instability in invariant theory - Annals of Math. 108(2)*

Definition. A *length* on Γ(G) is a non-negative real valued function $\lambda \mapsto \|\lambda\|$

such that:

a)(G-invariance) $\|g \lambda g^{-1}\| = \|\lambda\|$ for all $\lambda \in \Gamma(G)$ and $g \in G$

b)(inner product) For any maximal torus $T \subset G$, there exists a positive definite integral valued bilinear form $\langle \, , \, \rangle$ on $\Gamma(T)$ such that $\|\lambda\|^2 = \langle \lambda, \lambda \rangle$, for any $\lambda \in \Gamma(T)$.

In particular, by the G-invariance, the inner product should be invariant the action of the Weyl group of T on $\Gamma(T)$.

In particular, if the Weyl group is transitive on a $\mathbb{Z}$-basis of $\Gamma(T)$, this inner product is **unique** (up to a constant factor).

Definition. Suppose that $v \in V$ is unstable. For each nonzero $\lambda \in \Gamma(G)$, we define

$$\mathrm{speed}(v,\lambda) = \frac{\mu(v,\lambda)}{\|\lambda\|}$$

Set

$$\mathrm{Speed}(v) = \sup_{\lambda \in \Gamma(G)} \mathrm{speed}(v,\lambda)$$

and

$$\Xi(v) = \{\lambda \in \Gamma(G) : \mathrm{speed}(v,\lambda) = \mathrm{Speed}(v)\}$$

which is the so-called *optimal set*.

Goal: We would like to characterize $\Xi(v)$.

Polyhedral interpretation

There exists a "perfect pairing" between $X(T)$ (the character group) $\Gamma(T)$ (the set of one-parameter subgroups of T), seen as $\mathbb{Z}$-modules,

which is given by the bilinear map $(\chi, \lambda) \in X(T) \times \Gamma(T) \mapsto \chi(\lambda)$ (evaluation of the character on λ).

The inner product $\langle\,,\,\rangle$ (used to define the length) stablishes an isomorphism $\nu\colon \Gamma(T) \sim X(T)$, defined by the equality

$$\nu(\lambda)\rho = \langle \lambda, \rho \rangle, \qquad \forall \rho \in X(T)$$

which allows us to extend the **length function** to the character group.

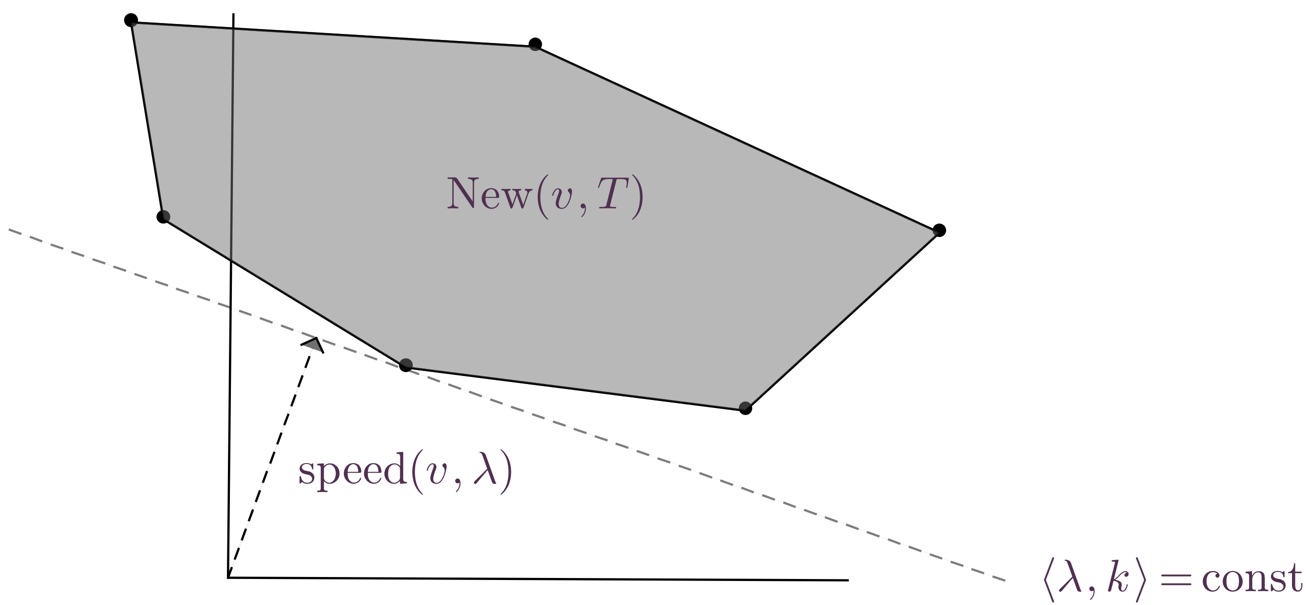
How to "see" the $\mathrm{speed}(v, \lambda)$?

Choose any maximal torus T which contains λ, and let

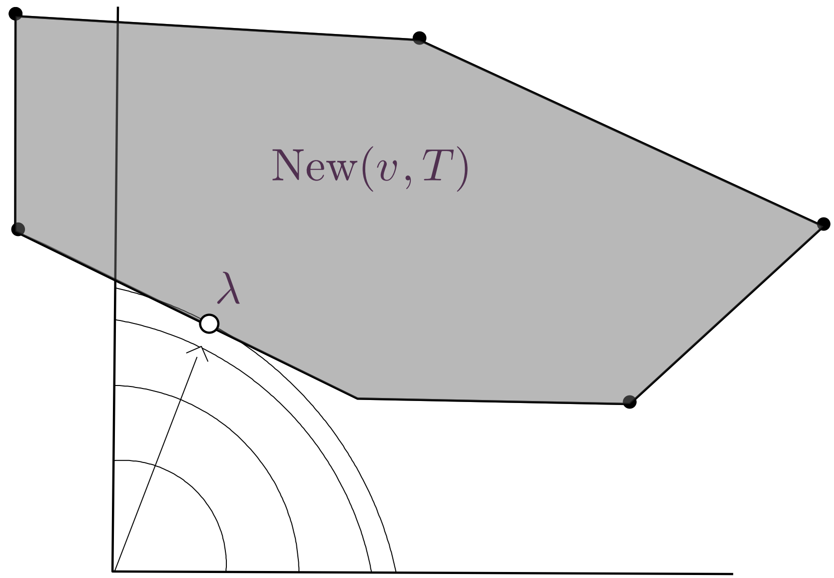
$$\mathrm{New}(v, T)$$

be the Newton polyhedron of v with respect to T (i.e. we decompose $v = \sum_\chi v_\chi$ and consider the convex enveloppe of the support).

By identifying $X(T)$ with $\mathbb{Z}^n$ (and assuming that each basis element has length one)...



For a *fixed maximal torus*, the speed in maximized by taking the “nearest point” on the polyhedron.



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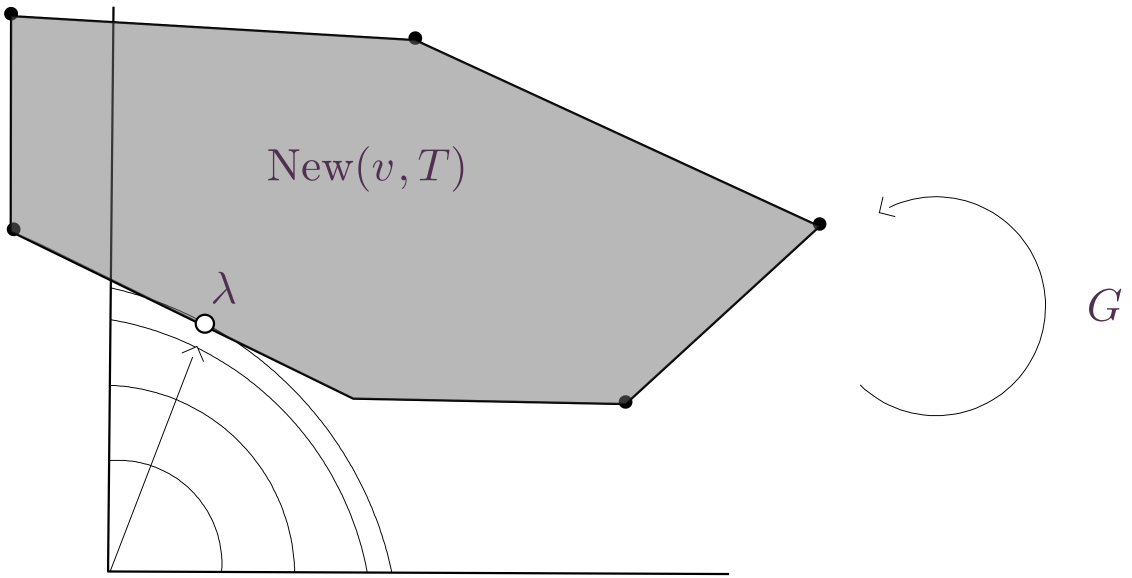

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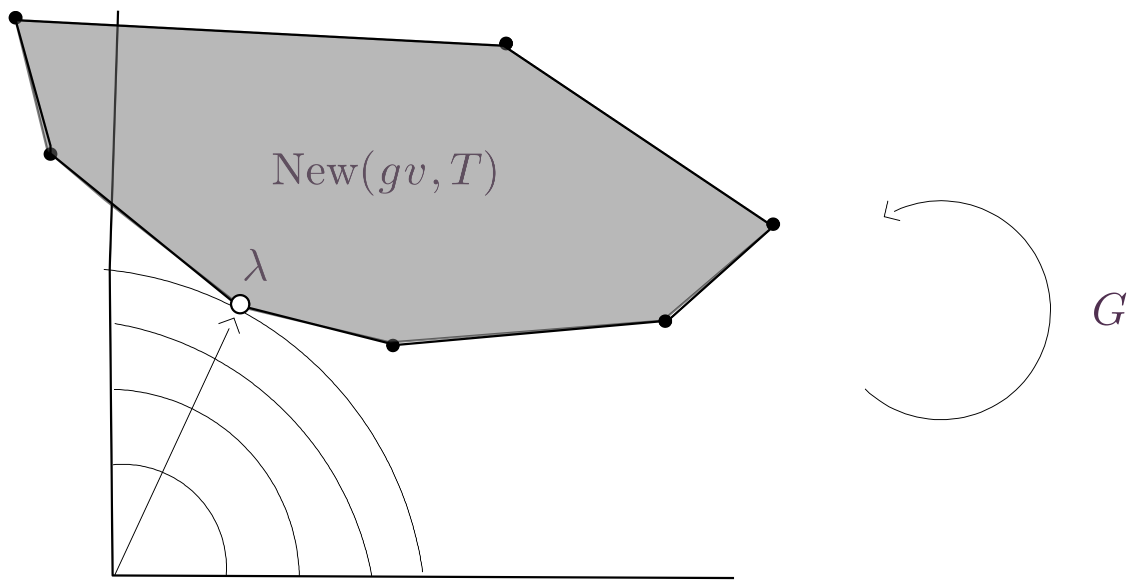

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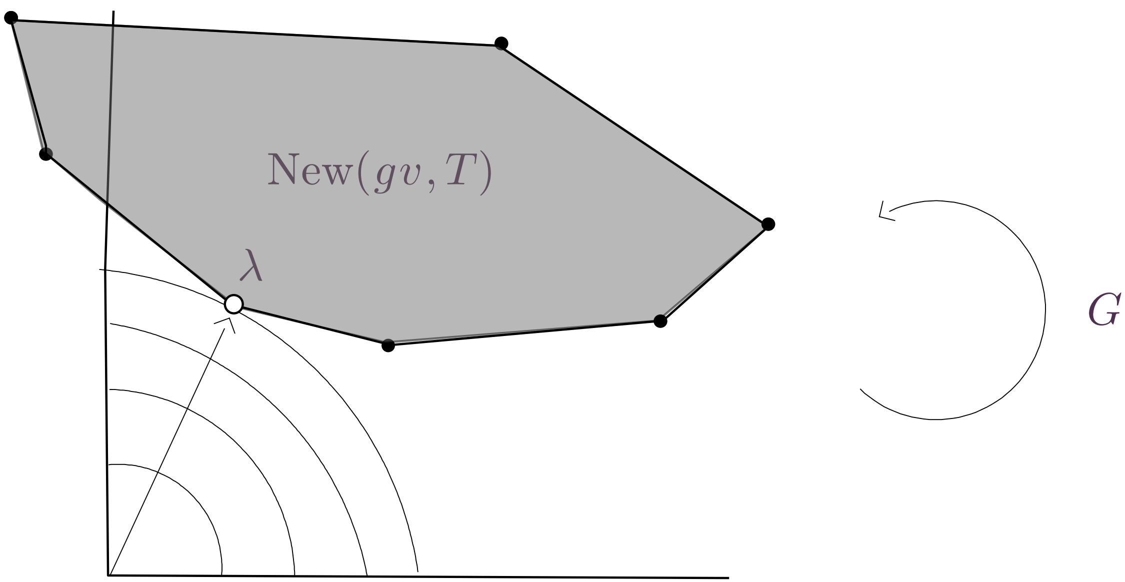


Theorem of Kempf. ()

1)(Existence) The set $\Xi(v)$ is non-empty (i.e. the $\sup$ of the speed is attained)

2)(Uniqueness of the optimal set up to parabolics) For any $\lambda \in \Xi(v)$, we have

$$\Xi(v) = \{g\lambda g^{-1}: \quad g \in \mathrm{Gr}_{\geqslant 0}(G, \lambda)\} = \mathrm{Par}(G, \lambda) \cdot \lambda$$

(i.e. all elements of $\Xi(v)$ define **precisely** the same filtration of V).

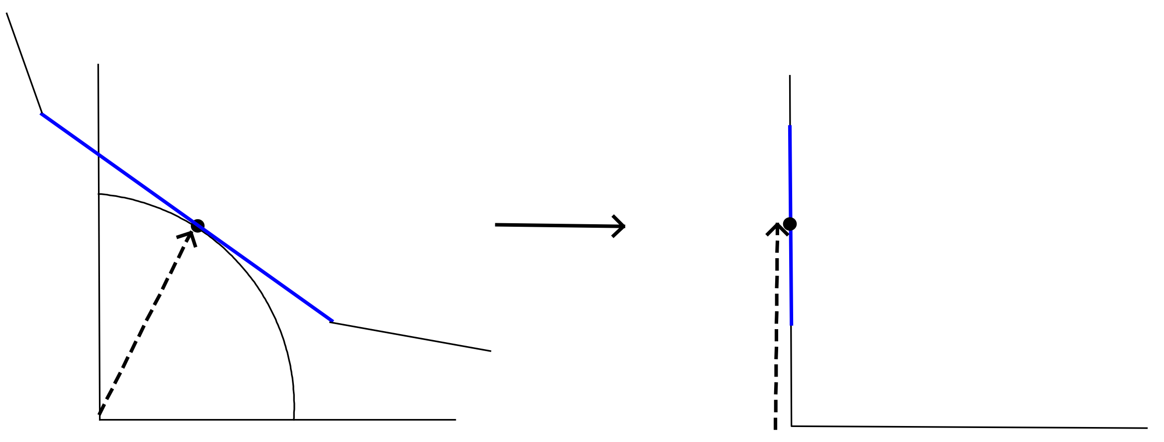
We would like to adapt this to the context of differential operators.

Basically: Let Φ be a germ of differential operator at $p \in M$

1) Define the Speed(Φ) as the main invariant.

2) The local strategy consists in blowing-up with the filtration defined by $\Xi(\Phi)$.

Combinatorial effect of a weighted blowing-up on the nearest point.



But we have to take care of the translations, and prevent the *compensation phenomena*.

Is there an analog of the stabilization procedure.

Theorem (Kirwan [1984], Ness [1984]) Let v be an unstable vector in V. Then, a one-parameter subgroup $\lambda \in T$ is optimal (i.e. lies in $\Xi(v)$) if and only if the projection

$$v_k \in \mathrm{Gr}_k(V, \lambda)$$

is *semi-stable* with respect to the action of the "slice subgroup" $\mathrm{Gr}_0(G, T) \subset G$, which is also reductive.

In fact, this result allows to define an algebraic stratification of the null-cone

$$\mathcal{N}_G(V) = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_s \cup \{0\}$$

in terms of the speed, so-called *Hesselink stratification*.

In our context, a similar result would completely prevent full compensation phenomena.

Example: For $\lambda(t)$ defined by $\lambda(t)\cdot x_i \to t x_i$ (in some coordinate system) we obtain

$$G(\lambda)/\langle\lambda\rangle = \mathrm{PSL}(n,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^{n-1})$$

(the automorphism group of the projective space)

To deal with these, we need some analog of Geometric invariant theory for non-reductive groups.