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Integrability Theorem (Sussman): There exists a leaf of $\mathcal{F}$ through each point $p \in M$.

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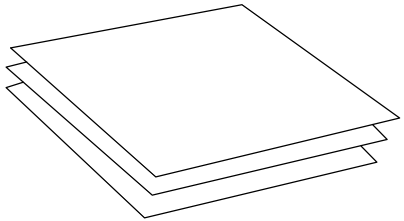

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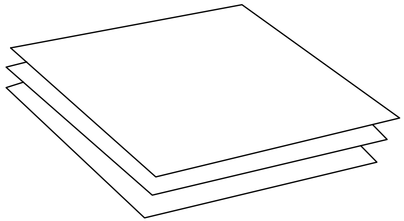

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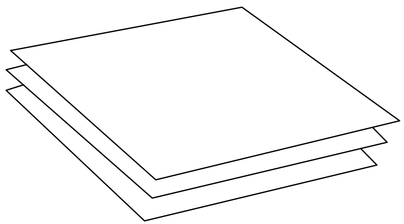

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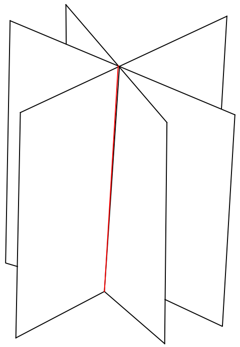

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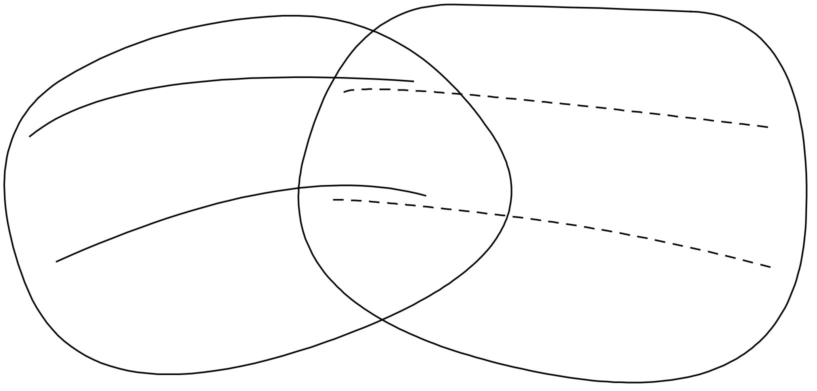
Remark: In general, we cannot expect to have a single global generator for a foliation.



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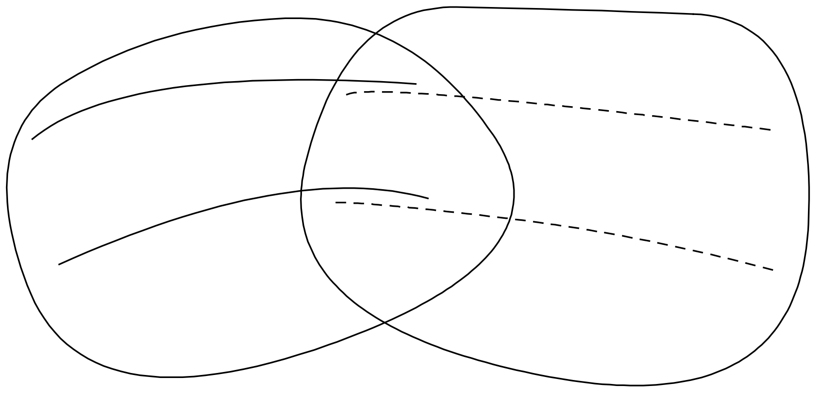

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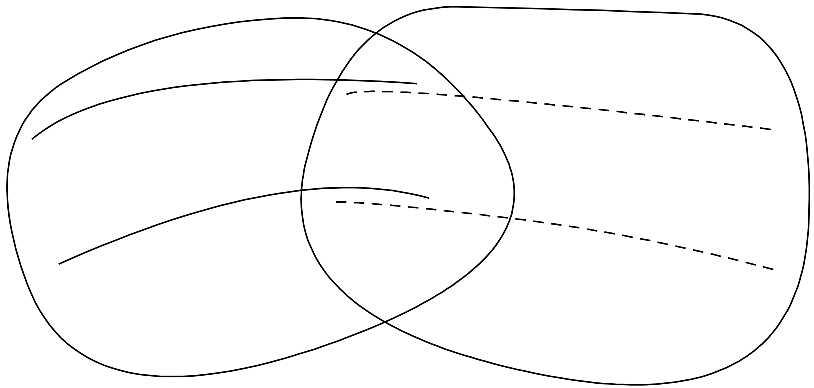

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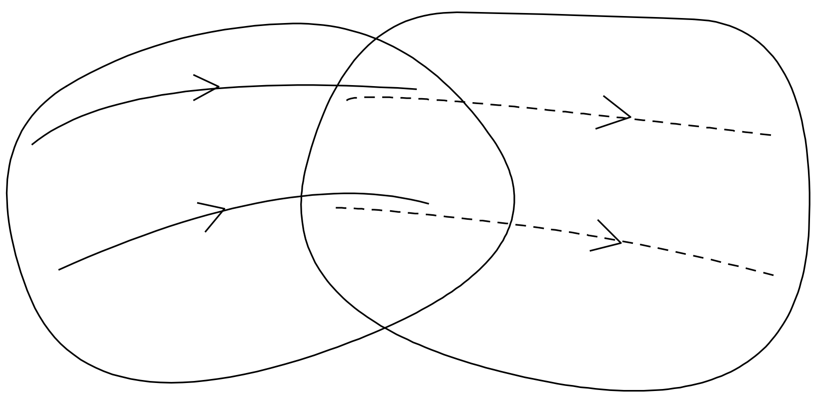

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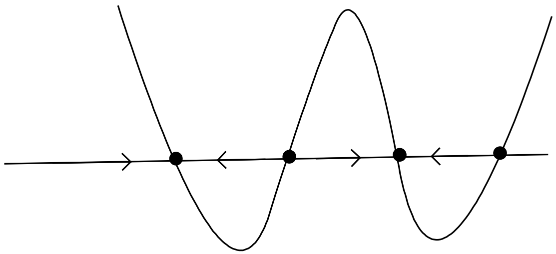

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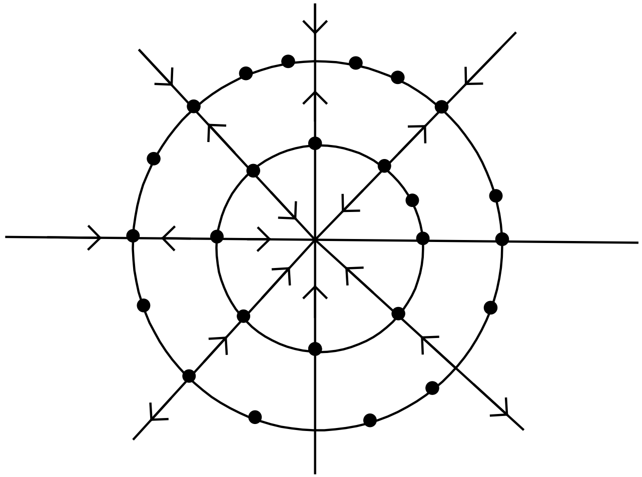

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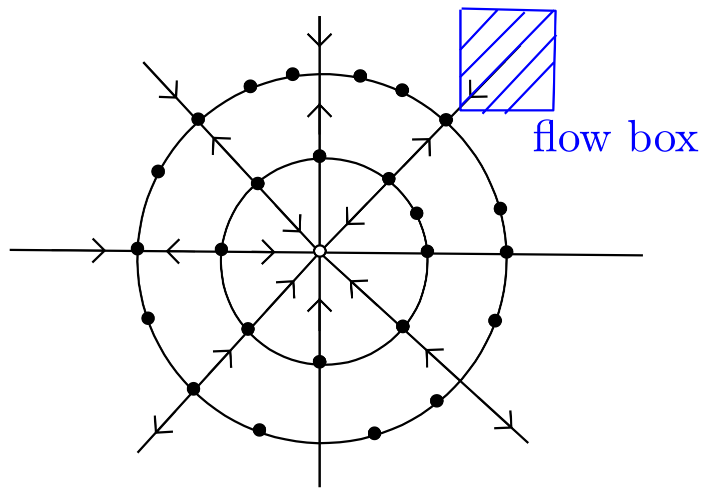

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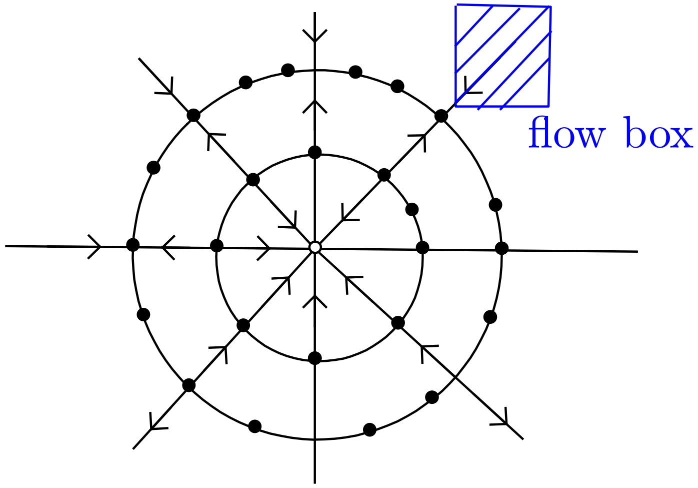

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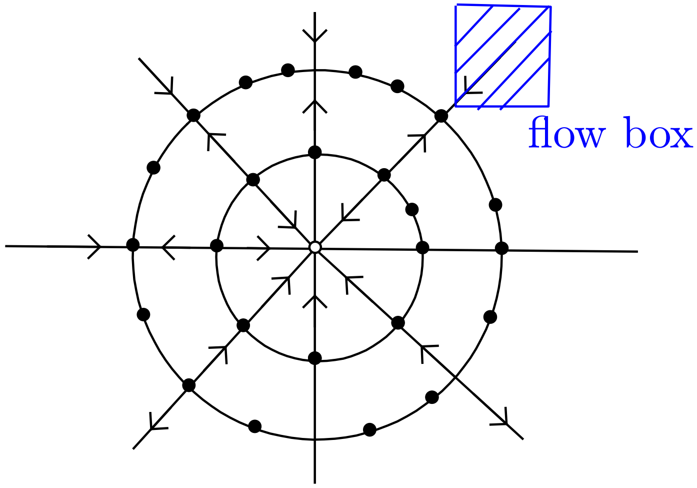

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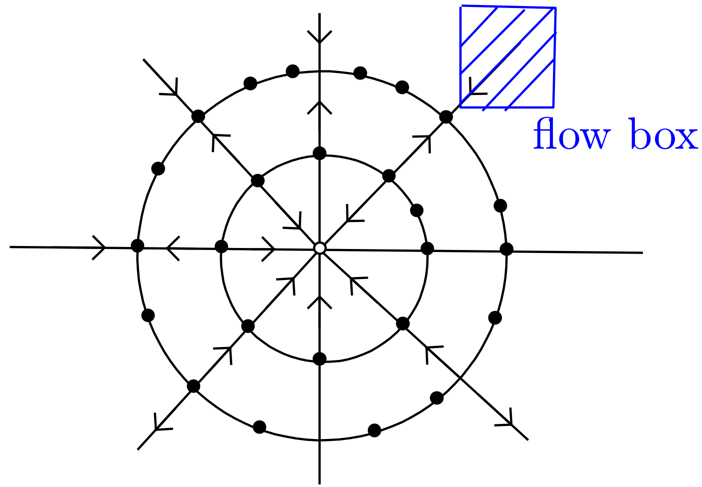

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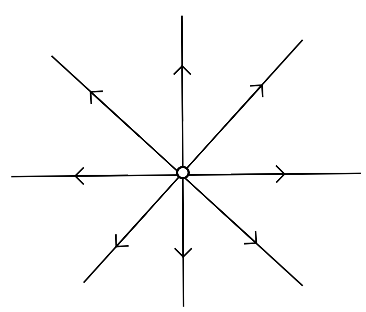

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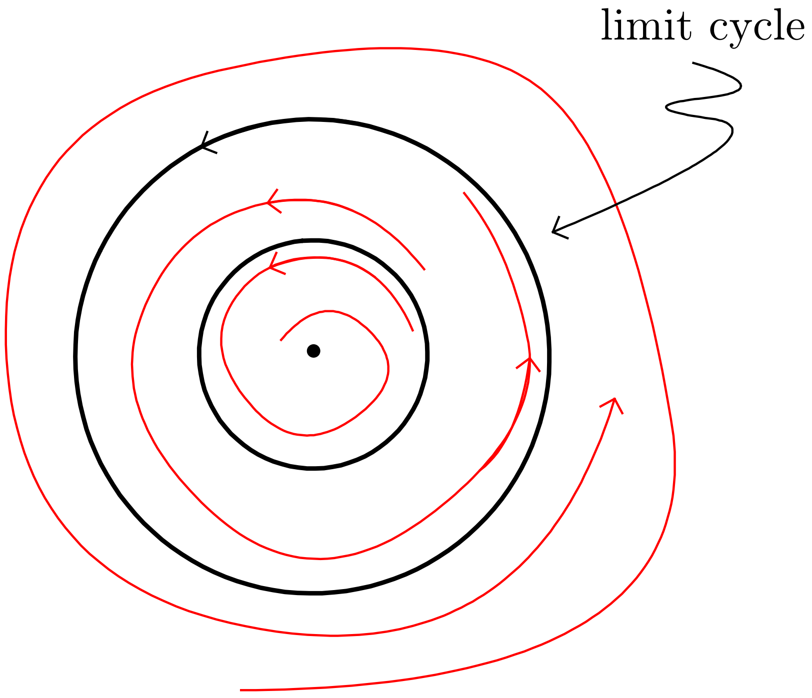

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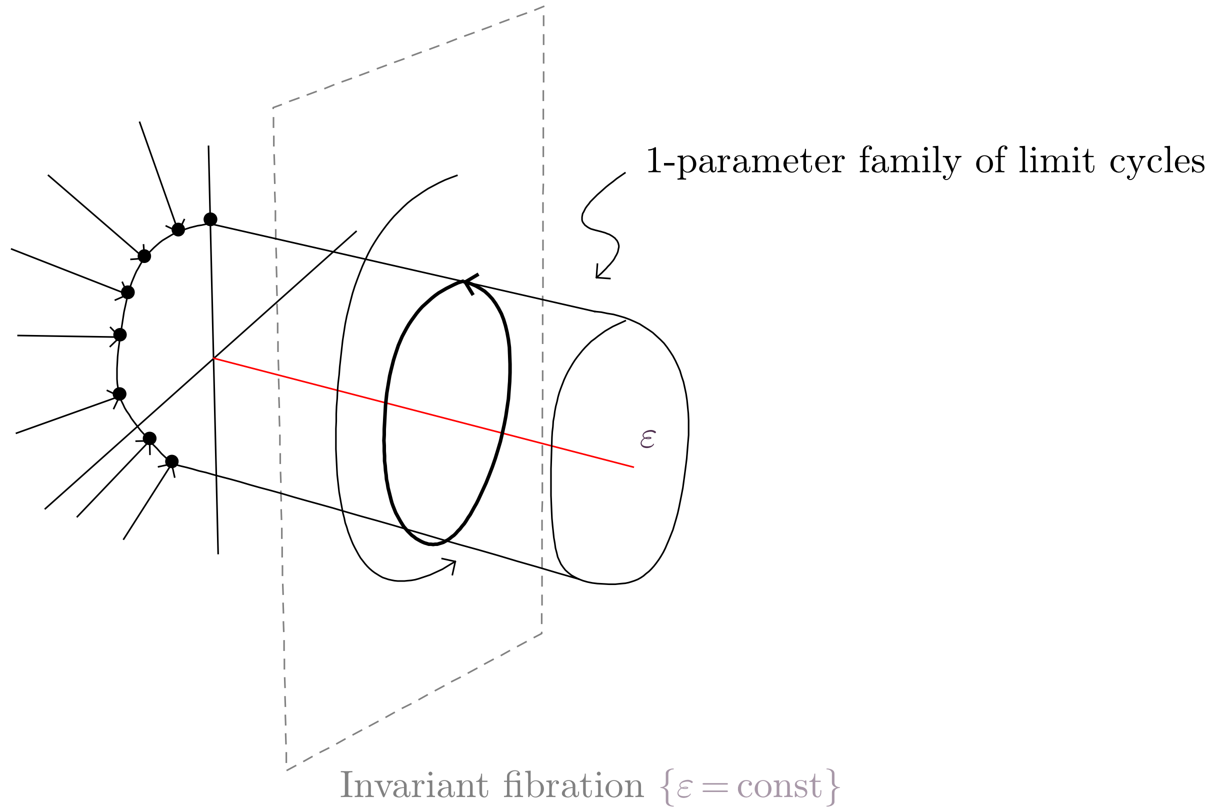

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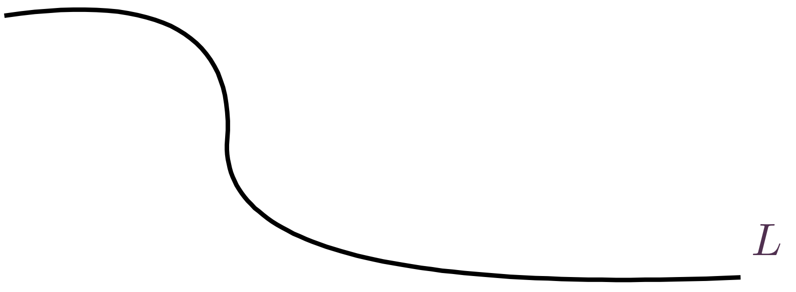
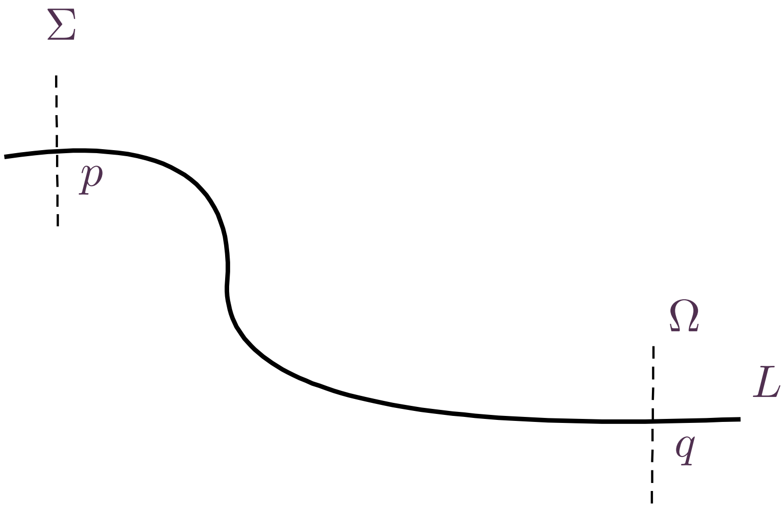
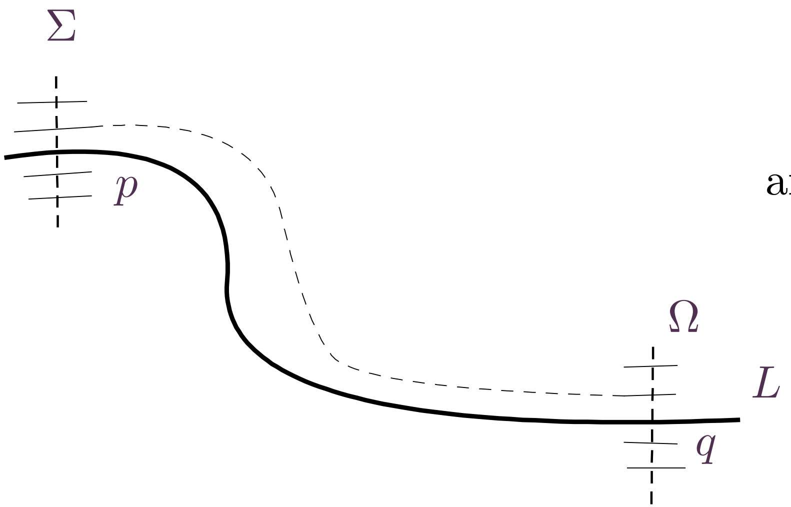


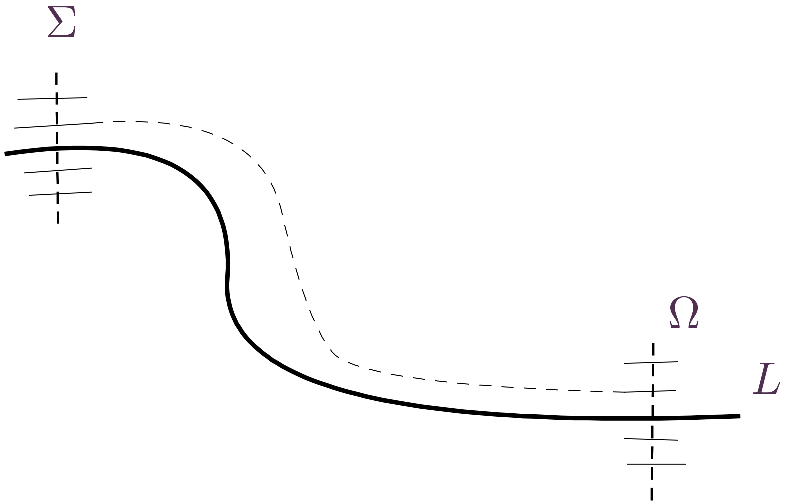
L



Σ
p
Ω
L
q



any path $p \rightarrow q$ on L can be lifted to nearby leafs



$$\mathrm{hol}\colon (\Sigma, p) \to (\Omega, q)$$

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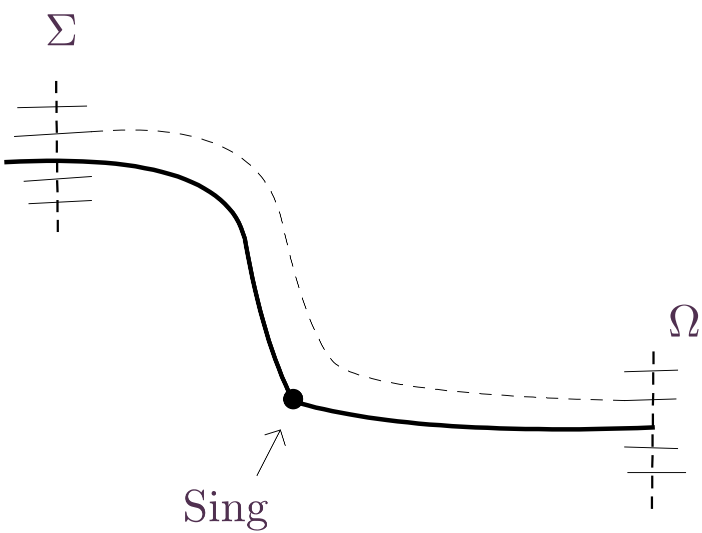
Adding a singularity on the path…



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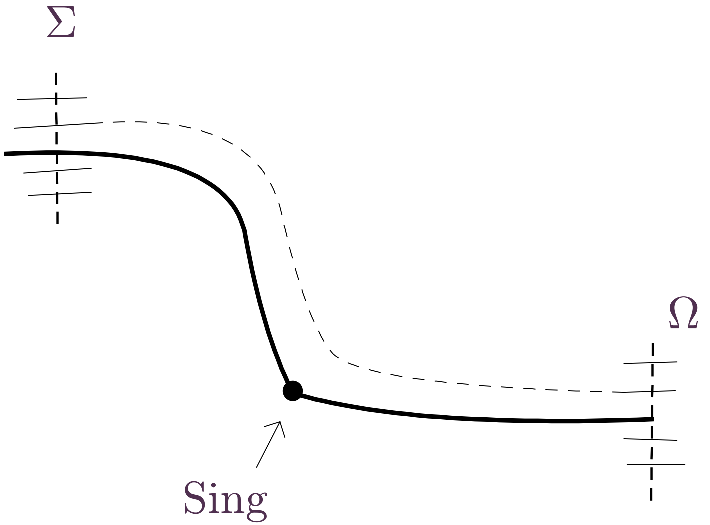

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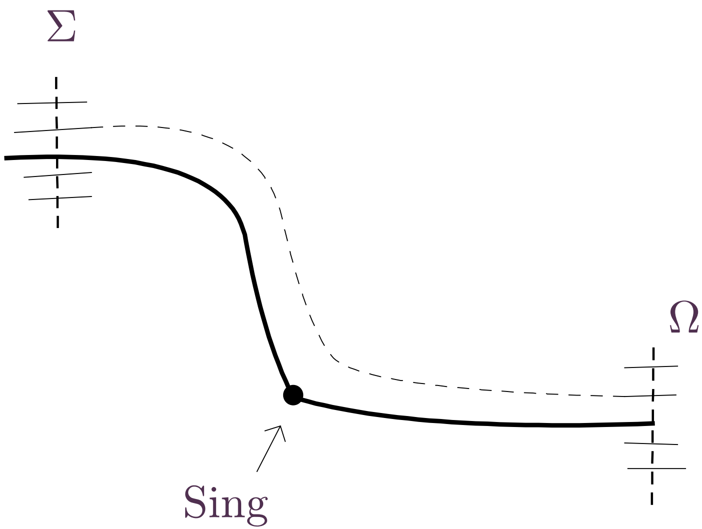

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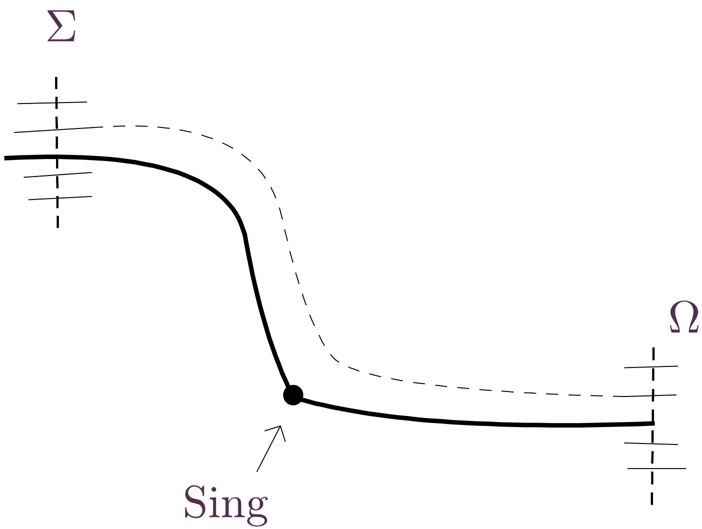

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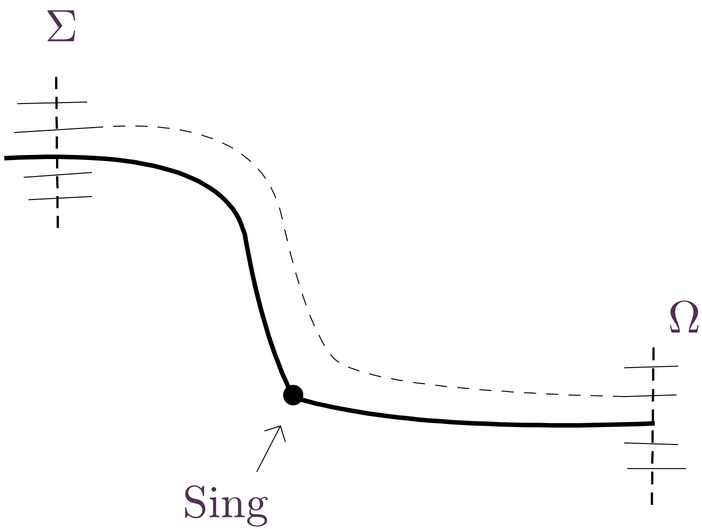

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Moreover, ∂_s and ∂_n are derivations of $\hat{\mathcal{O}} = \varprojlim J^k$ (see Jean Martinet - Exposé Bourbaki'81).

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Poincaré-Dulac normalisation: (over $\mathbb{C}$) Suppose that $\partial(\mathfrak{m}) \subset \mathfrak{m}$. Then, there exists formal coordinates $(x_1, \ldots, x_n)$ which diagonalize the semi-simple part of ∂, namely such that

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Definition. A germ of vector field ∂ is ***elementary*** if:

- either $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ (i.e. in appropriate local coordinates $\partial = \frac{\partial}{\partial x}$)
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where k ranges over the subset $\mathbb{Z}^n \setminus \{0\}$ such that $\langle \lambda, k \rangle = 0$. These are the **resonant monomials**.

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The Poincaré-Dulac Theorem says that, up to a formal change of coordinates, we can write

$$\partial = \underbrace{\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)}_{\partial_s} + \underbrace{\sum_{k \geqslant 1} (x y)^k \left(a_k\, x\frac{\partial}{\partial x} + b_k\, y\frac{\partial}{\partial y}\right)}_{\partial_n}$$

where $u = x y$ is the generator of the subring $\ker(\partial_s)$. By further reductions, we can write

$$(1 + F)\left(\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) + \frac{u^n}{1 + \rho u^n}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right) \quad \text{or} \quad (1 + F)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

for some $F \in \mathbb{C}[[u]]$ of order $\geqslant 1$, $n \geqslant 1$ and $\rho \in \mathbb{C}$.

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is a first integral of the vector field (namely, $\partial I = 0$). It is an element of $\mathbb{R}_{\mathrm{an,exp}}$.

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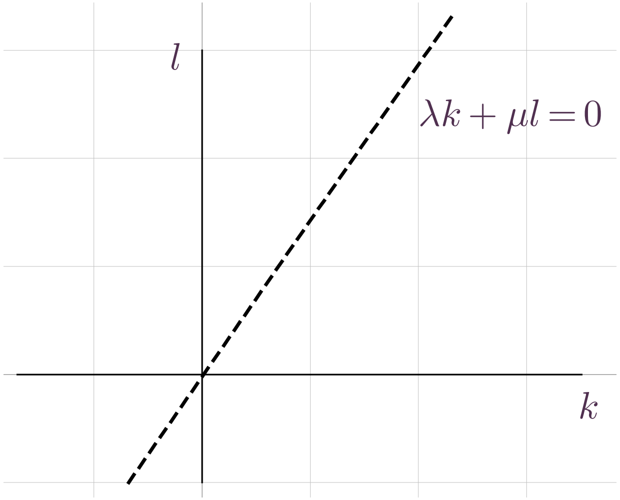

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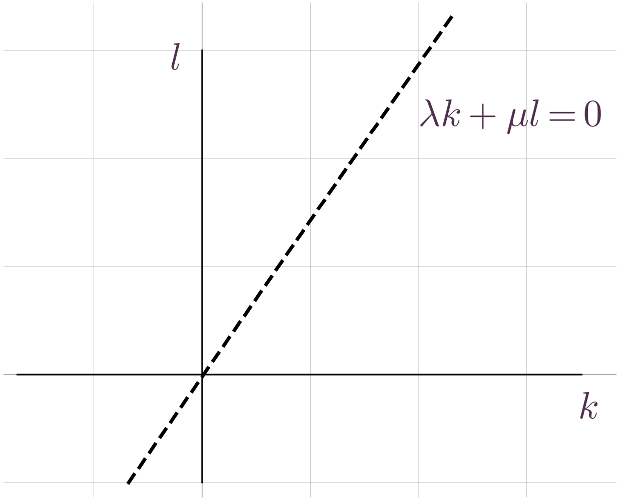

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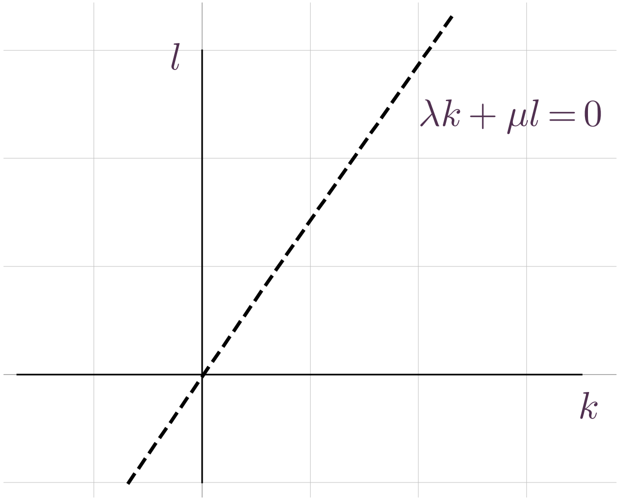

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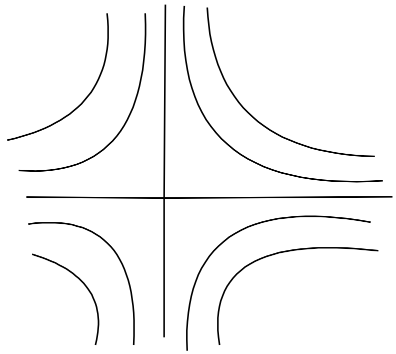

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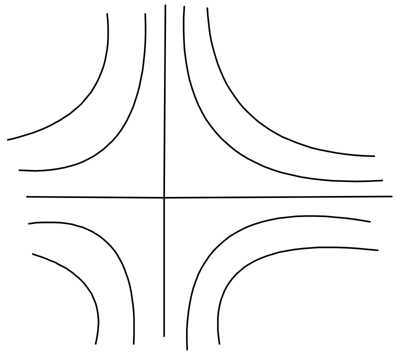

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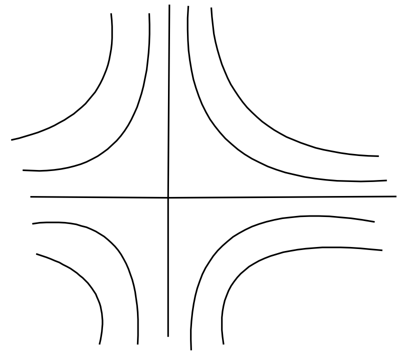

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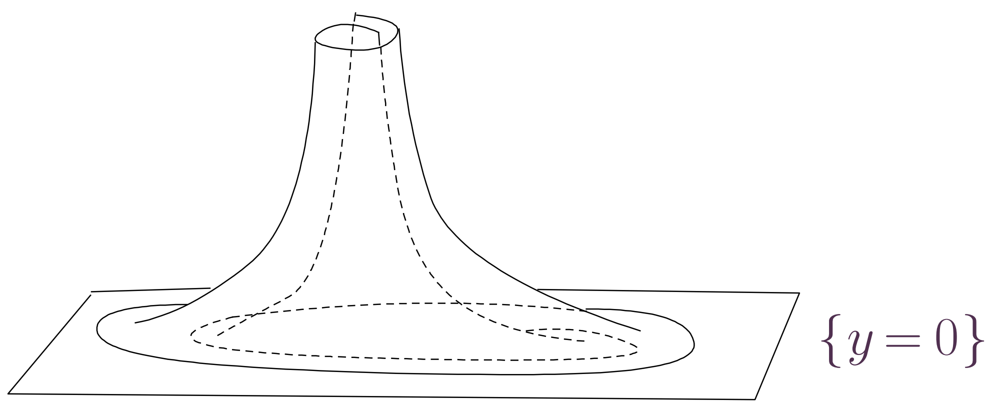

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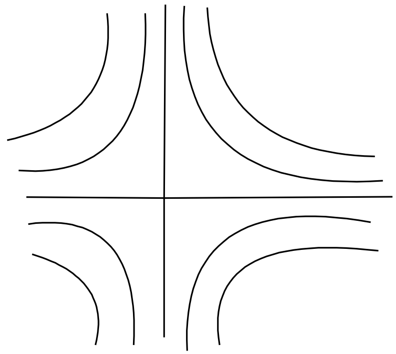

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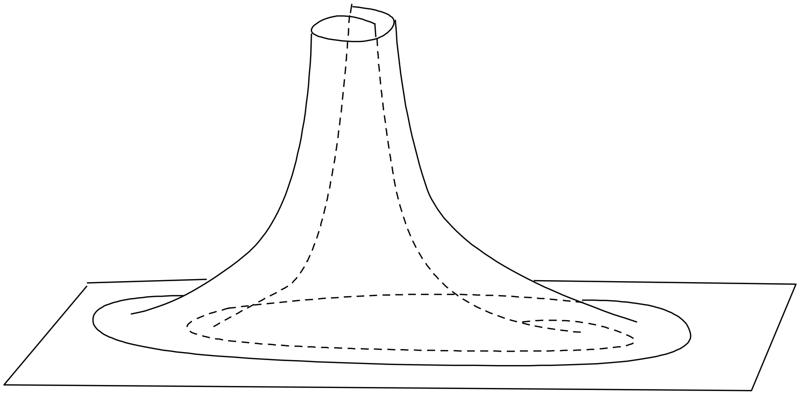

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Over $\mathbb{C}^2$: There are several **rigidity phenomena**

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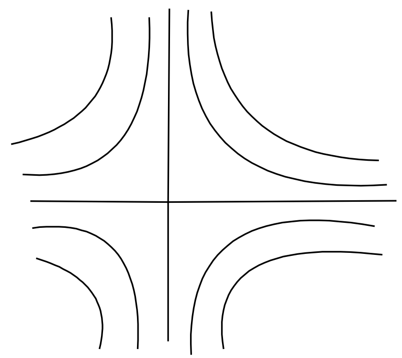

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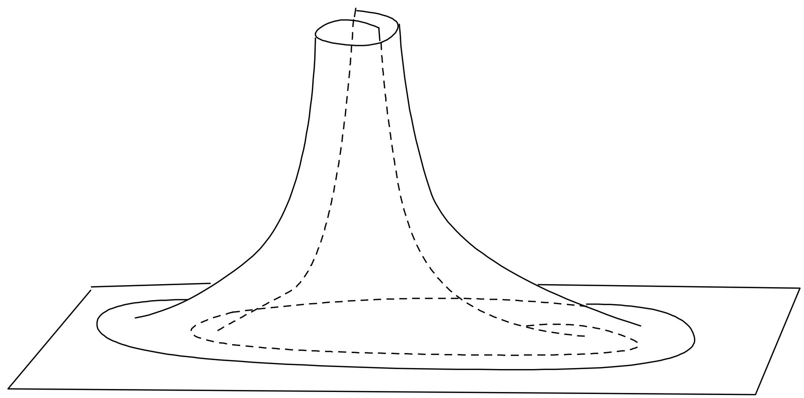

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Over $\mathbb{C}^2$: There are several **rigidity phenomena**

E.g. Some analytic invariants are topologically determined (for instance, linearizability).

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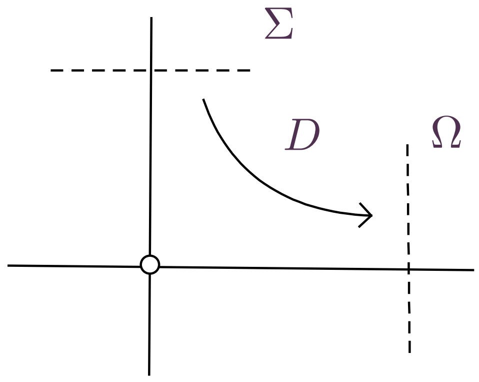

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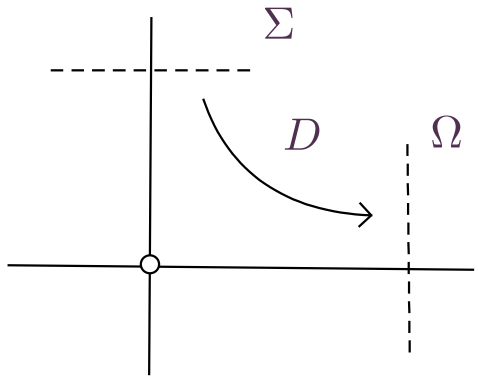

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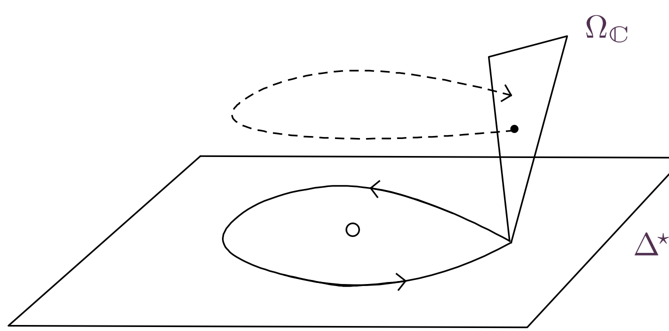

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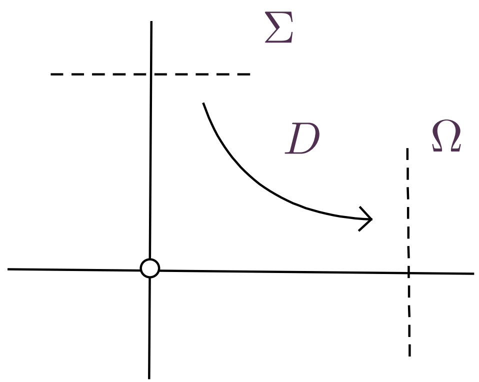

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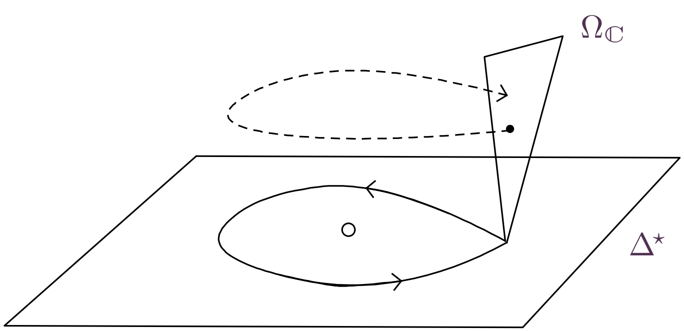

"The" Holonomy map

We can recover the (orbital) analytic class of the saddle from the analytic class of one of these maps (once we fix the ratio μ/λ)

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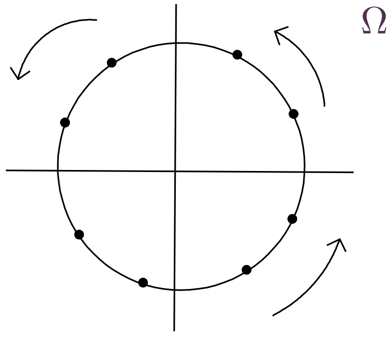
Definition: Two germs of vector fields $\partial, \tilde{\partial}$ are **orbitally analytic equivalent** if there exists a unit $u \in \mathbb{C}\{x\}$ such that ∂ is analytically conjugated to $u\,\tilde{\partial}$.

Dynamics of the complex holonomy map as an element of $\mathrm{Diff}(\mathbb{C}, 0)$



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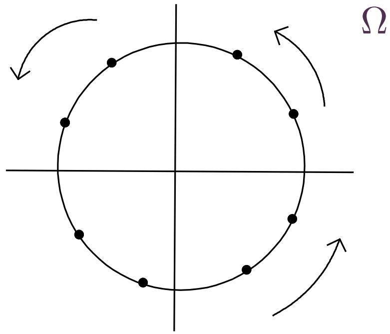

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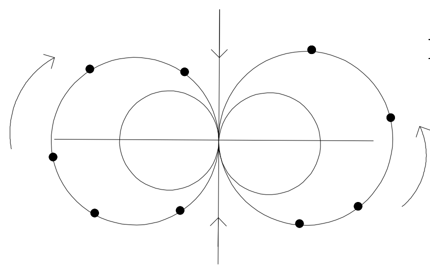

parabolic fixed point

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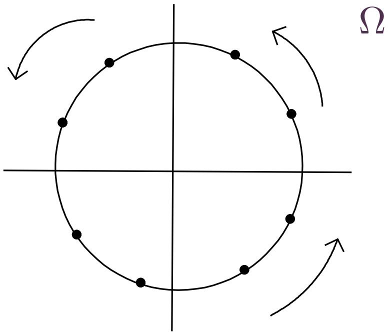

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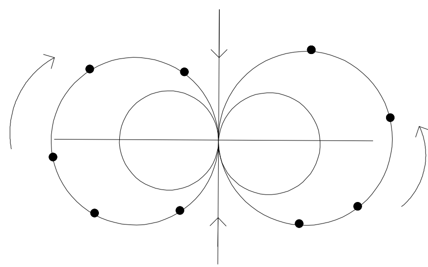

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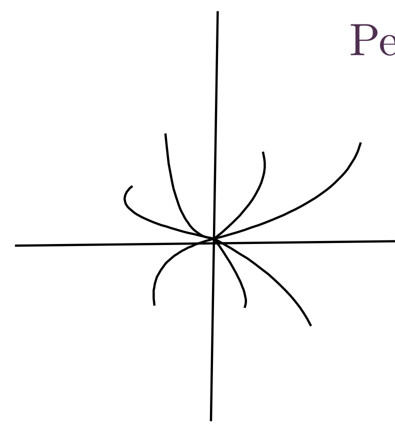

Perez – Marco's hedgehogs

Classification Problem: “Describe” the orbits of the action of $\mathrm{Aut}(\mathbb{C}\{x\})$ on $\mathrm{Der}(\mathbb{C}\{x\})$ by conjugation

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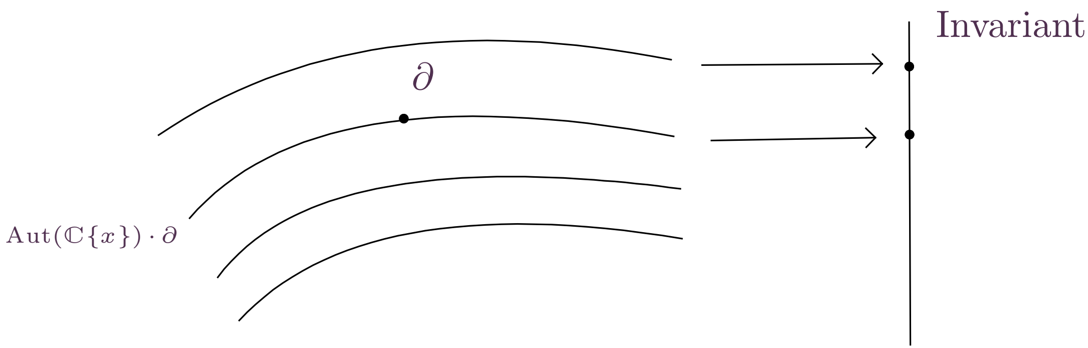

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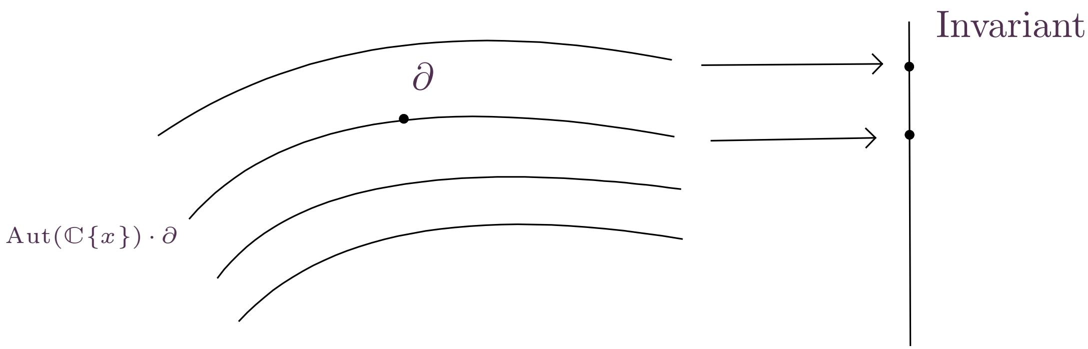

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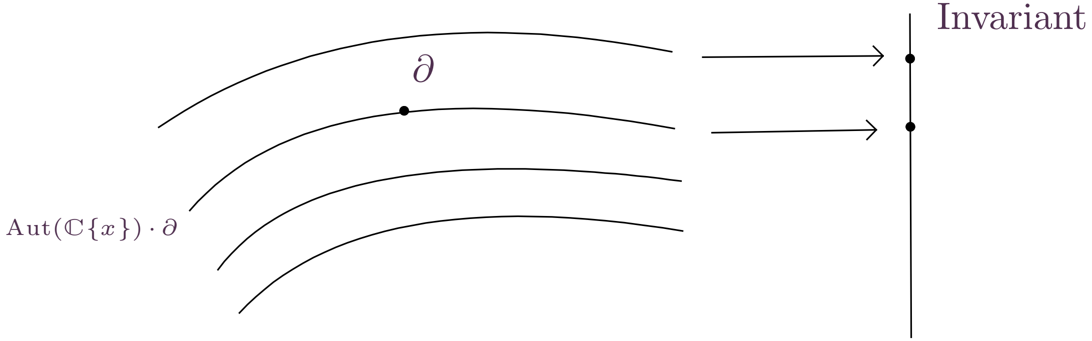

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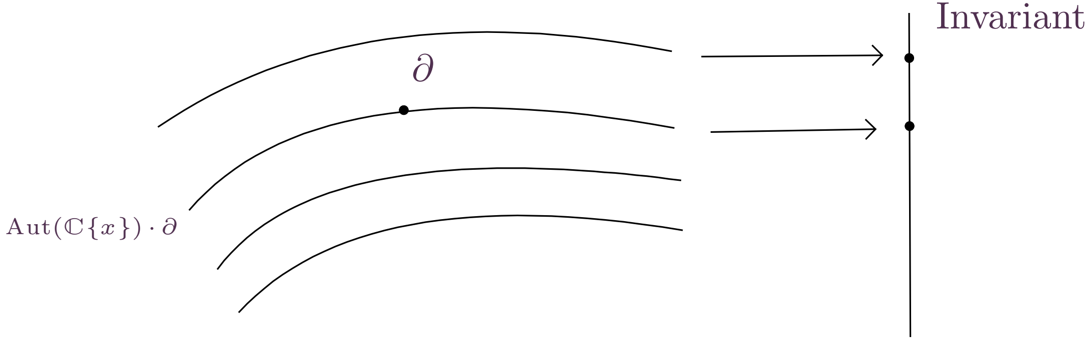

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This problem is much less understood for vector fields higher dimensions.

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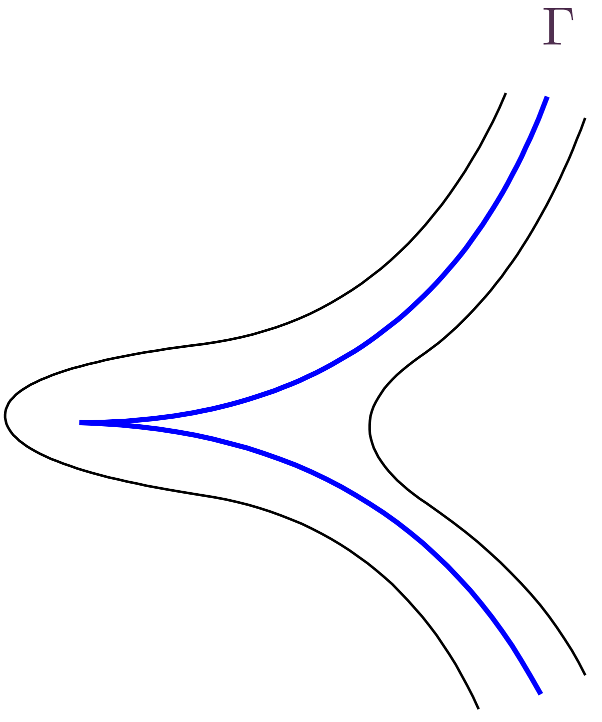
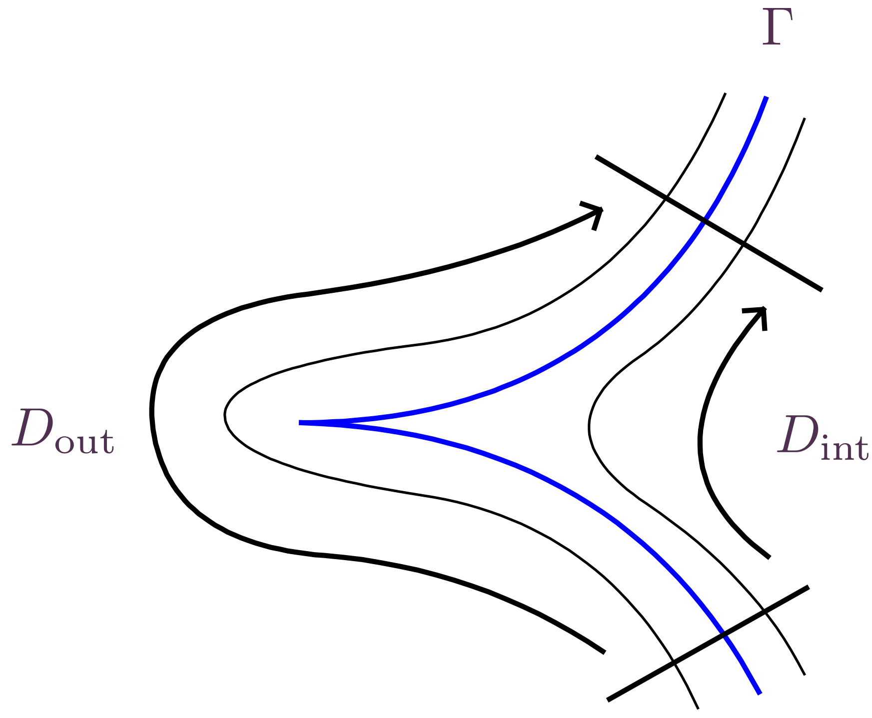
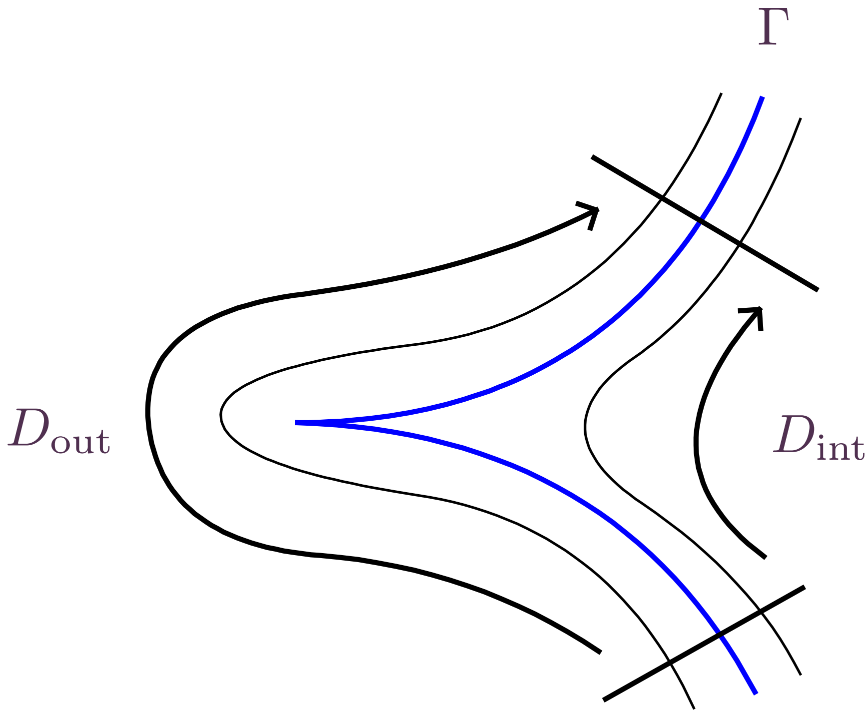
The cusp $\Gamma = \{f = 0\}$ is an invariant curve.

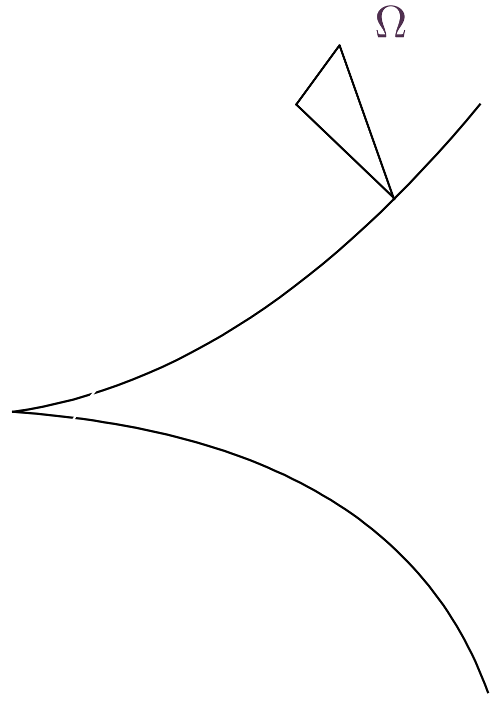

Γ

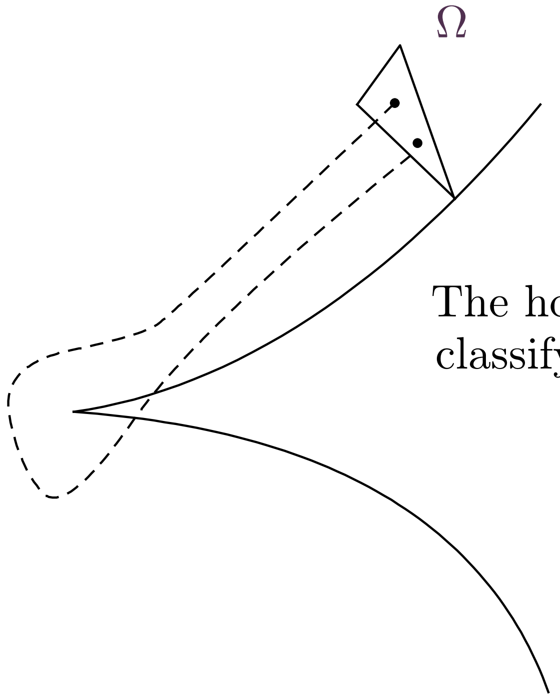


Γ
D_{out}
D_{int}



There are two **distinct** corner transition maps.


Ω



The holonomy map **does not** classify the singularity

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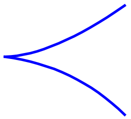

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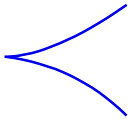

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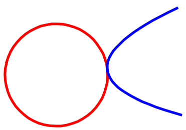

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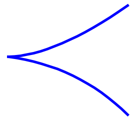

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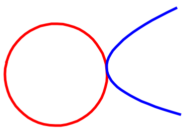

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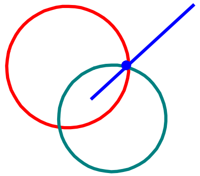

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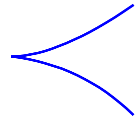

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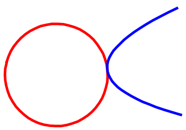

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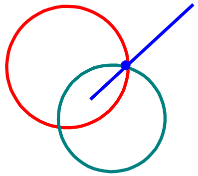
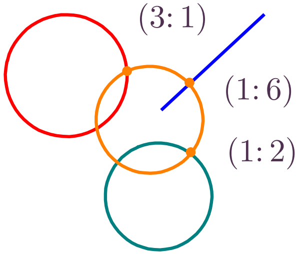


Blow-up 3: $x \to x\,, \qquad y \to x\,y$

$$d(x^6 y^3 (y - 1))$$



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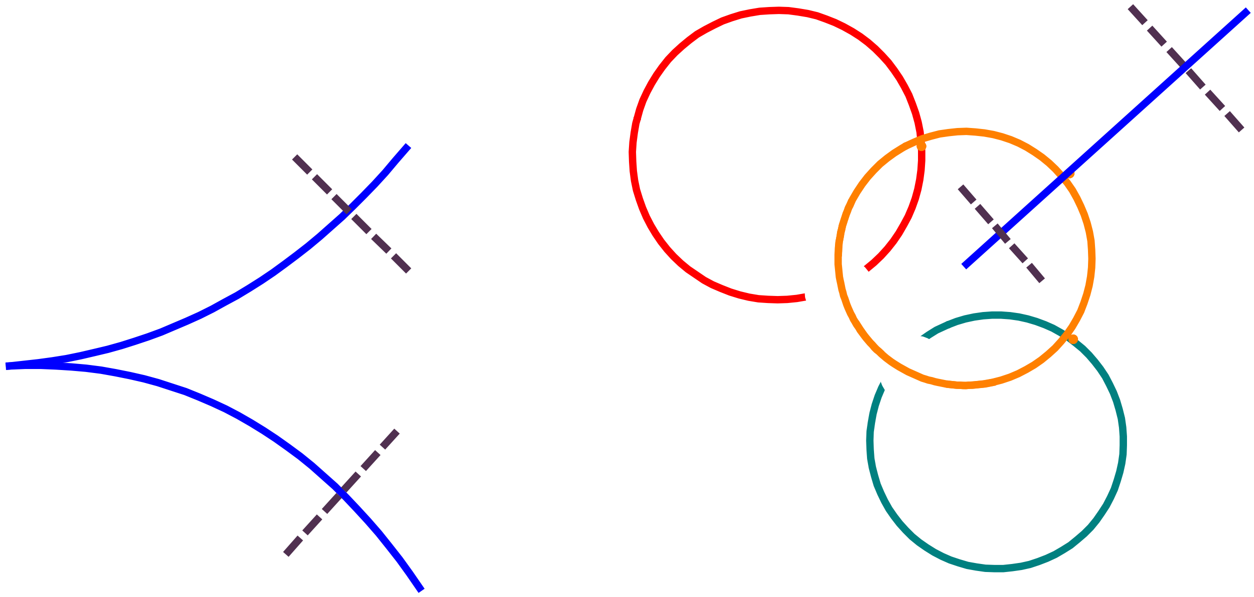

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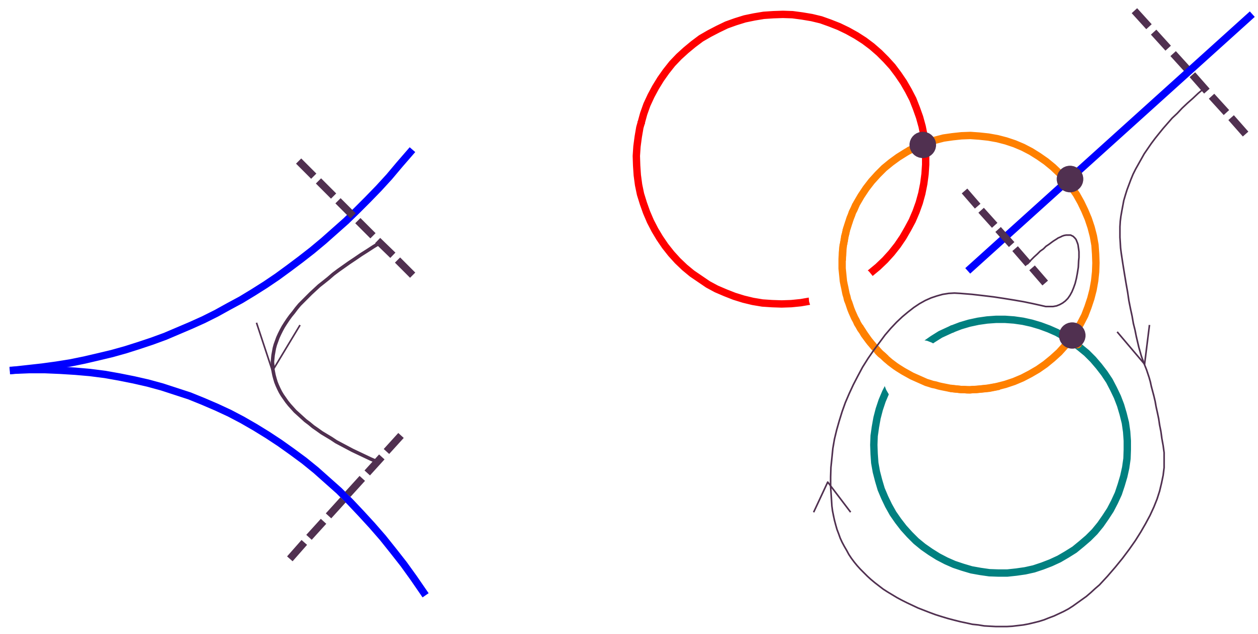
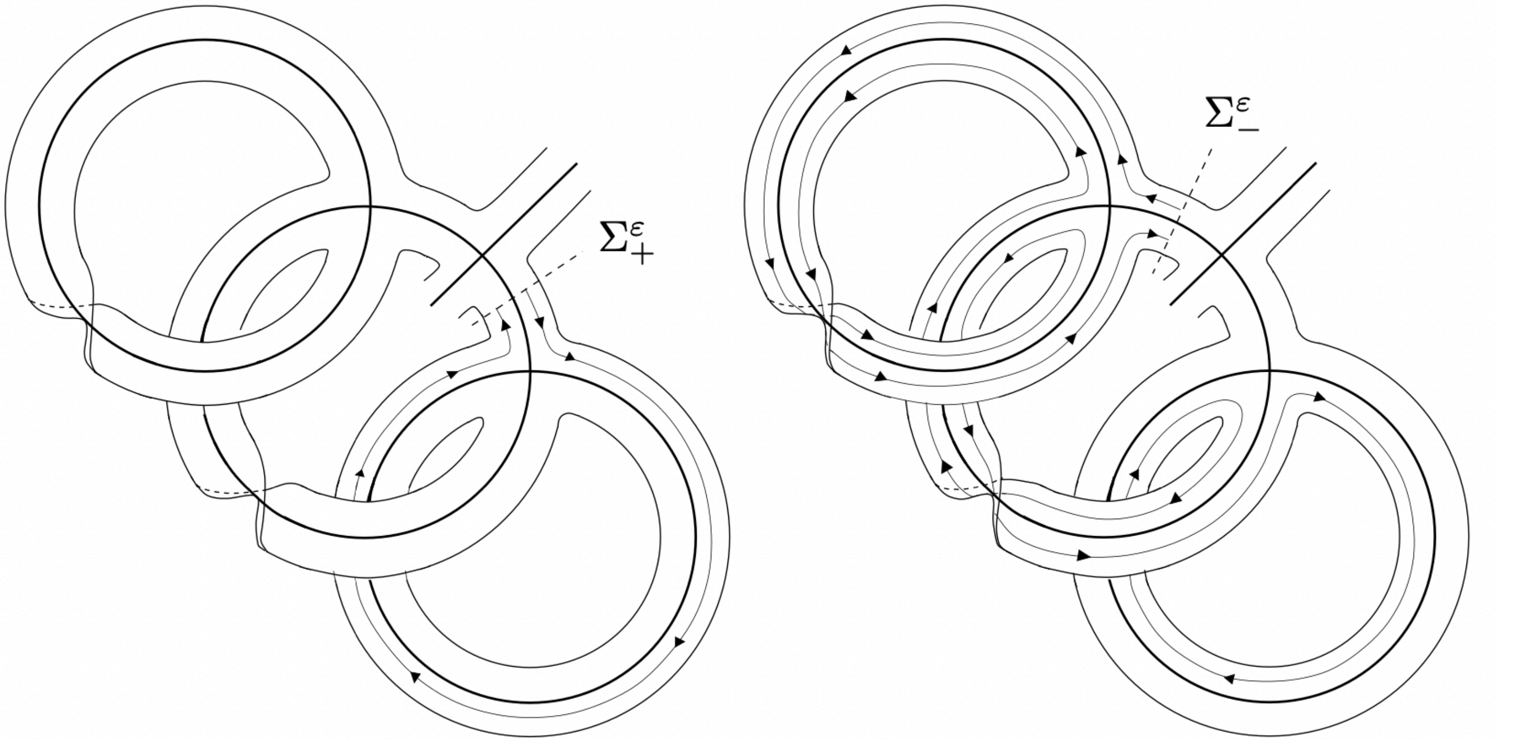
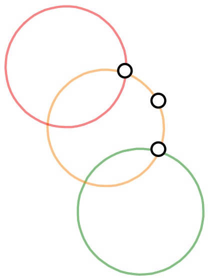
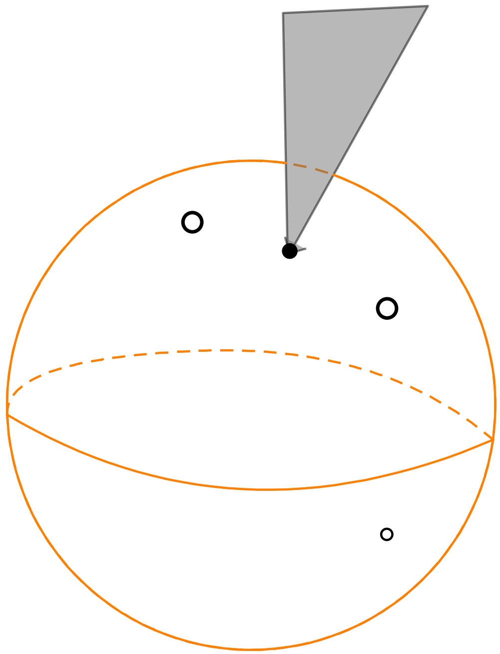


The foliation is now organized in a neighborhood of the exceptional divisor..


Σ^{ε}_{+}
Σ^{ε}_{-}

Can we recover the analytic moduli from the transverse behaviour?





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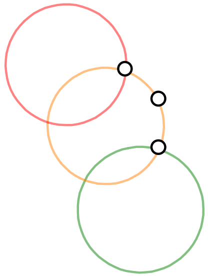

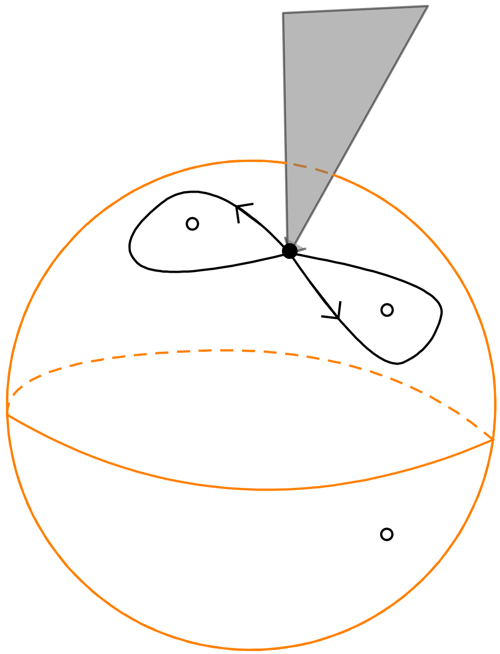

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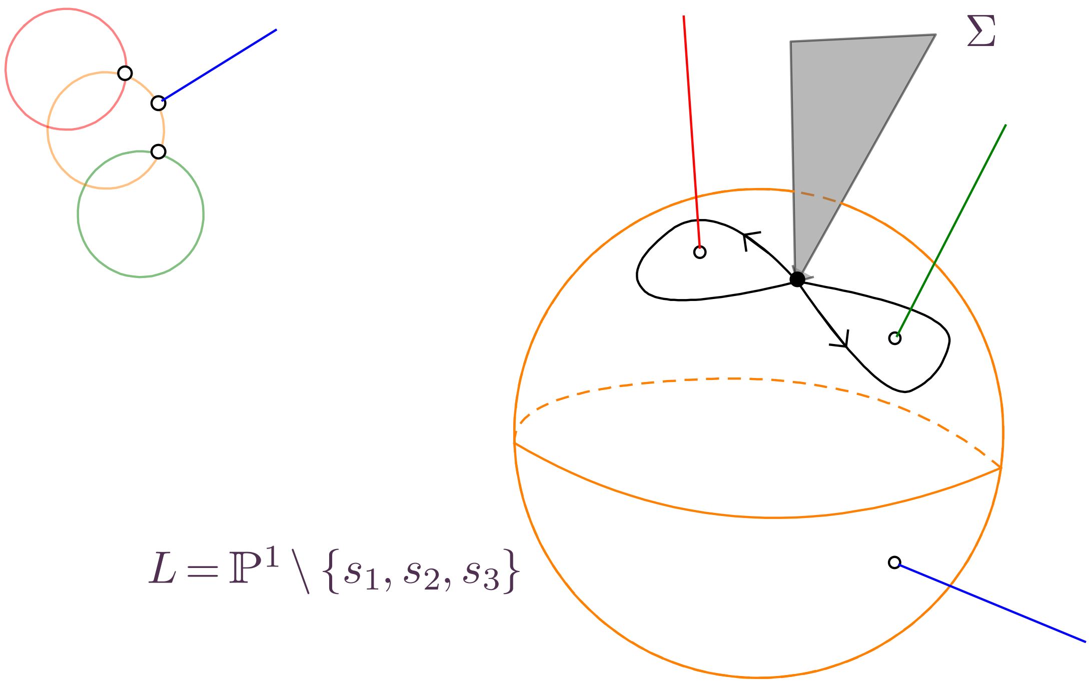

(Moussu) The vanishing holonomy $\mathrm{Hol}(\mathcal{F}, L) = \langle f, g \in \mathrm{Diff}(\mathbb{C}, 0) \mid \; f^2 = g^3 = \mathrm{id} \rangle$ characterizes the analytic class of the germ of foliation.

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Alternatively,

$$p \in \mathrm{Nilp}(M, \mathcal{F}) \iff \forall k \in \mathbb{N} \exists n \in \mathbb{N} : (\partial_k)^n = 0$$

where $\partial_k: J^k \to J^k$ is the induced derivation on the k^{th} jet.

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We further say that $\mathcal{F}$ is **tightly adapted** to D if there exists an index i such that

$$\partial(\langle f^i \rangle) \not\subset \langle f^{i+1} \rangle$$

In other words, for $E=(x_1 \dots x_k=0)$,

$$\partial=\sum_{i=1}^{k} a_i \left(x_i \frac{\partial}{\partial x_i} \right) + \sum_{i=k+1}^{n} a_i \frac{\partial}{\partial x_i}$$

with $a_1, \dots, a_n \in \mathbb{C}\{x\}$ such that $\langle a_1, \dots, a_n \rangle \not\subset \langle x_i \rangle$, for each $i=1, \dots, k$.

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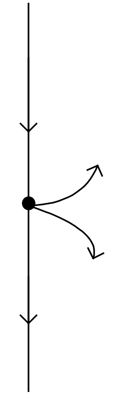

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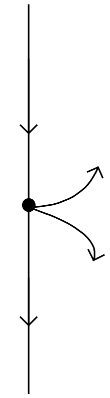

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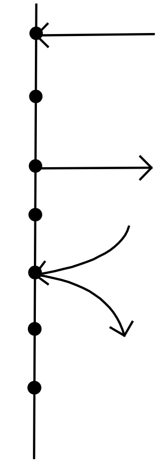

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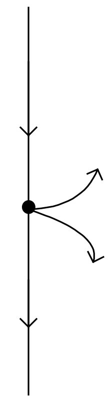

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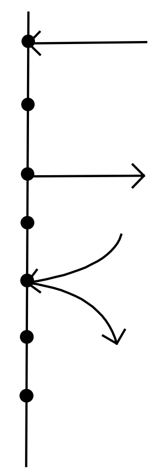

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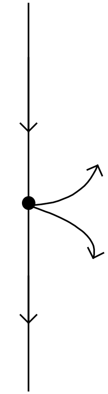

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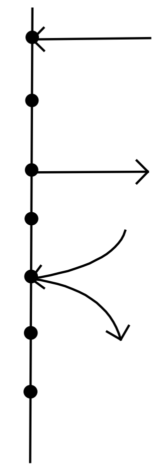

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How to compute the transform of a foliation by blowing-up?

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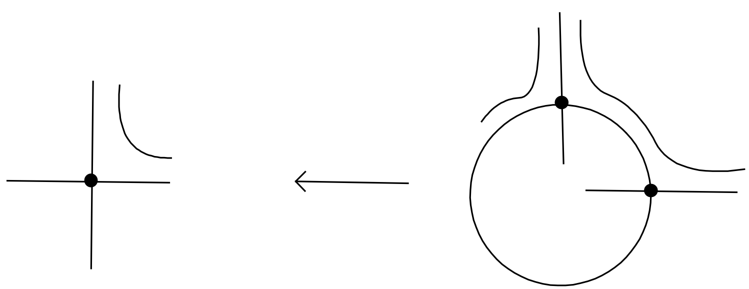

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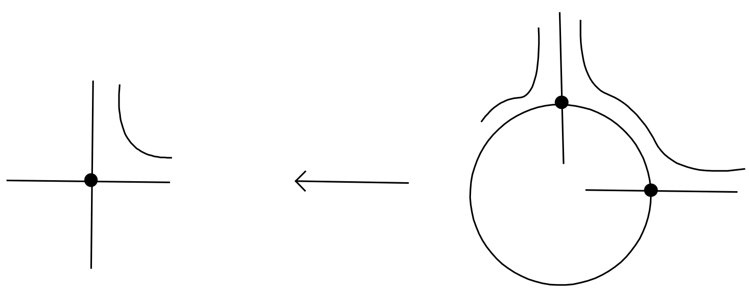

We can never get rid of saddle points...

Example: node

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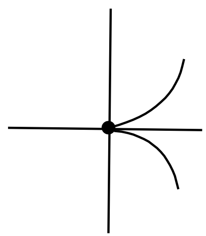
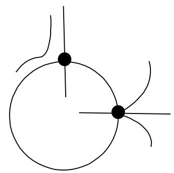


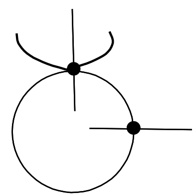
$\rho > 1$



$\tilde{\rho} = \rho - 1$

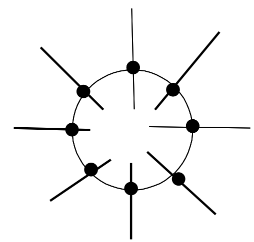
$$\rho = \rho_0 + \cfrac{1}{\rho_1 + \cfrac{1}{\cdots}}$$

$\rho < 1$



$\tilde{\rho} = \frac{1}{\rho} - 1$

$\rho = 1$



(dicritical situation)

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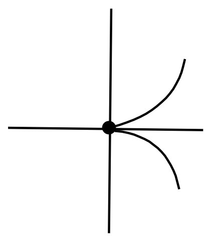

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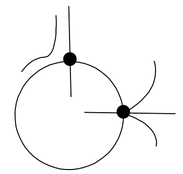

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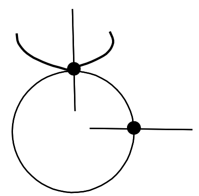

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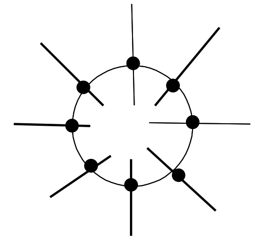

(dicritical situation)

We can never get rid of a node if $\rho\notin\mathbb{Q}$.

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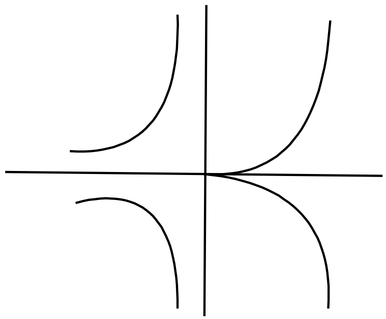
$$\text{First integral} \qquad h = (x^m y)\exp\left(\frac{1}{k x^k}\right)$$



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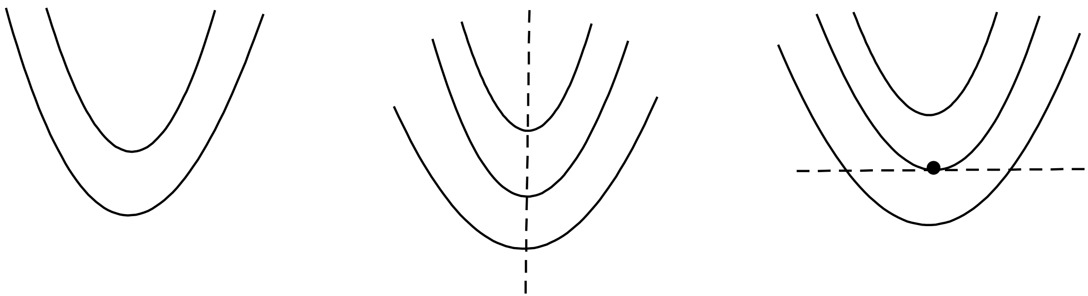

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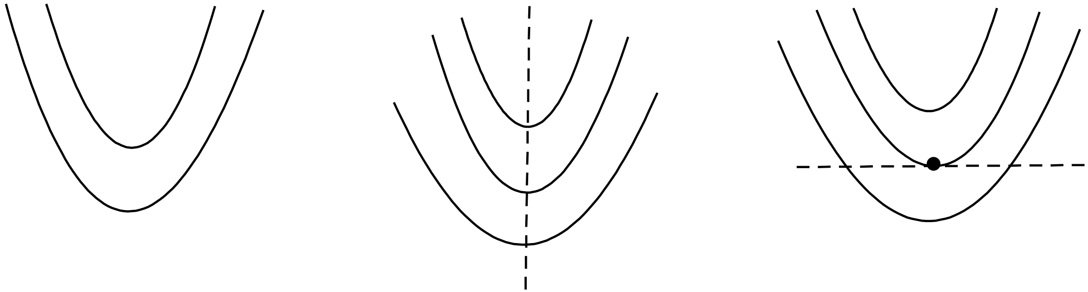

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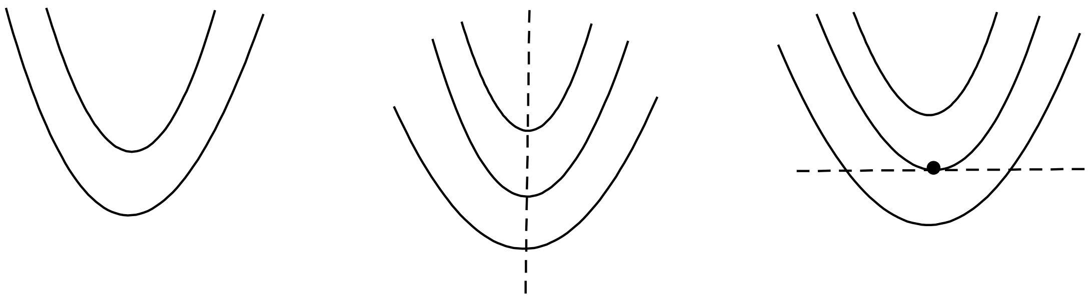

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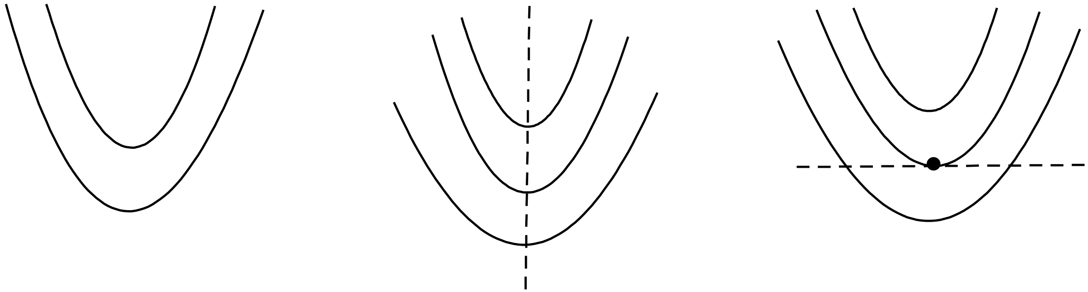

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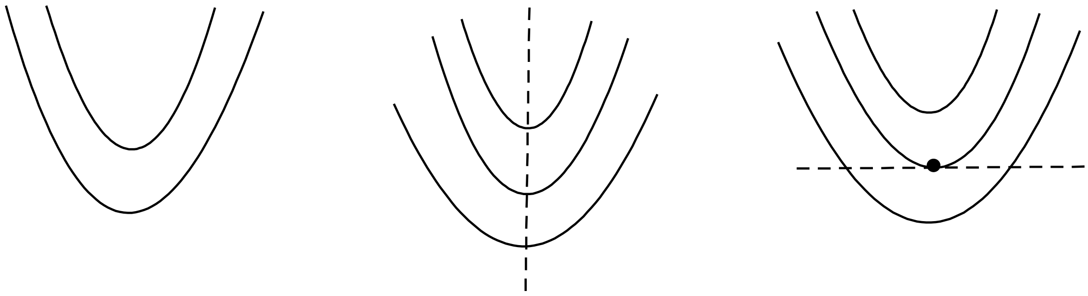

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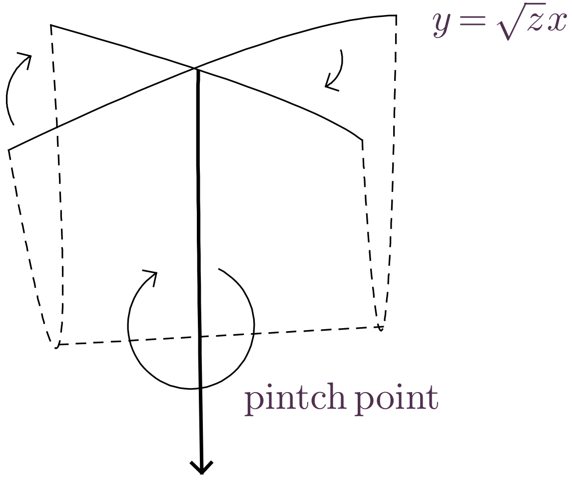

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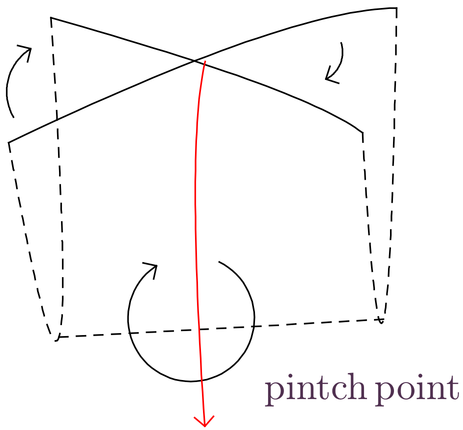

Formal expansion of the “handle”

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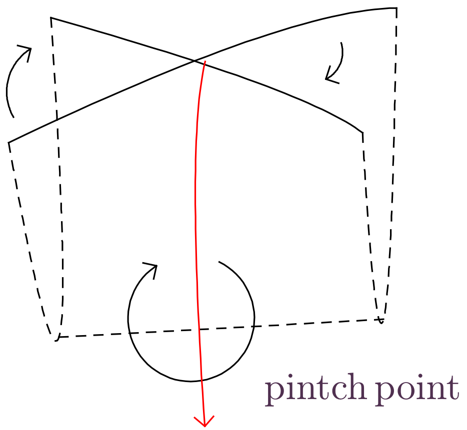

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We cannot take the handle as a blowing-up center because it is non-analytic.

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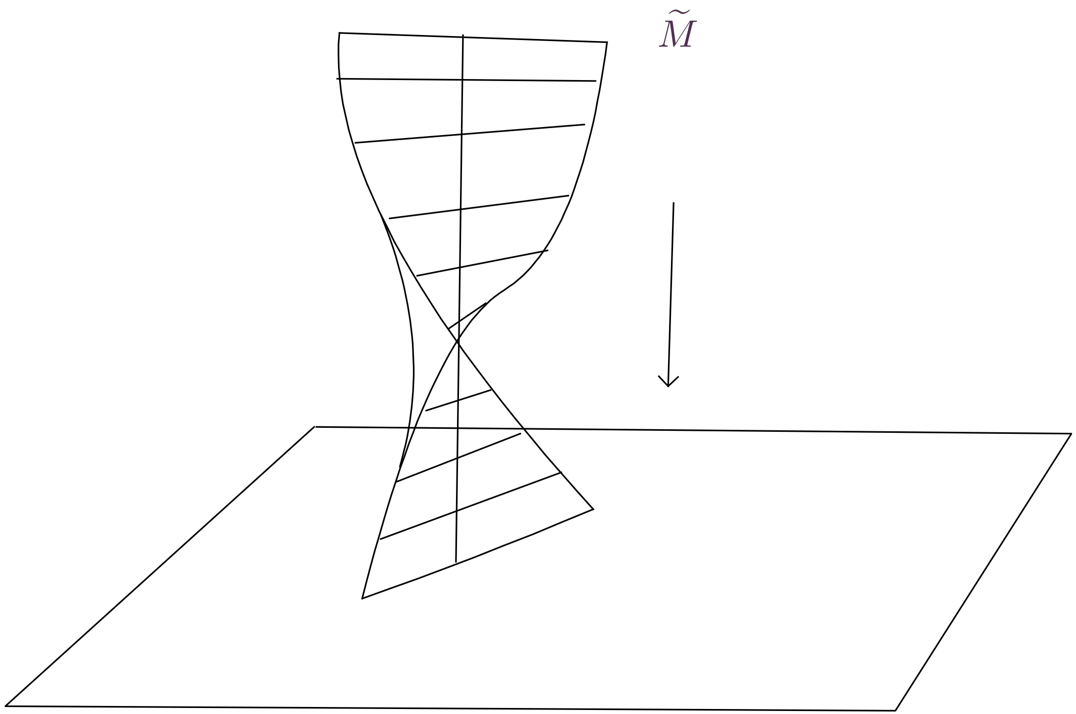
and the projection $\pi\colon \widetilde{M} \to \mathbb{C}^n$ is the weighted blowing-up of the origin in $\mathbb{C}^n$.



$\widetilde{M}$

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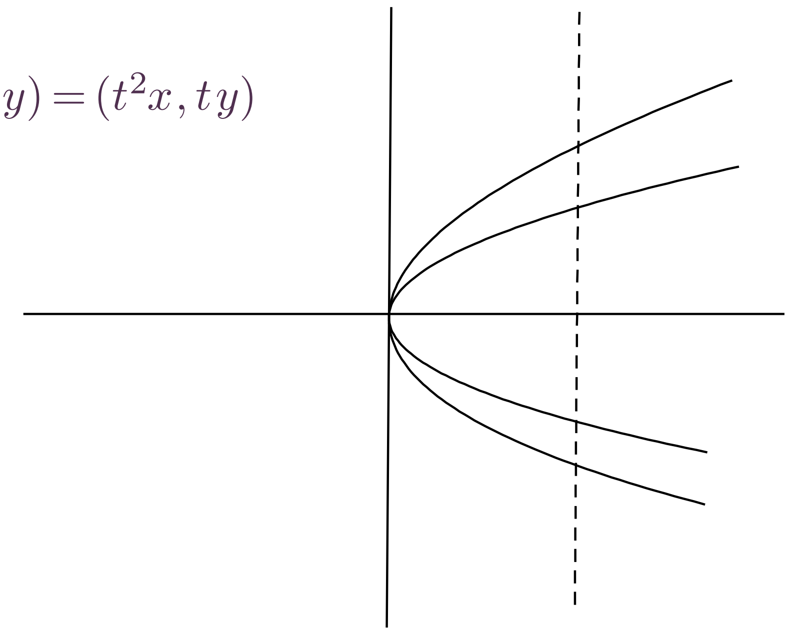
Example

$\omega = (2, 1)$

$t \cdot (x, y) = (t^2 x, t y)$



We have to take into account the quotient by $\mathbb{Z}/2\mathbb{Z}$.

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The charts of a weighted-blowing up

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An **orbifold** is a pair $(M, \mathcal{U})$ where M is paracompact Hausdorff topological space and $\mathcal{U}$ is a maximal orbifold atlas on M.

A sub-variety $Y \subset M$ is a **sub-orbifold** if for each point $p \in Y$ there exists a local chart (U, G, ϕ) such that $\phi^{-1}(Y \cap U)$ is a G-invariant submanifold of U.

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$X = \operatorname{Spec} \mathbb{C}[x, y]^G$ (ring of invariants)

$$\mathbb{C}[x, y]^G = \mathbb{C}[x^2, xy, y^2]$$

$$X = \operatorname{spec} \mathbb{C}[u, v, w]/(v^2 - uw)$$

X is the quadratic cone.

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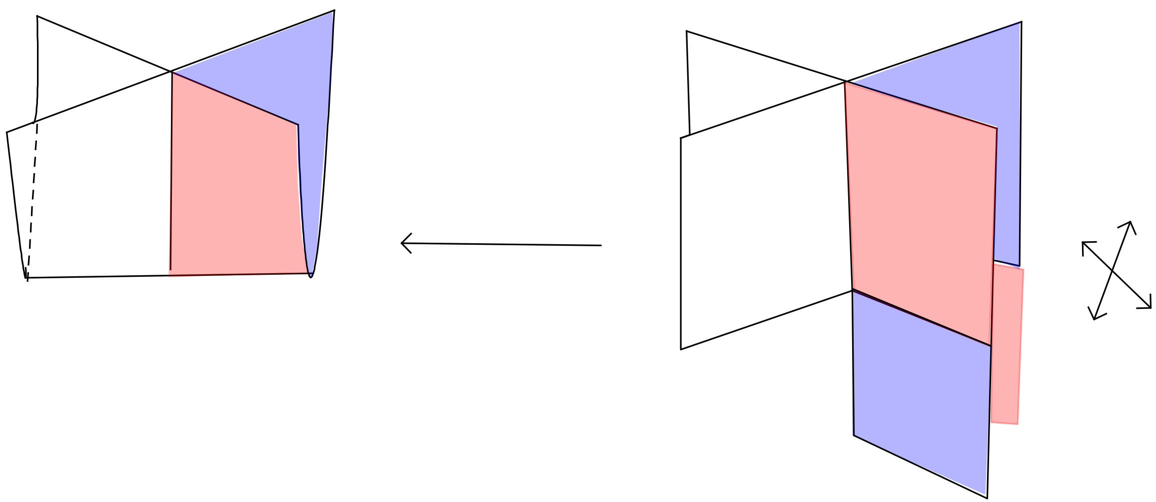

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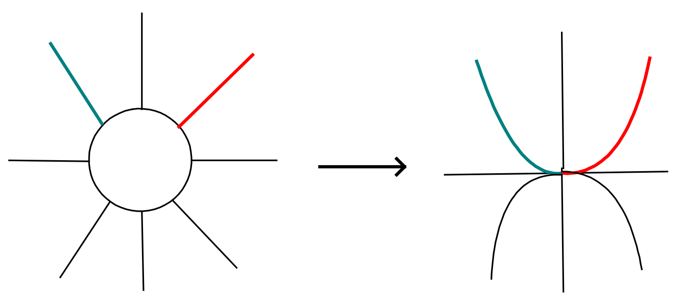

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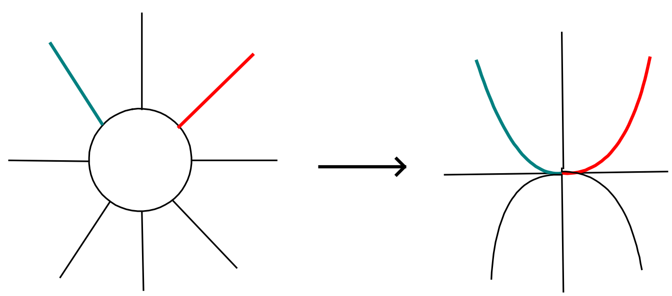

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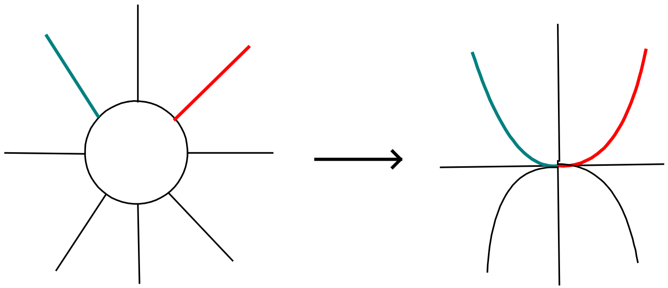

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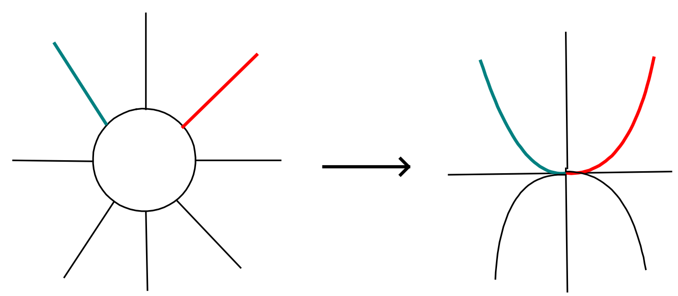

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(c.f. Melrose’s “Analysis on manifolds with corners” - online)

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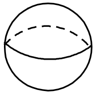

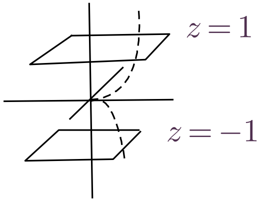

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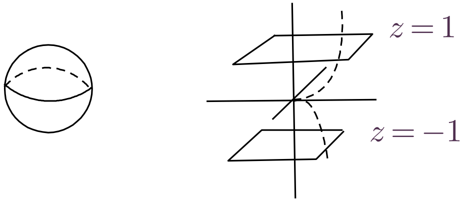

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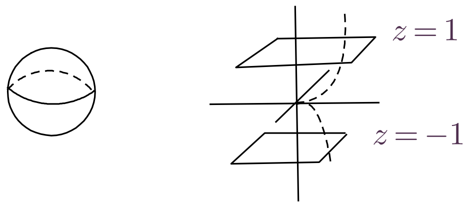

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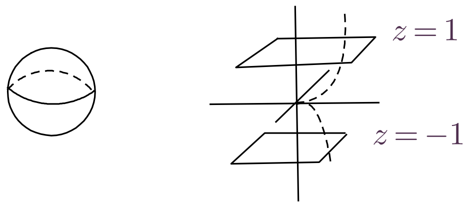

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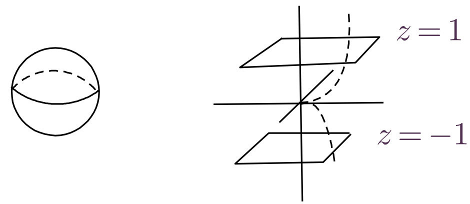

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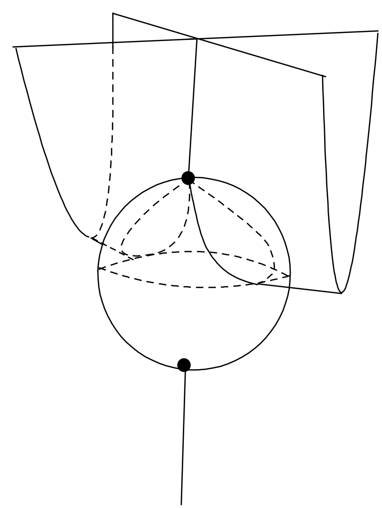

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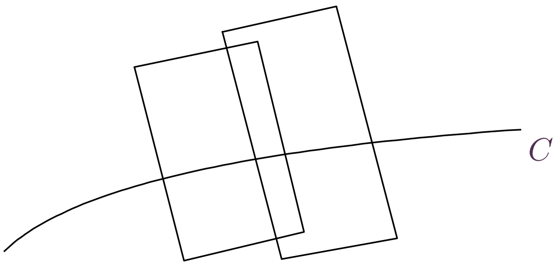

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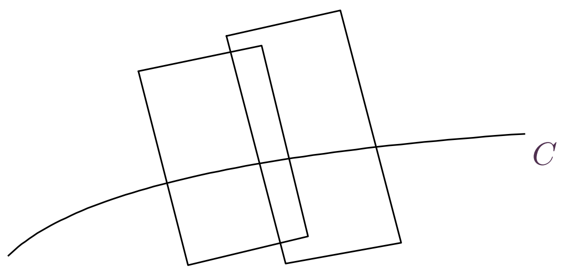

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$$\mathcal{O}_p = \mathcal{O}_0 \supset \mathcal{O}_1 \supset \mathcal{O}_2 \supset \cdots \qquad \mathcal{O}_k \cdot \mathcal{O}_l \subset \mathcal{O}_{k+l},$$

such that in appropriate coordinates $(x_1, \ldots, x_n)$, we have $x_1 \in \mathcal{O}_{\omega_1}, .., x_n \in \mathcal{O}_{\omega_n}$.

In other words, $\mathcal{O}_k$ is the subring of functions of quasi-homogeneous weight $\geqslant k$.

In order to define a quasi-homogeneous blow-up along a submanifold (suborbifold) $C \subset M$, we need to require the existence of a global trivialization of C

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such that $F_i F_j \subset F_{i+j}$ and such that, for each point p on the support, the stalk of this filtration coincides with a quasi-homogeneous filtration as defined above.

Example: $C = Z(x, y) \subset \mathbb{C}^3$

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More generally, all automorphisms obtained by integrating the Lie algebra (over $\mathbb{C}$) generated by

$$\left\{ x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^l\frac{\partial}{\partial y}, y^m\frac{\partial}{\partial x} \quad \middle| \quad m \geqslant 1, l \geqslant \beta \right\}$$

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The solution curves of ∂ are precisely the orbits of the torus action $t \cdot (x, y) = (t x, t^n y)$.

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The divisor $\{x=0\}$ is contained in the nilpotent locus. We factor out x and write

$$\partial_1 = x\,y\,\frac{\partial}{\partial x} + 3(1-y^2)\frac{\partial}{\partial y} + \Delta_1$$

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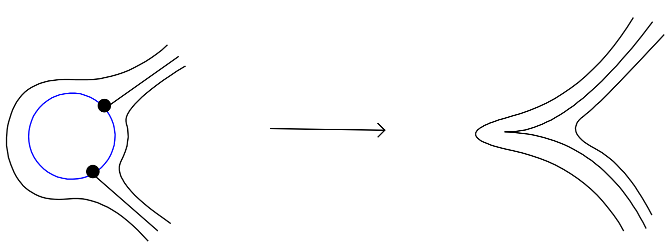

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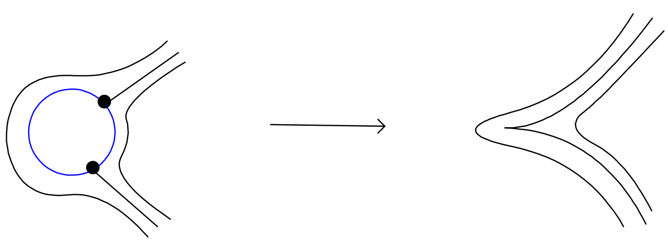

The resulting perturbation Δ is of quadratic order along E (does not change the eingenvalues at the singular point)

Local symmetries of the foliated orbifold

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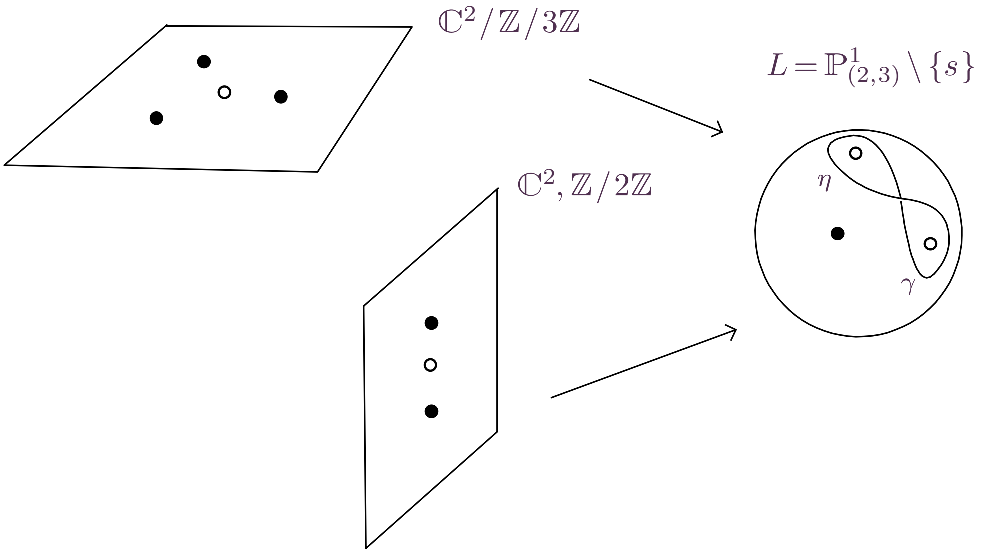

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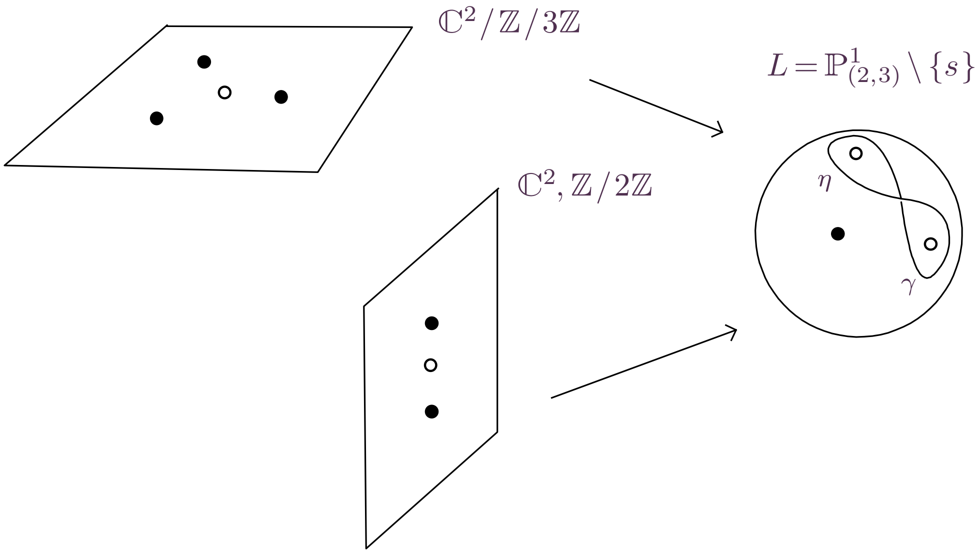

The fundamental group of the (orbi-)leaf L is

$$\pi_1(L) = \{\gamma, \eta, \rho | \, \gamma^2 = \eta^3 = 1, \, \rho = \gamma\eta\}$$

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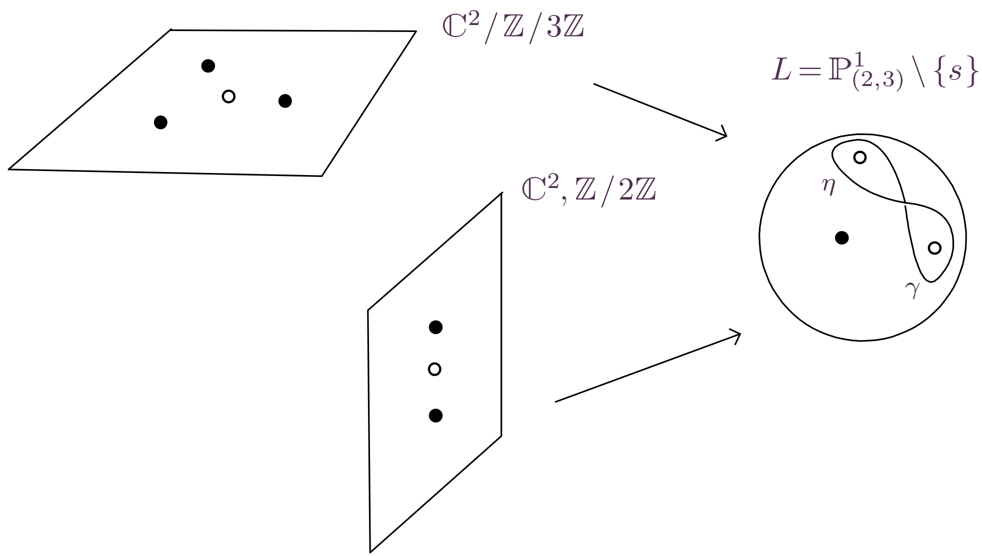

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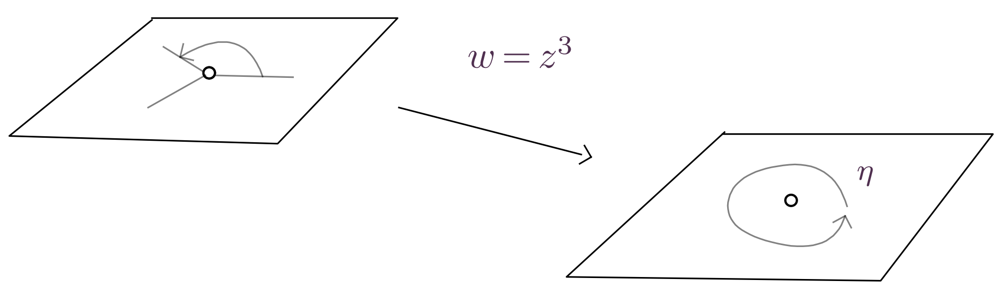

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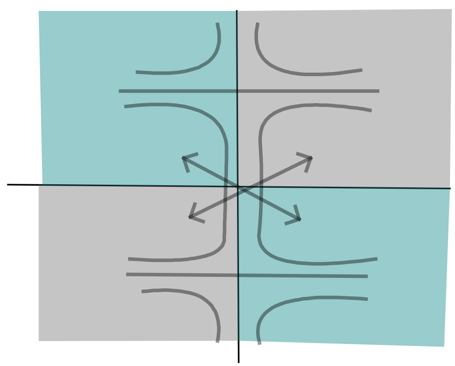

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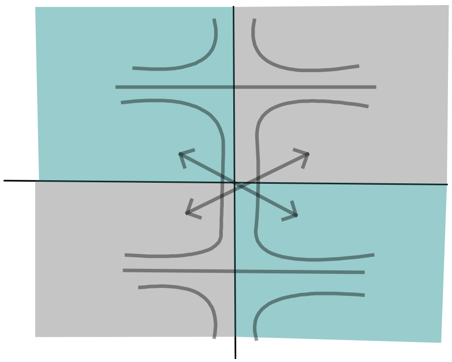
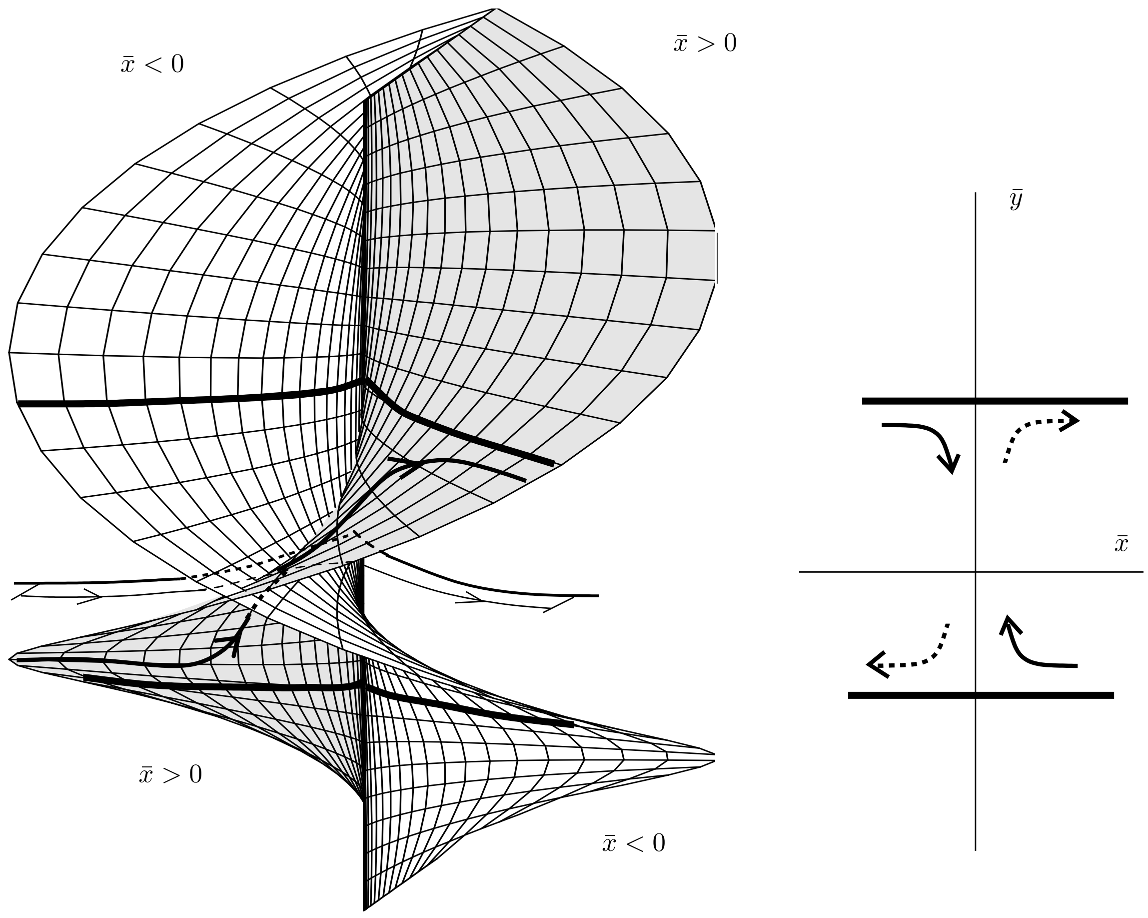


Other chart

$$\partial_2 = 2(1-x^3)\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y}$$

$$g \cdot x = \xi^{-2}\,x\,, \qquad g \cdot y = \xi\,y\,, \qquad\quad (\xi^3 = \mathrm{id})$$

$$g \cdot \partial_2 = \xi^2 \partial_2$$



$\bar{x} < 0$
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Elimination of nilpotent points in dimension two - Classical proof

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Suppose that the germ is singular. We can assume that $a, b \in \mathbb{C}\{x, y\}$ have no common factor and consider

$$m(0) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(a, b)} \geqslant 1, \quad \mu(0) = \min_{k} \{(J^k a, J^k b) \neq (0, 0)\}$$

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- If $l(0) \geqslant 2$ then $m(\tilde{p}_j) < m(p)$

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- If $l(0) \geqslant 2$ then $m(\tilde{p}_j) < m(p)$
- If $l(0) = 1$ then this is a special case which has to be treated separately...

Example of "special case".

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$\mu = 1, m = M \geqslant 3$

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We can reorder the expansion and write the monomial expansion

$$\partial = \sum_{k \in \mathbb{Z}^n} x^k L(\mu_k)$$

where, we recall, each $L(\mu) = \sum \mu_i x_i \frac{\partial}{\partial x_i}$ is a diagonal vector field, i.e. an element of the $\mathbb{C}$-maximal toral subalgebra

$$\mathfrak{t} = \left\langle x_1 \frac{\partial}{\partial x_1}, \dots, x_n \frac{\partial}{\partial x_n} \right\rangle$$

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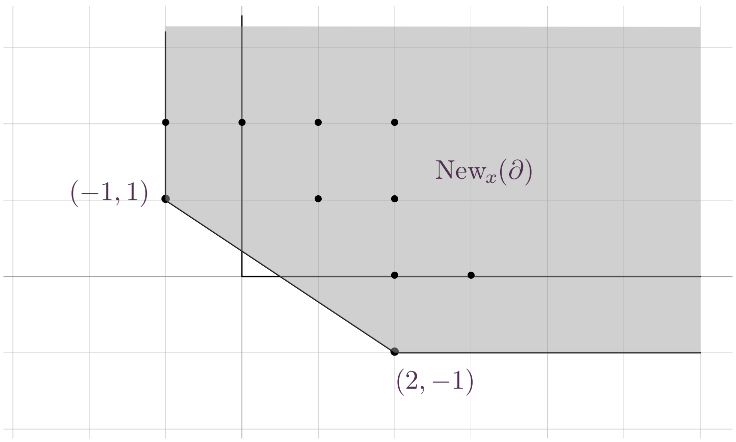

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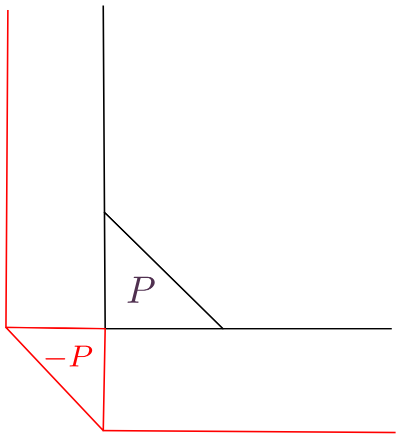

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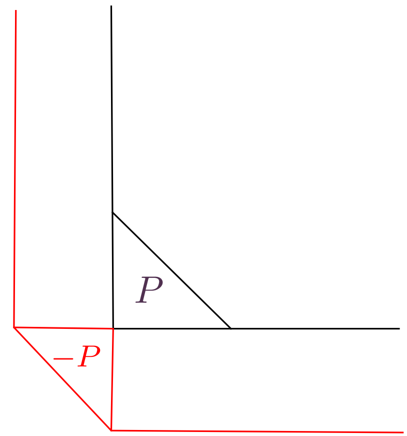

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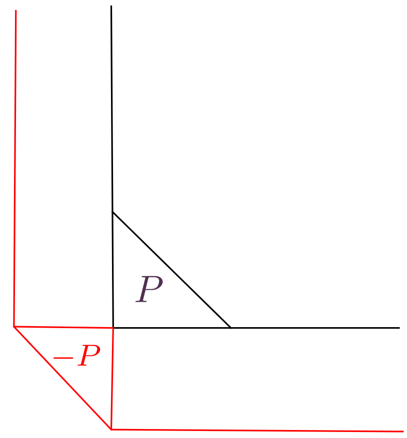

2) The hypersurface $(x_i = 0)$ is invariant by ∂ if and only if $\mathrm{supp}_x(\partial) \subset \{k : k_i \geqslant 0\}$.

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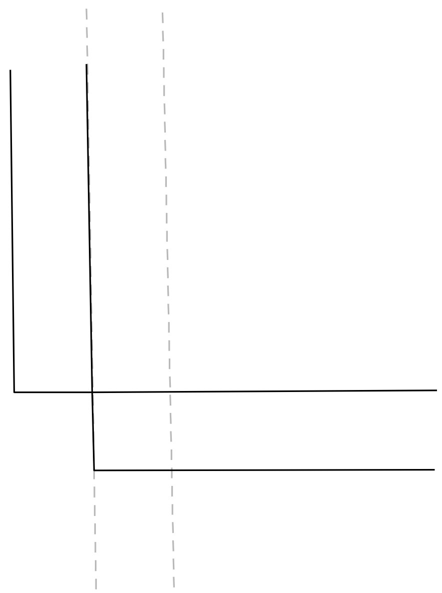

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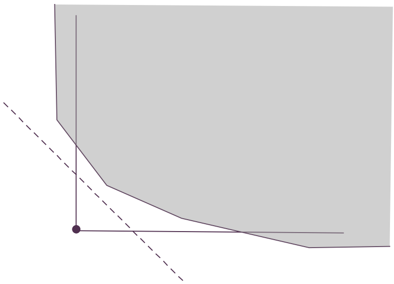

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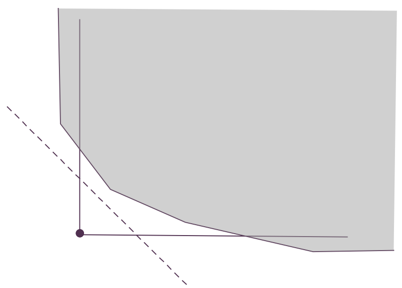

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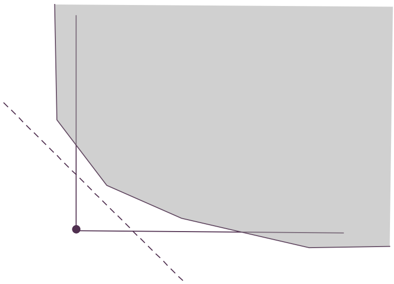

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$$\mathrm{New}_x(\partial) \subset H = \{\langle \omega, \cdot \rangle \geqslant \alpha\}$$

(indeed, if some $\omega_i < 0$ then for $v \in \mathrm{supp}_x(\partial)$, $\langle \omega, v + t e_i \rangle \to -\infty$ as $t \to +\infty$).

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And, naturally $\partial \in \mathrm{Gr}_\alpha$, $f \in \mathrm{Gr}_\beta \Longrightarrow \partial f \in \mathrm{Gr}_{\alpha+\beta}$.

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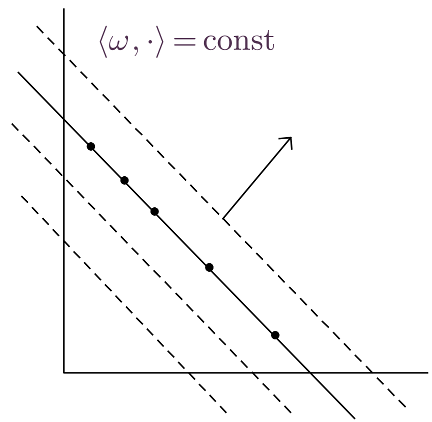

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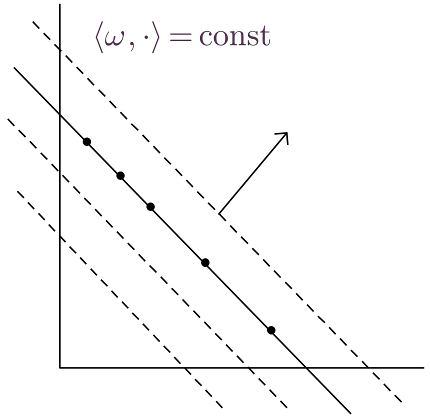

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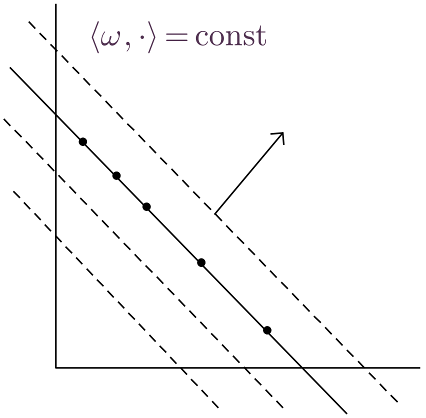

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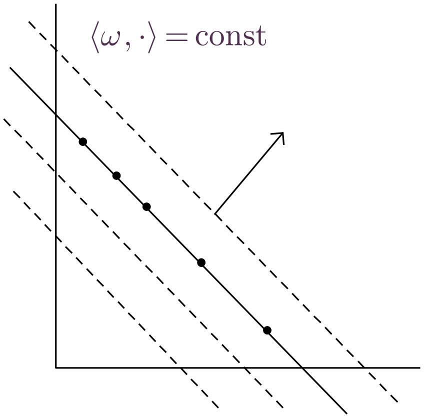

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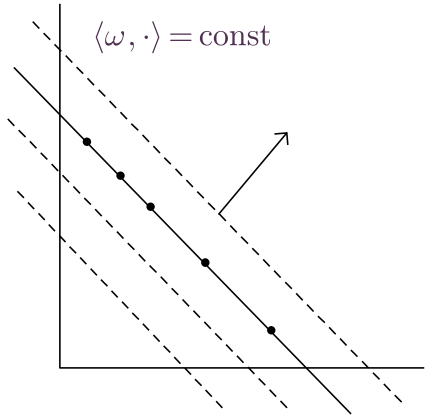

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$$\mathrm{supp}_x(\partial) \subset \{k \colon \langle \omega, k \rangle \geqslant \alpha\} \Longrightarrow \partial \in \mathrm{Gr}_{\geqslant \alpha}(\mathrm{Der}, \lambda)$$

Since this is a filtration, $\partial^2 \in \mathrm{Gr}_{\geqslant 2\alpha}$,.., $\partial^r \in \mathrm{Gr}_{\geqslant r\alpha}$ for all $r \geqslant 1$.

and if if $f \in \mathrm{Gr}_{\geqslant \beta}(\mathcal{O}, \lambda)$ then $\partial^r(f) \in \mathrm{Gr}_{\geqslant r\alpha + \beta}(\mathcal{O}, \lambda)$.

As a consequence, for $\mathfrak{m} = \langle x_1, \ldots, x_m \rangle$ the maximal ideal, for each s there exists a $r \geqslant 1$ such that

$$\partial^r(\mathfrak{m}^s) \subset \mathfrak{m}^{s+1}$$

(because for $k \in \mathbb{Z}^n_{\geqslant 0}, |k| \geqslant \langle \omega, k \rangle / \max\{\omega_i\}$). Hence, ∂ is nilpotent.

Reciprocally, assume that ∂ is nilpotent. Then, $\partial(\mathfrak{m}) \subset \mathfrak{m}$ and $\partial_S = 0$. There exists a local coordinate system such that $\partial|_{J^1} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{pmatrix}$, i.e. such that

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where $\varepsilon_i \in \{0, 1\}$. In other words, in the logarithmic basis, we obtain

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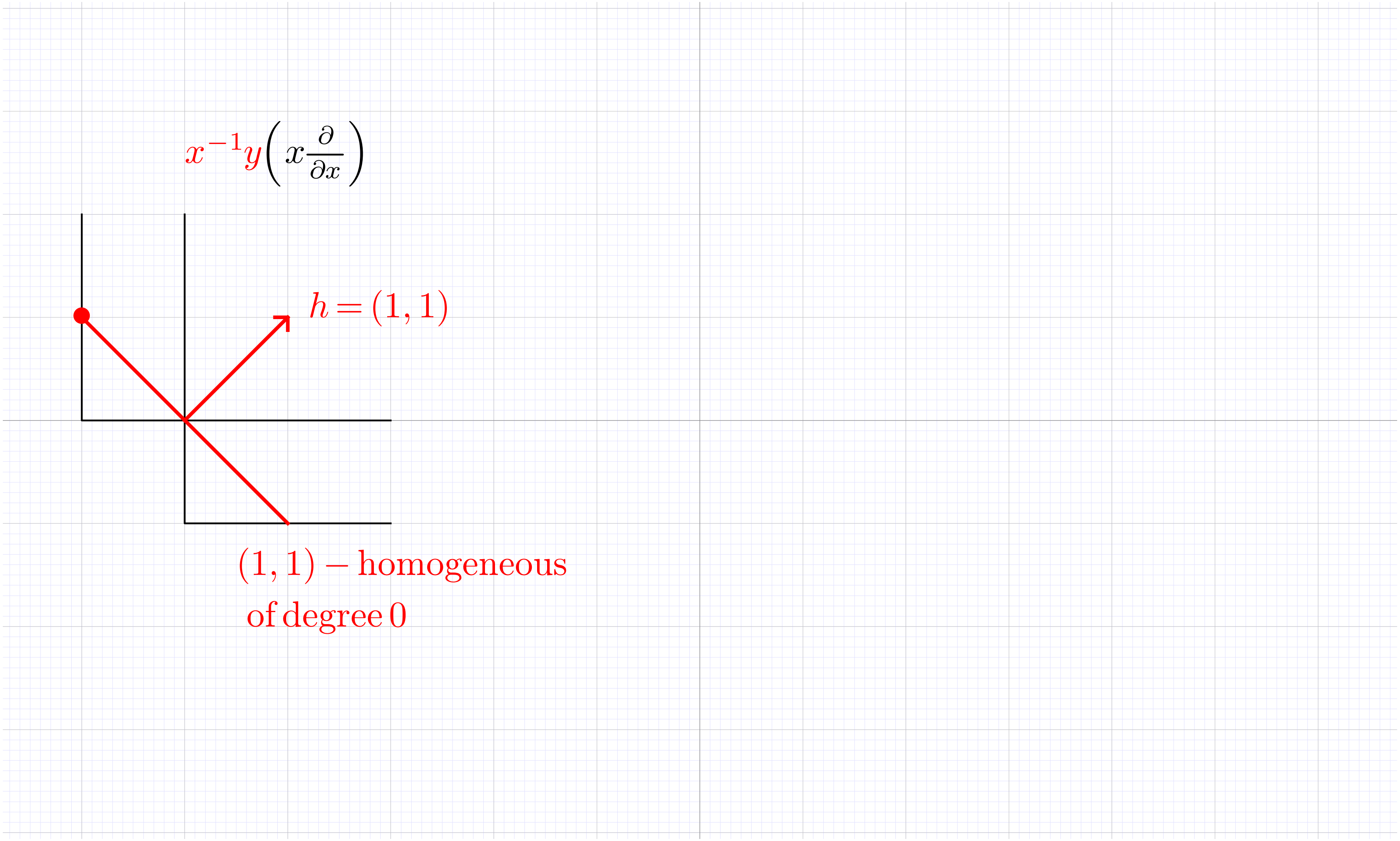

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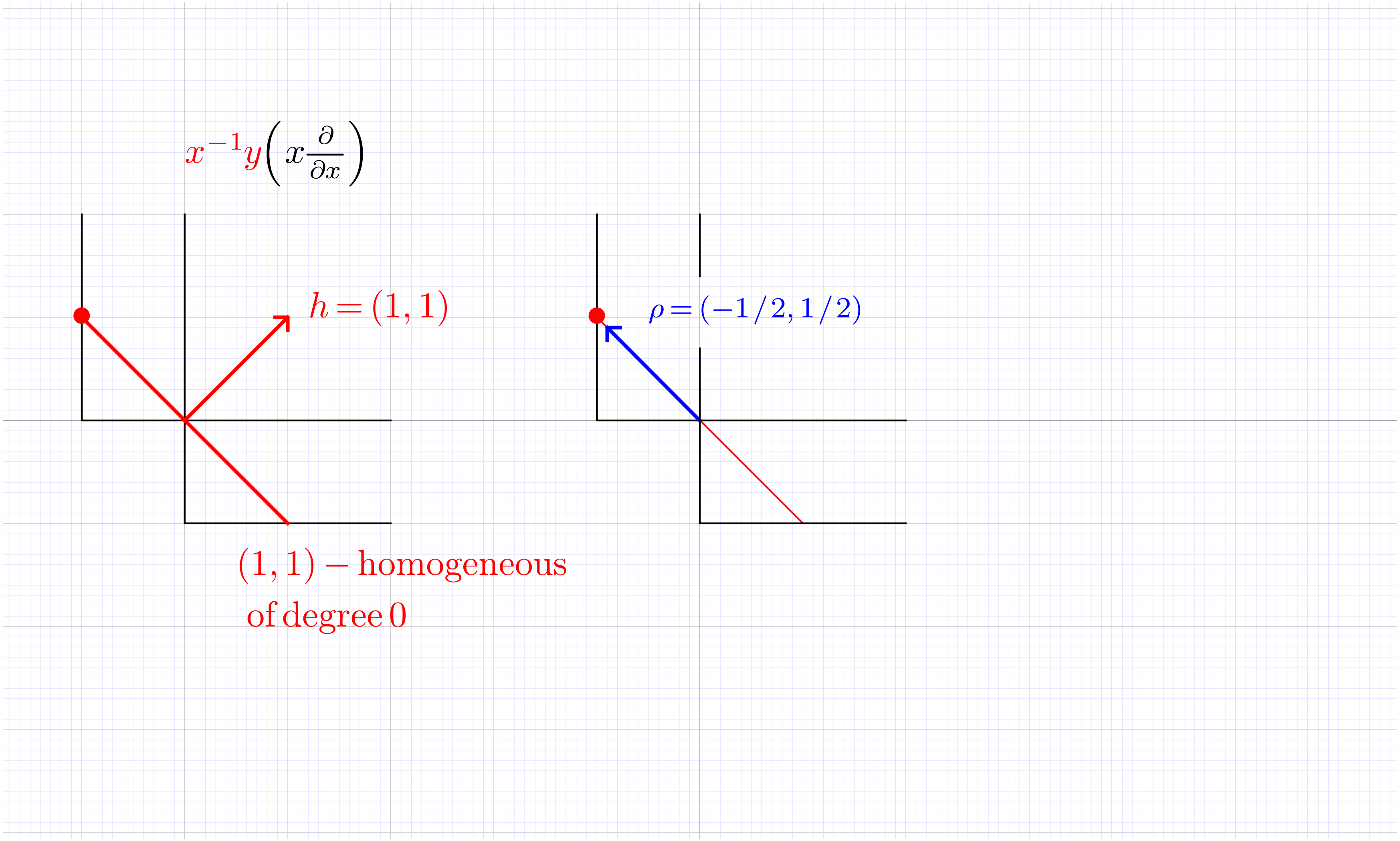

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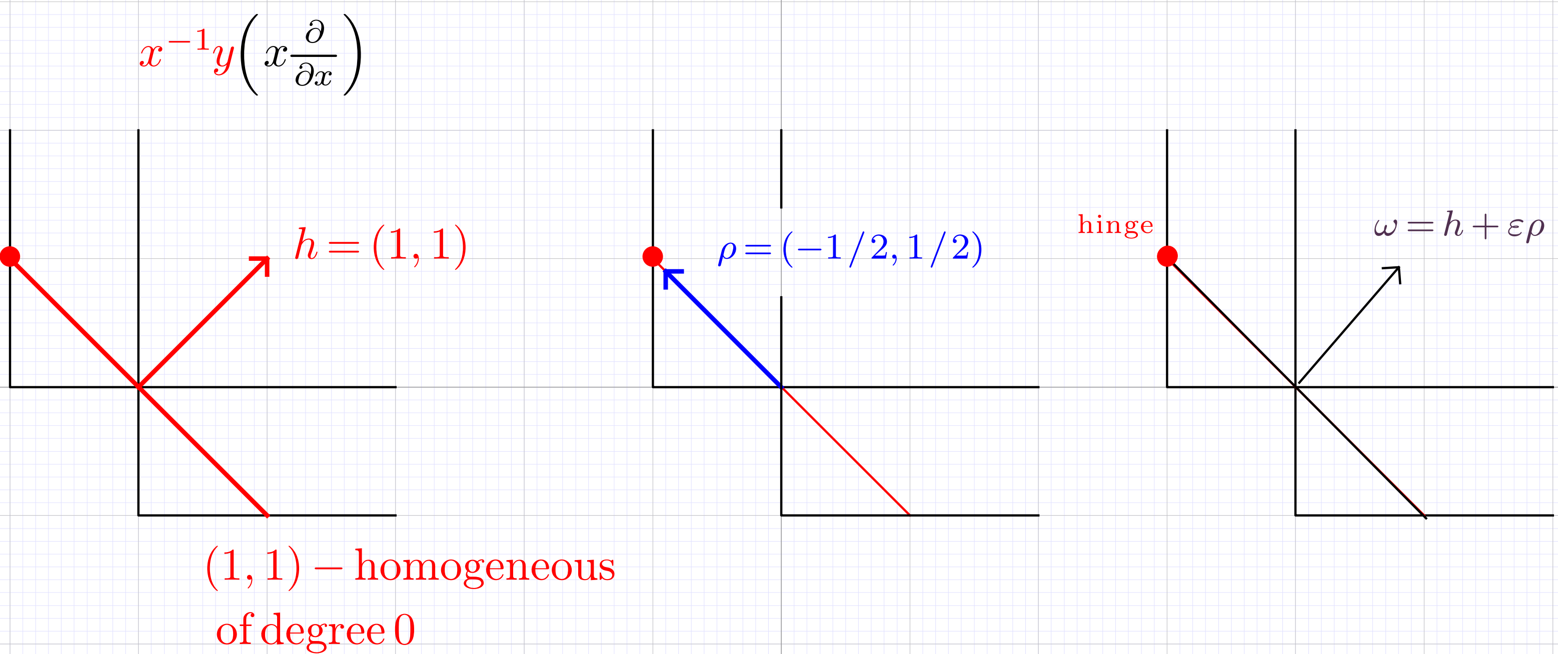

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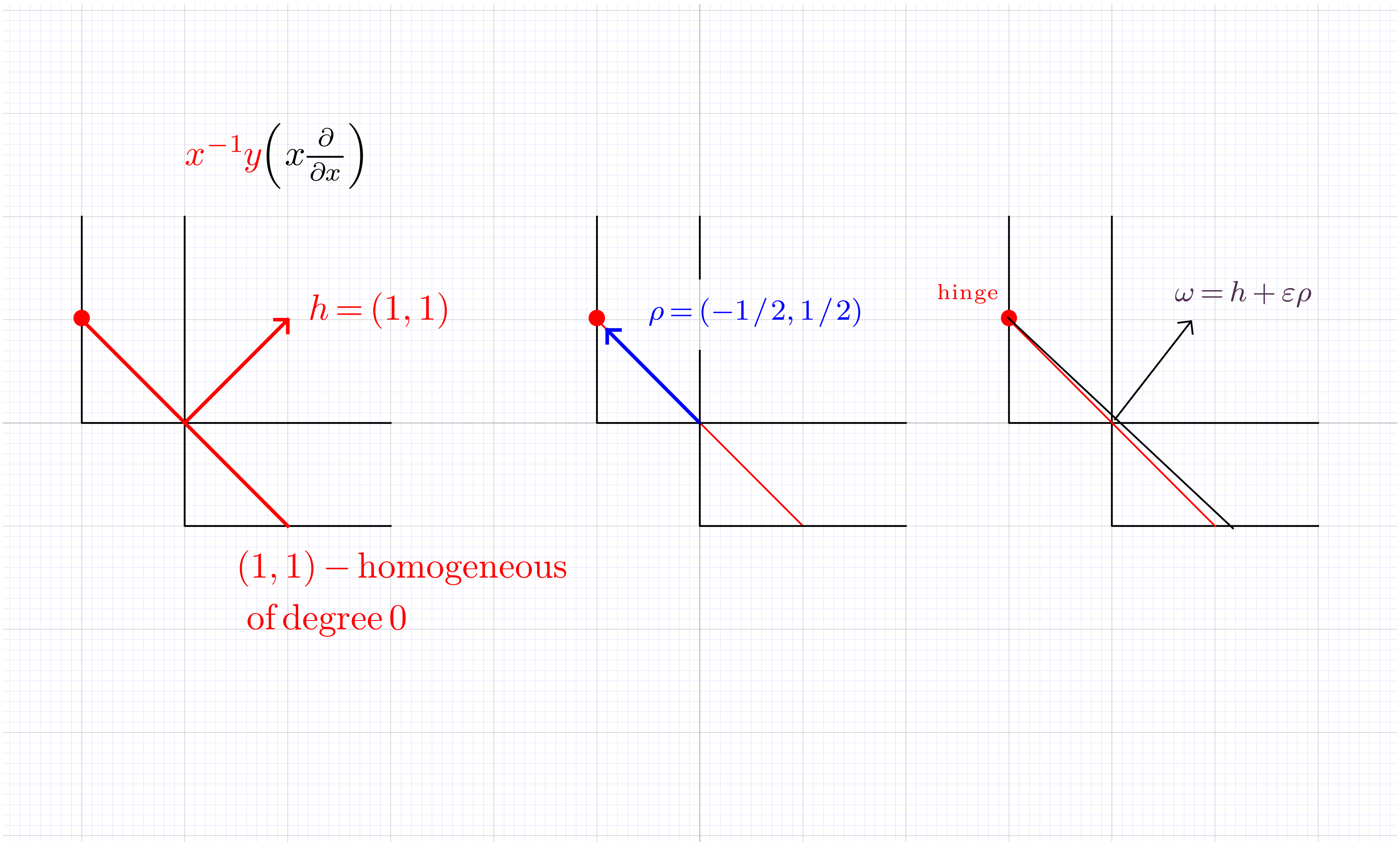

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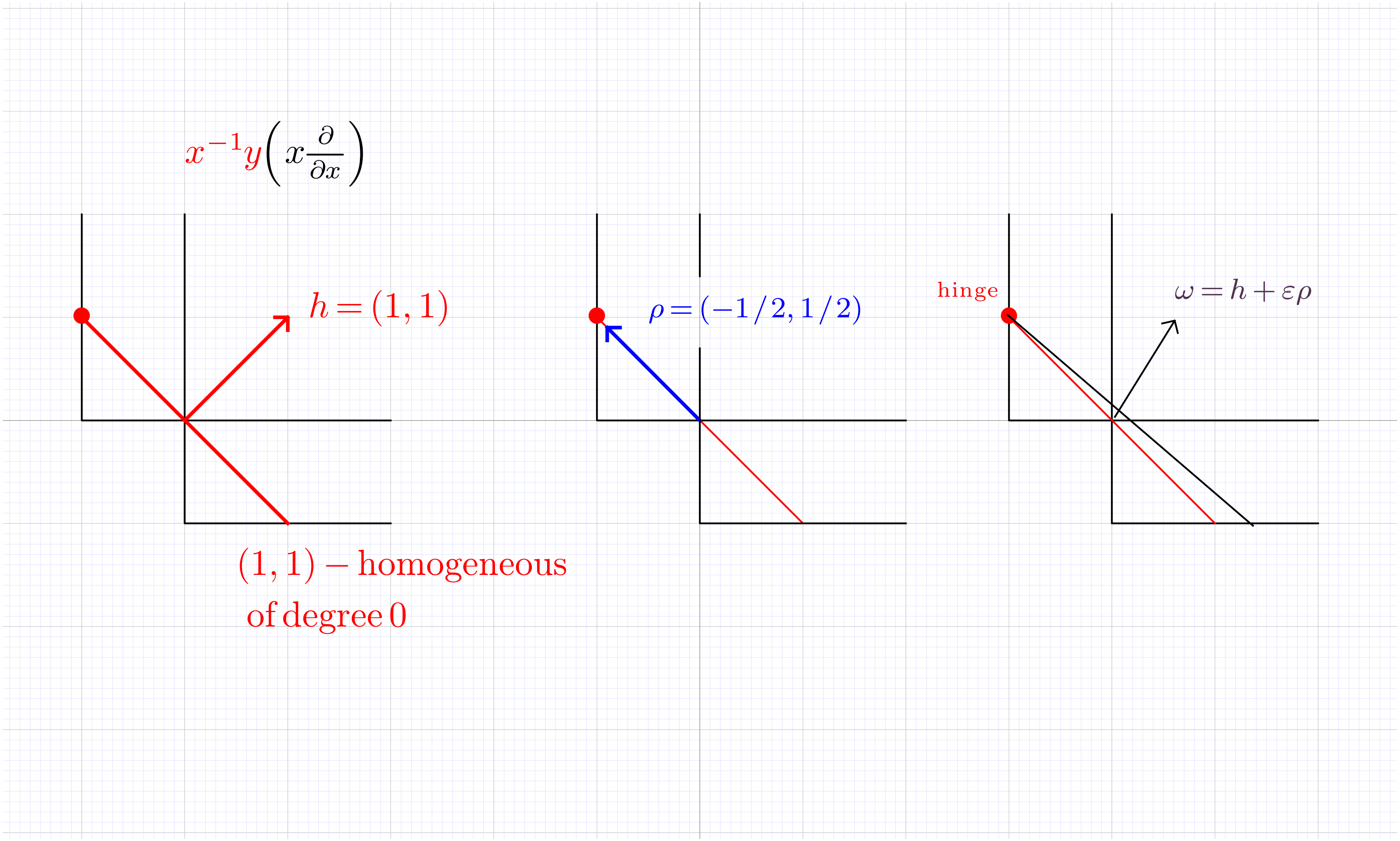

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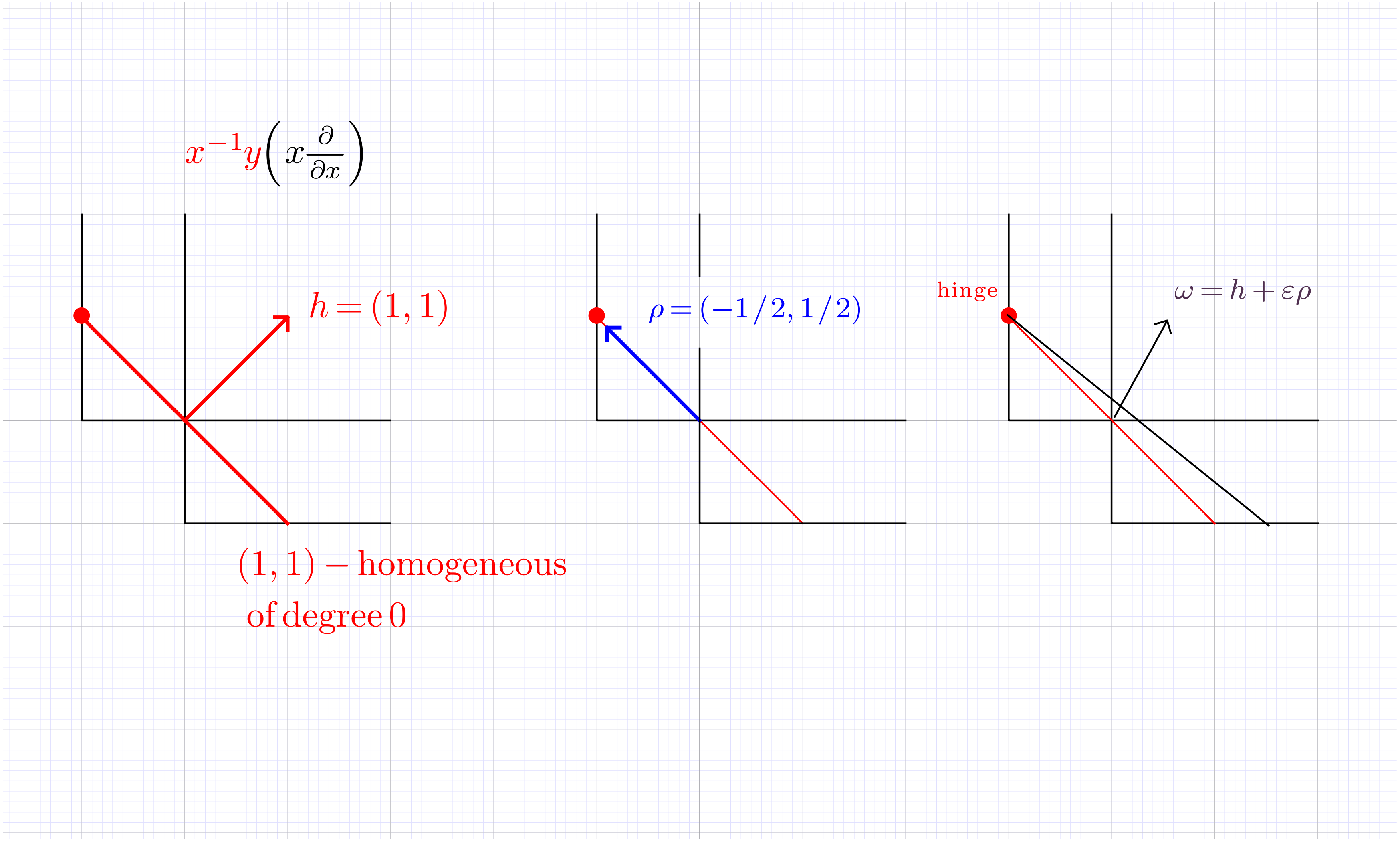

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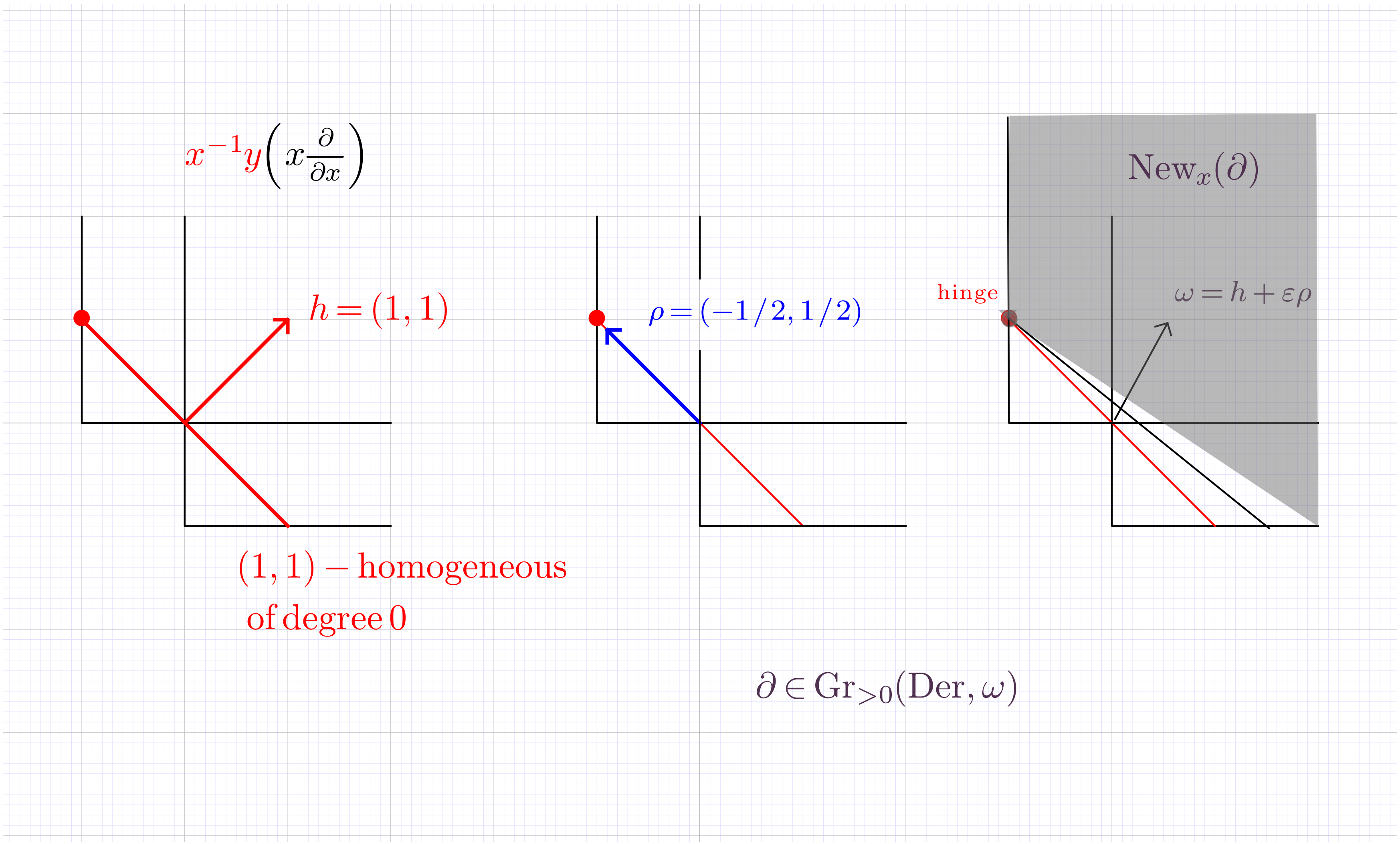

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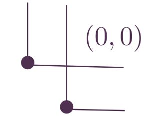
The case $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ is even easier. In fact, $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ if and only if

$$\exists i \in \{1, \ldots n\}: \quad -e_i = (0, \ldots, -1, \ldots, 0) \in \mathrm{New}_x(\partial)$$



Example: $\partial = y\,\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$. The graduation defined by the one parameter group

$$t\cdot(x,y) = (t^2x, t^3y)$$

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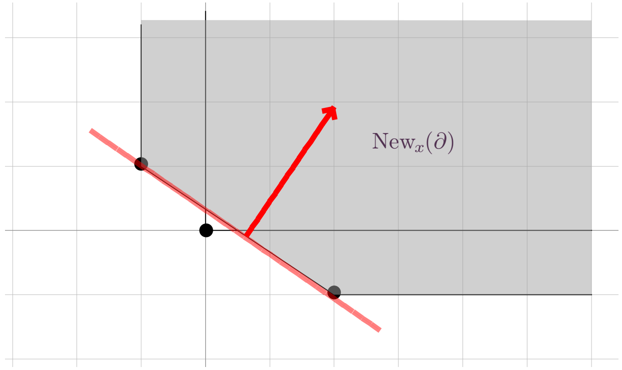

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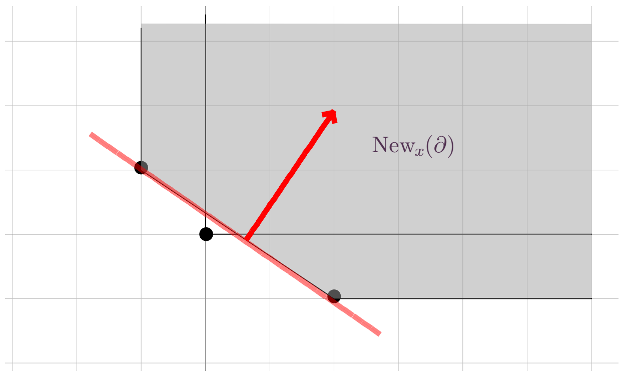

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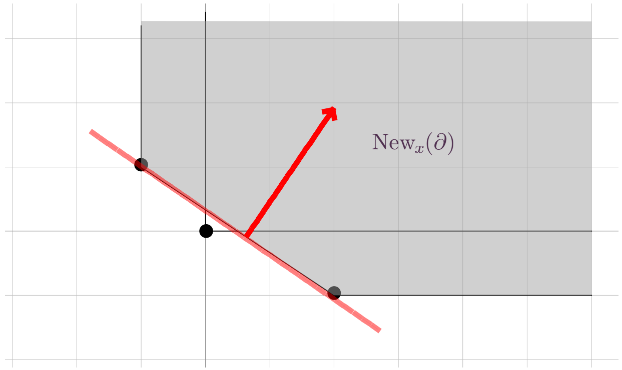

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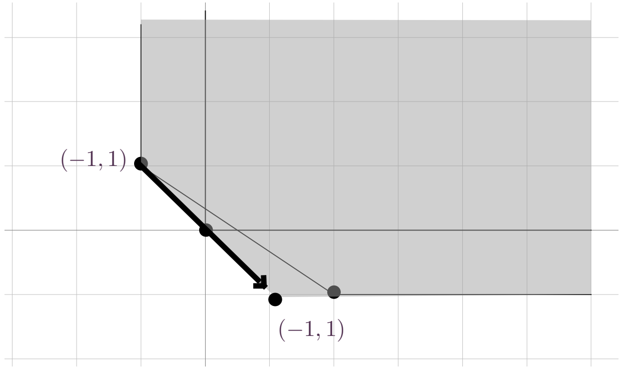

In these new coordinates, $0 \in \mathrm{New}_{(x,y)}(\partial)$.

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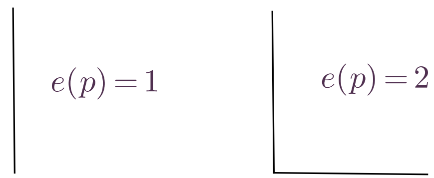

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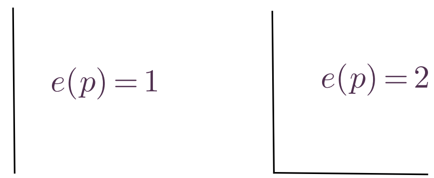

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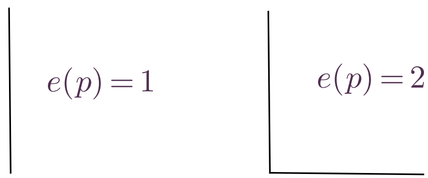

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To simplify, we will assume that $e(p) = 1$ for all points $p \in \mathrm{Nilp}(M, \mathcal{F})$.

(otherwise it suffices to slightly modify the invariant by including $e(p)$ lexicographically).

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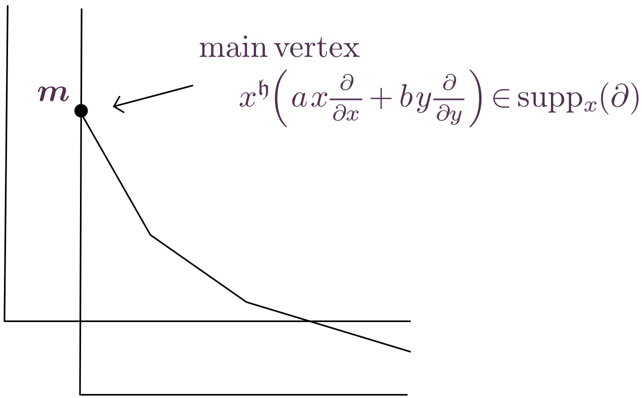

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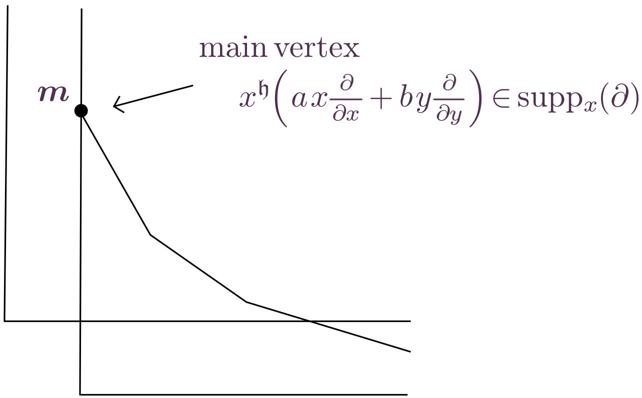

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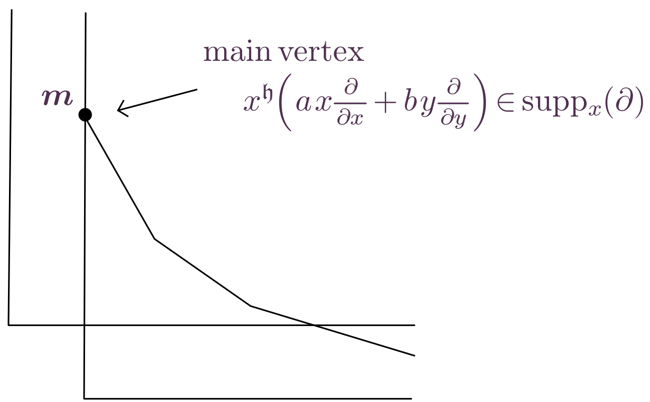

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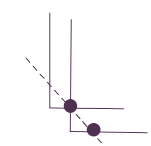
Indeed, if $0 \in \mathrm{New}(\partial)$ then the initial (1,1)-homogeneous part of ∂ would be either

$$b\frac{\partial}{\partial y} \quad (\text{case } h = -1), \quad \text{or} \quad ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cxy^{-1}\left(y\frac{\partial}{\partial y}\right) \quad (\text{case } h = 0)$$



where for some constants a, b, c such that $(a, b) \neq (0, 0)$.

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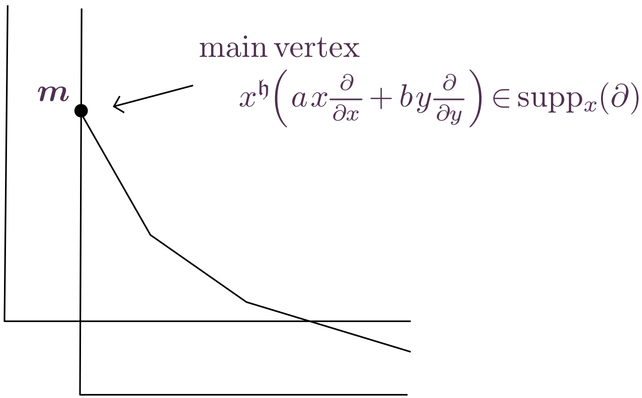

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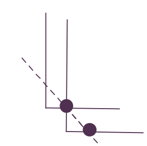
Indeed, if $0 \in \mathrm{New}(\partial)$ then the initial (1,1)-homogeneous part of ∂ would be either

$$b\frac{\partial}{\partial y} \quad (\text{case } h=-1), \quad \text{or} \quad ax\frac{\partial}{\partial x}+by\frac{\partial}{\partial y}+cxy^{-1}\left(y\frac{\partial}{\partial y}\right) \quad (\text{case } h=0)$$



where for some constants a, b, c such that $(a, b) \neq (0, 0)$.

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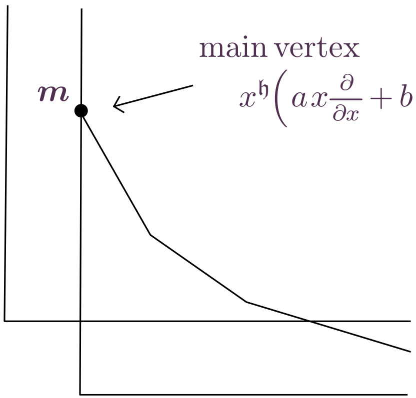
Suppose that $p \in E$. In adapted coordinates (x, y), the Newton polygon has the form



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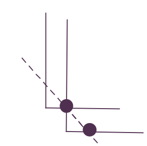

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In fact, the group of local automorphisms (preserving $x=0$) has the form

$$x \to x\,f(x,y), \quad y \to g(x,y)$$

f unit, $\partial g / \partial y(0,0) \neq 0$. Its Lie algebra is generated by vector fields with support in

$$x^k y^l\, x\frac{\partial}{\partial x}, \quad x^u y^v\, y\frac{\partial}{\partial y}, \quad k+l \geqslant 0, u+v \geqslant 0$$

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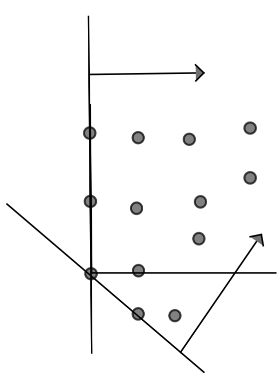

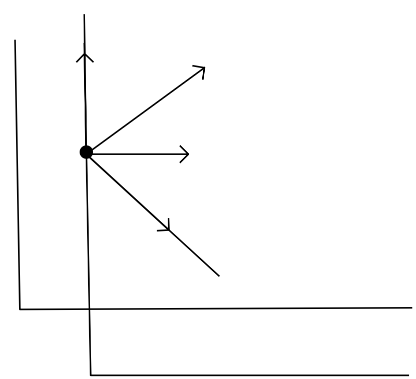

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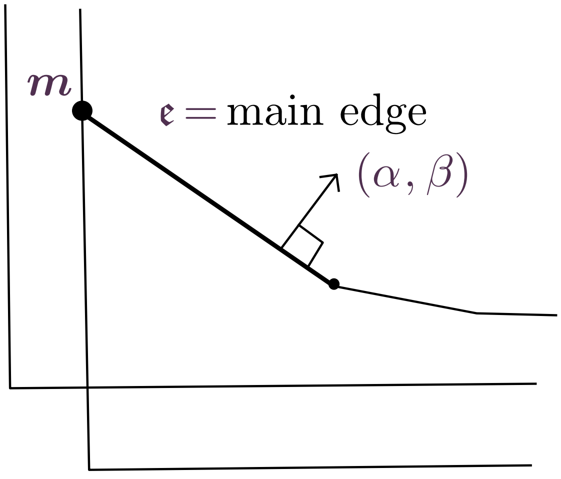

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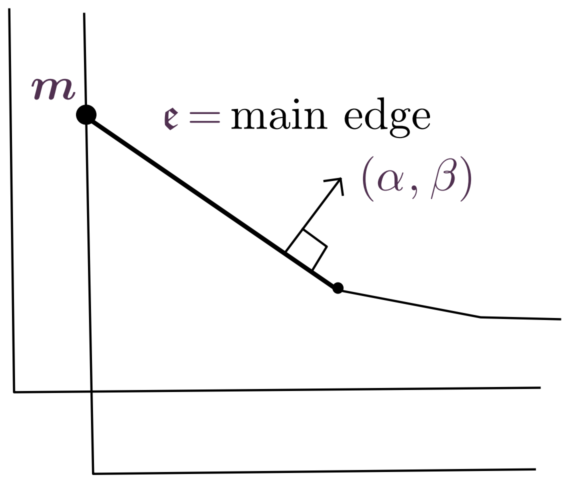

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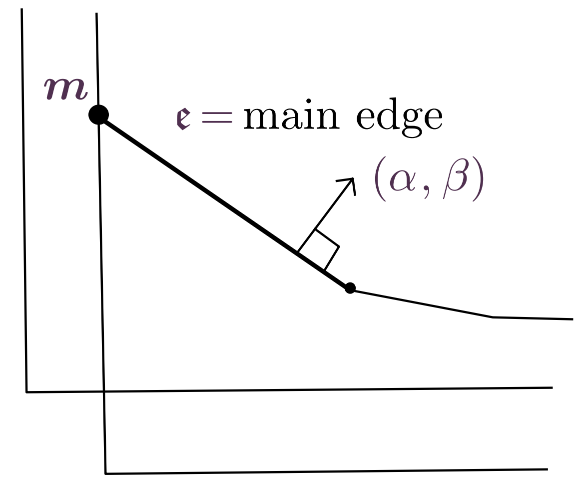

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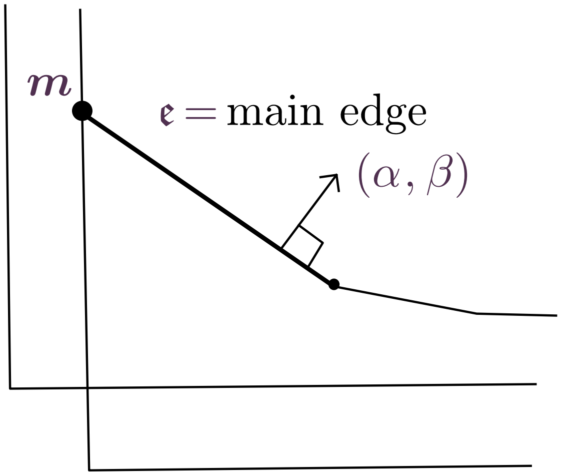

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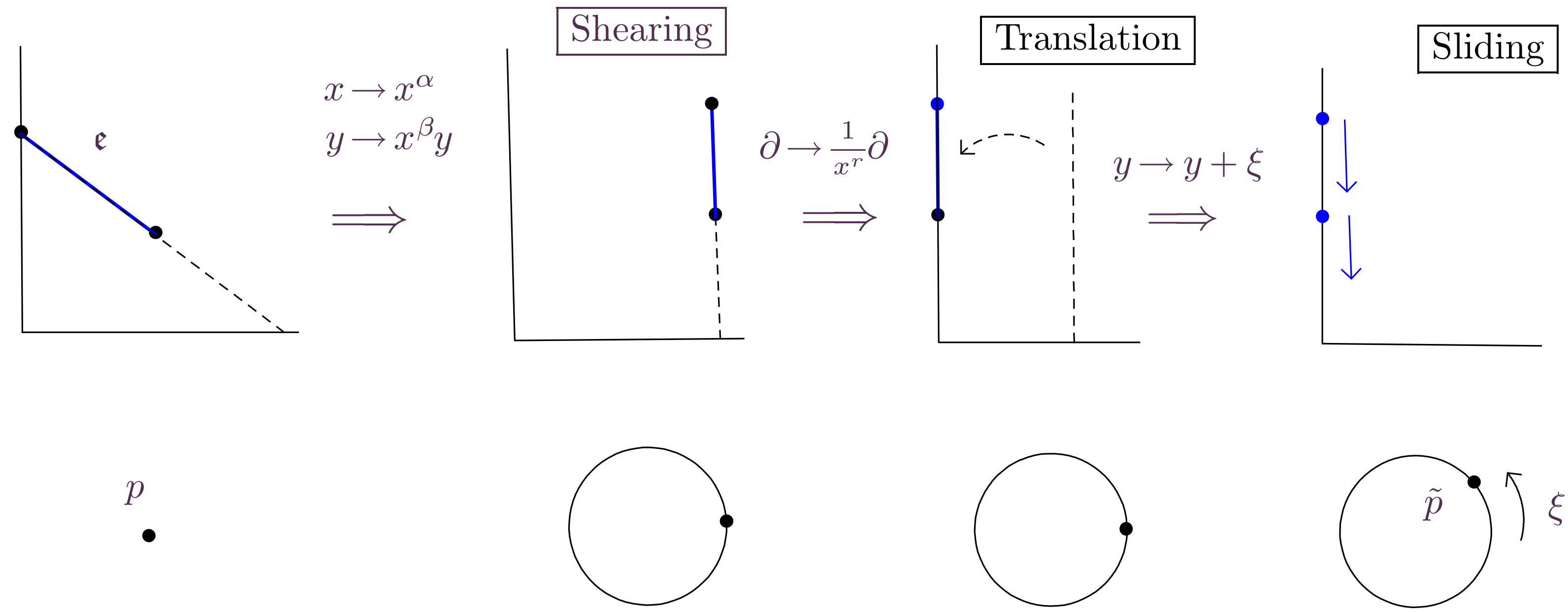
Let $\mathrm{wt}(\mathfrak{e})=\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial x}$ denote the irreducible weight-vector determined by $\mathfrak{e}$.

Action of the blowing-up with weight (α, β) on the polygon



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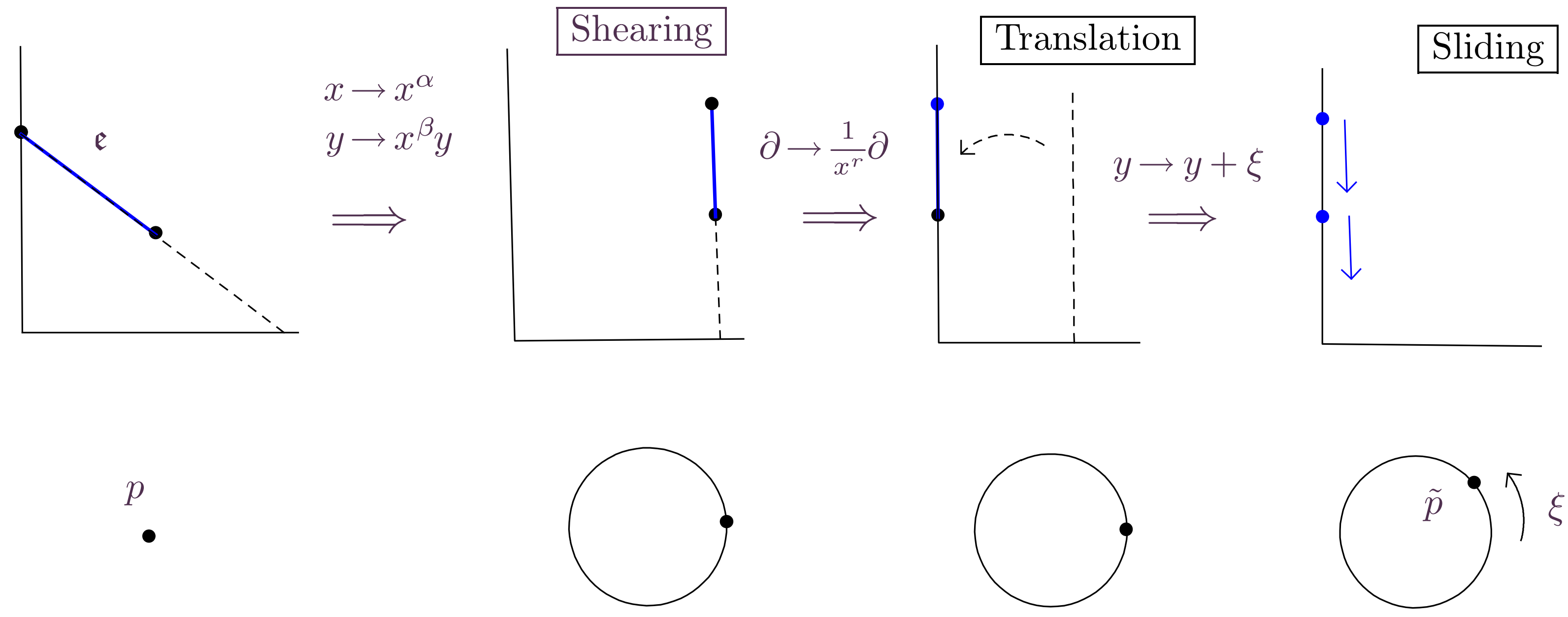

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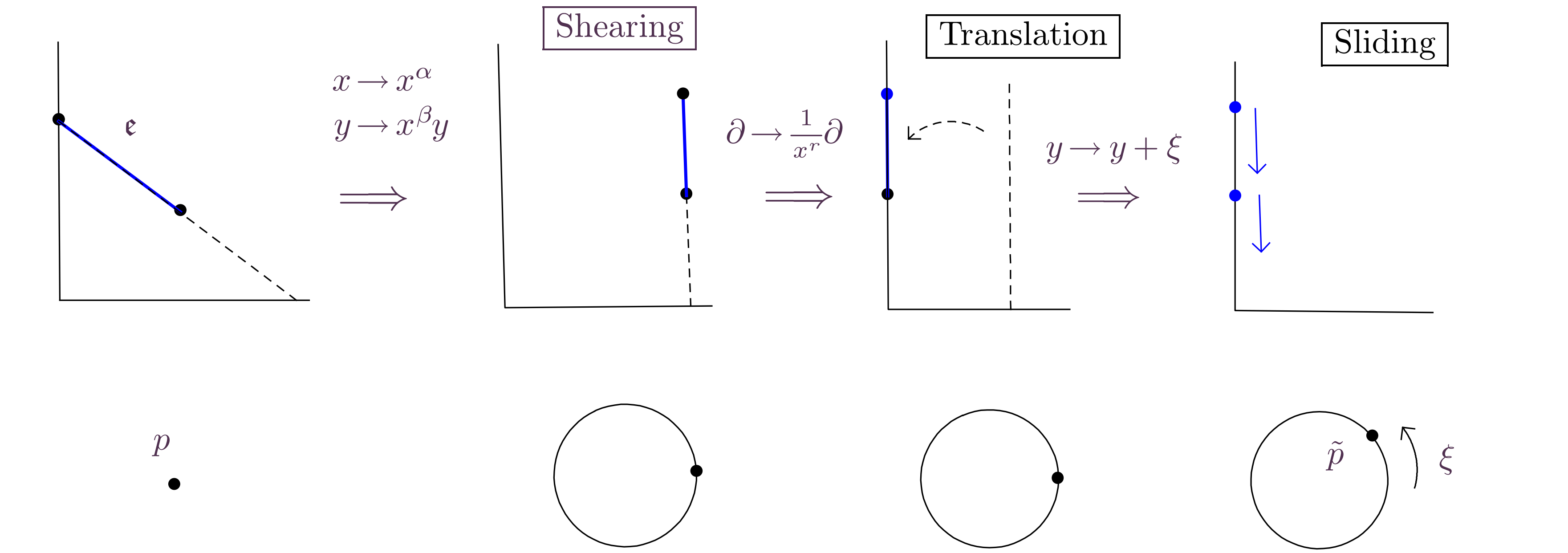

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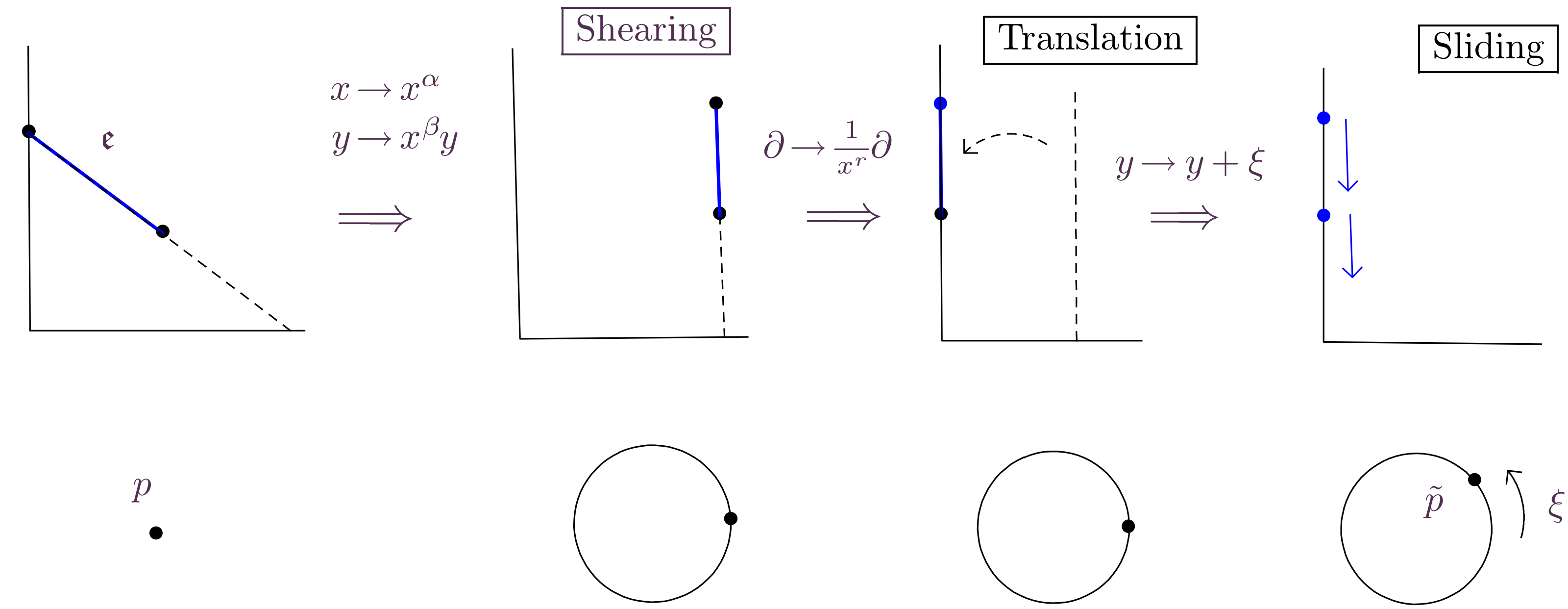

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We can have a full compensation phenomena in the "sliding phase".

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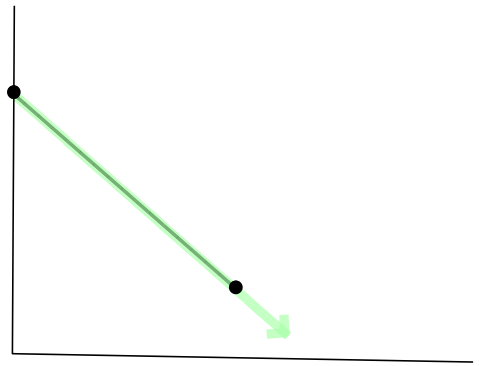

The above map slides the monomials in the direction of the main edge.

Theorem (Local resolution) Suppose that $\mathrm{New}_{(x,y)}(\partial)$ is edge stable, and let

$$\Phi\colon \tilde{M} \to M$$

be the blowing-up of $p \in \mathrm{Nilp}(M, \mathcal{F})$ with weight $\mathrm{wt}(\mathfrak{e})$. Then,

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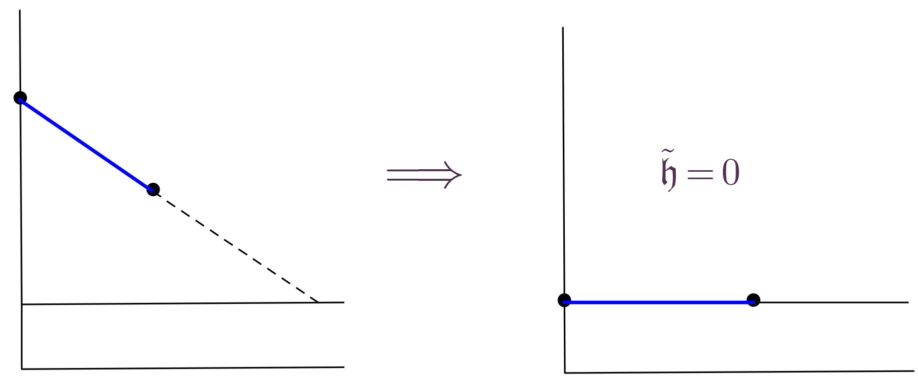

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Suppose that $\tilde{\mathfrak{h}} = \mathfrak{h}$. Then, there should exists a non-zero constant ξ such that the translation (in blowed-up coordinates)

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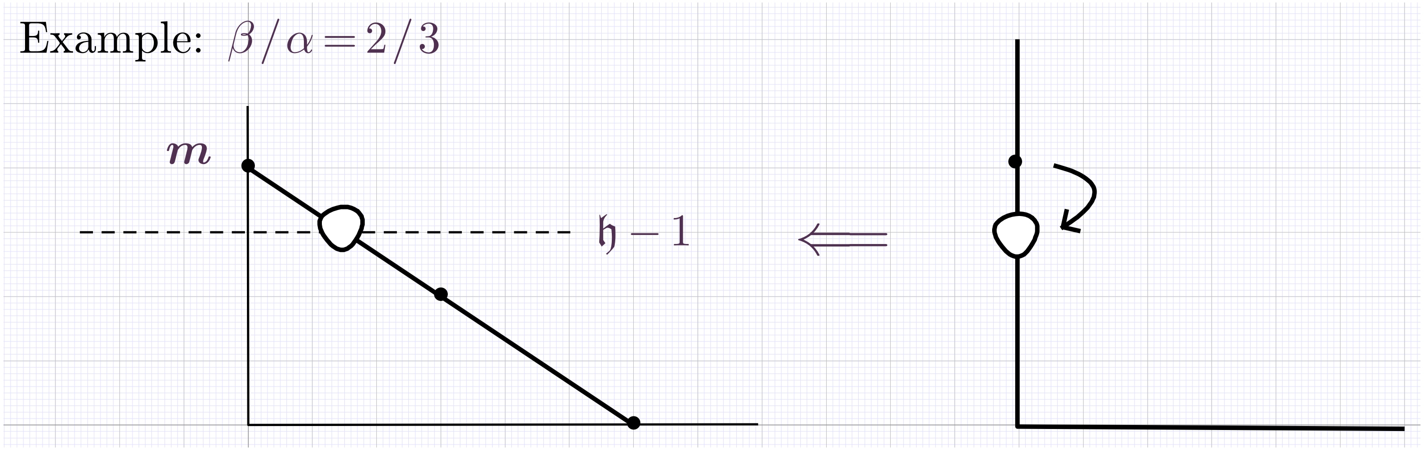
The assumption $\tilde{\mathfrak{h}} = \mathfrak{h}$ is equivalent to say that $\mathrm{New}_{(x,y)}(\partial)$ is edge-unstable, which contradicts the hypothesis of the Theorem.

In the former case $(i.e.\, \beta/\alpha \notin \mathbb{Z})$, the $\mathfrak{e}$-initial form of ∂ has a gap at height $\mathfrak{h}-1$.



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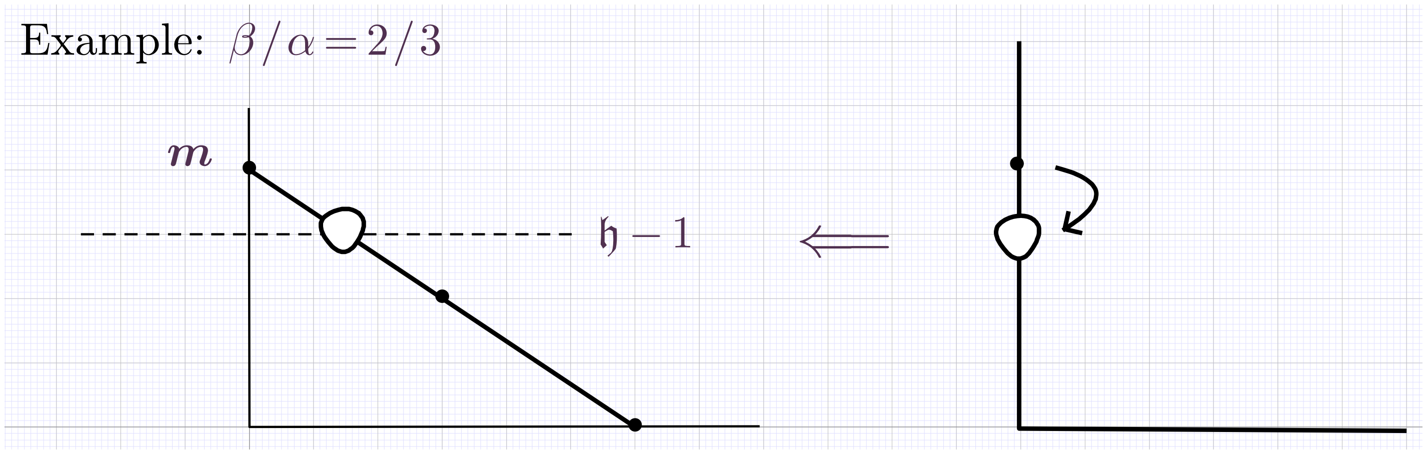

After blowing-up, followed by an arbitrary translation $y \to y+\xi$, we have

$$(y^{\mathfrak{h}})\left(\alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}\right) + \text{terms in } y^{\leqslant \mathfrak{h}-2} \longrightarrow (y+\xi)^{\mathfrak{h}}\left(\alpha x\frac{\partial}{\partial x} + \beta(y+\xi)\frac{\partial}{\partial y}\right) + \cdots$$

which gives a monomial on the support at height $\mathfrak{h}-1$.

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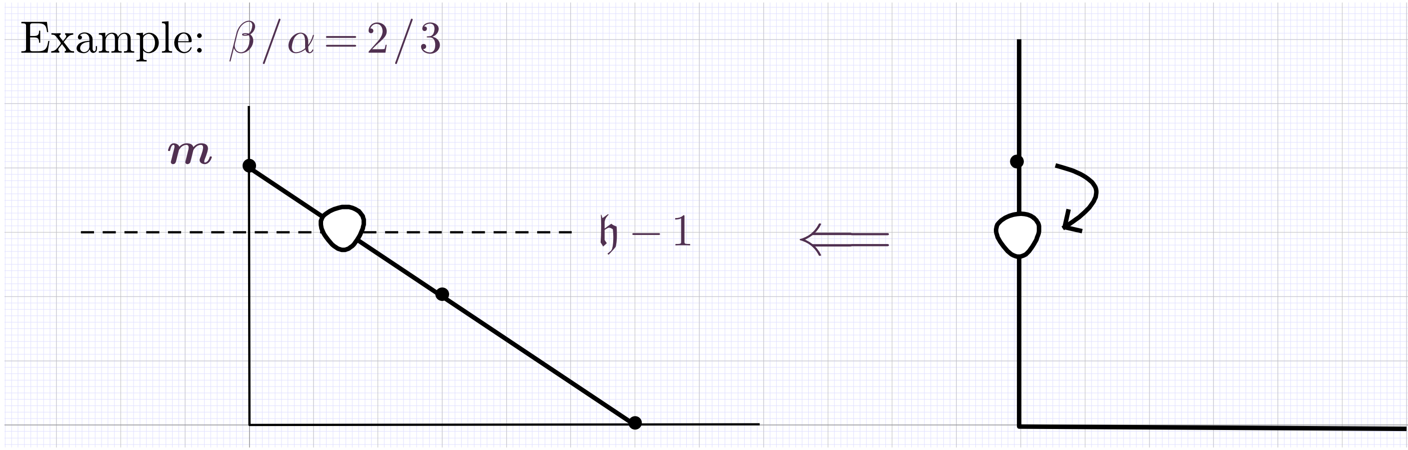

After blowing-up, followed by an arbitrary translation $y \to y+\xi$, we have

$$(y^{\mathfrak{h}})\left(\alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}\right) + \text{terms in } y^{\leqslant \mathfrak{h}-2} \longrightarrow (y+\xi)^{\mathfrak{h}}\left(\alpha x\frac{\partial}{\partial x} + \beta(y+\xi)\frac{\partial}{\partial y}\right) + \cdots$$

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(Abhyankar called this argument the "lazy Tschirnhaussen").

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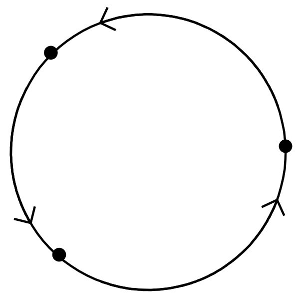
$$y \longrightarrow \zeta^{-\beta} y$$



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Proof: We start with an arbitrary adapted coordinate system (x, y_0).

1) If $\mathrm{New}_{(x,y_0)}(\partial)$ is edge-stable, we stop

2) If $\mathrm{New}_{(x,y_0)}(\partial)$ is edge-unstable, we choose a polynomial coordinate change $(x, y_0) \to (x, y_1)$, where

$$y_1 = y_0 + \xi_0\, x^{k_0}, \qquad k_0 = \beta_0 / \alpha_0$$

eliminates the main edge $\mathfrak{e}_0$.

We now consider the new coordinates (x, y_1) and apply the same argument. I claim that this procedure eventually stops with an edge stable situation.

Indeed, assume the contrary. Then, we end-up with an infinite sequence of coordinate changes

$$y_{i+1} = y_i + \xi_i x^{k_i}, \quad i \geqslant 1$$

where $\{k_i = \beta_i / \alpha_i\}$ forms an strictly increasing sequence of integers, corresponding to the successive slopes of the edges $\mathfrak{e}_i$.

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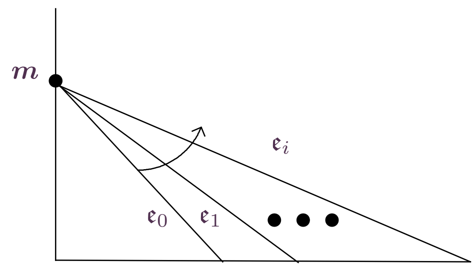

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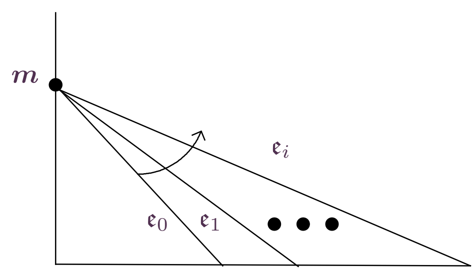

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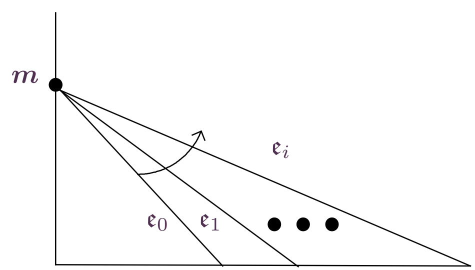

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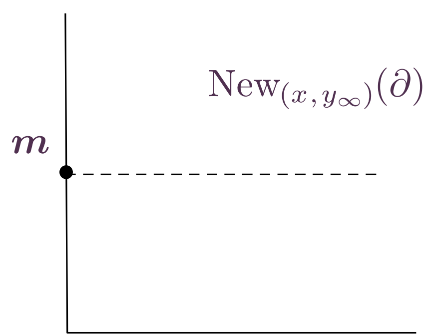
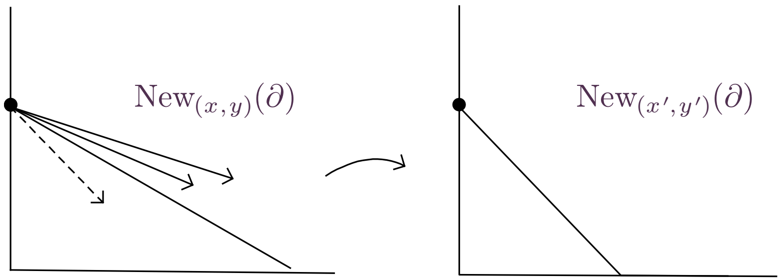
In these coordinates, 

i.e. $(\widehat{y_\infty} = 0) \subset \mathrm{Nilp}(M, \mathcal{F})$. Contradiction.

Uniqueness of the filtration. Suppose that $\mathrm{New}_{(x,y)}(\partial)$, $\mathrm{New}_{(x',y'')}(\partial)$ are edge stable



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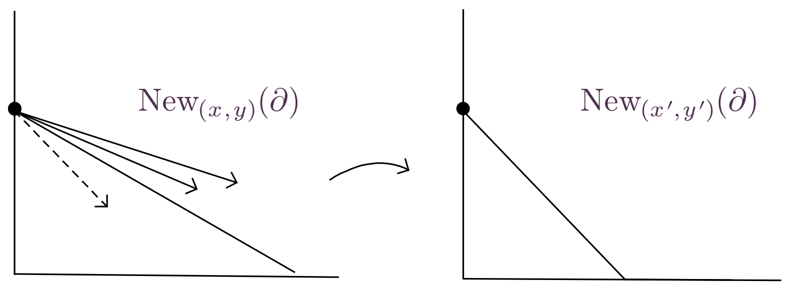

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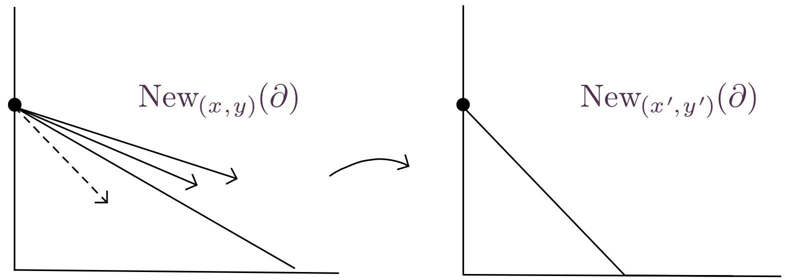

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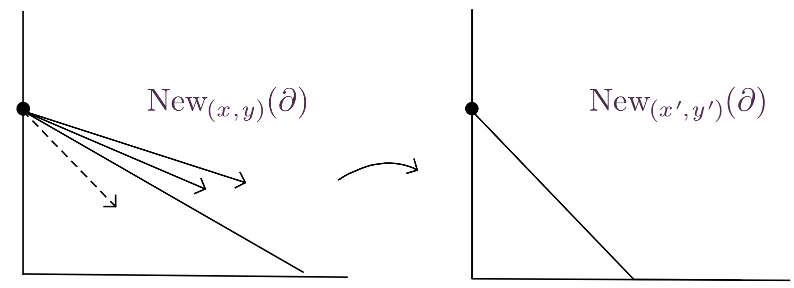

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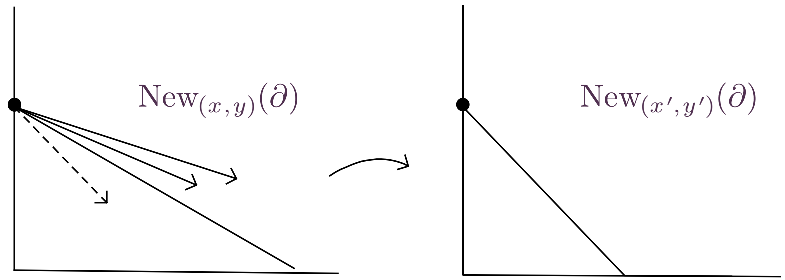

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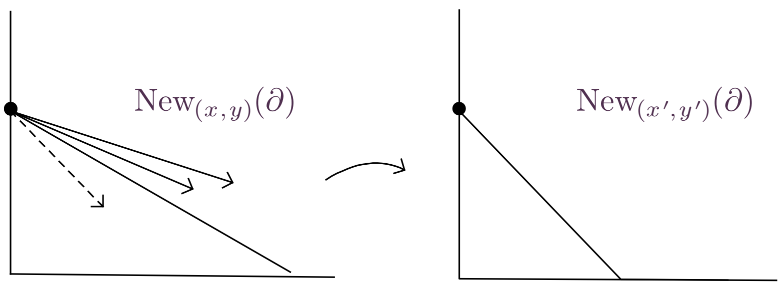

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is has a main edge $\mathfrak{e}'$ of slope $k < \beta/\alpha$ (because the action of $y \to y + \xi x^k$ on $\mathrm{New}_{(x,y)}(\partial)$ is effective).

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But this contradicts the fact that the **inverse transformation** $y = y_1 - \xi x^k$ eliminates the main edge.

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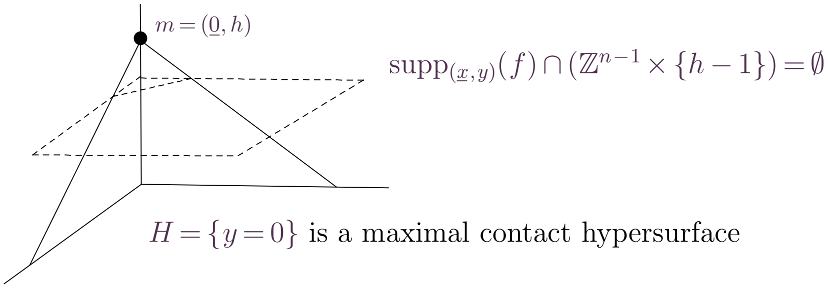

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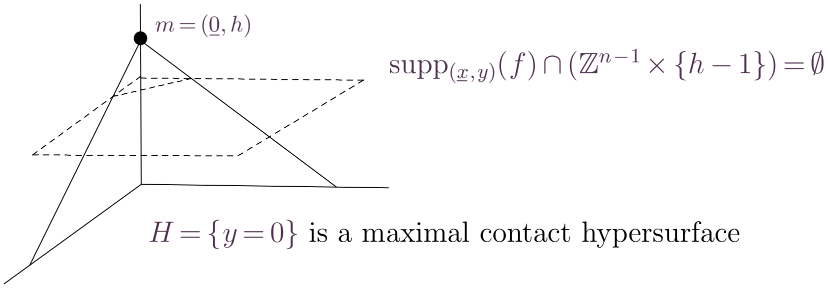

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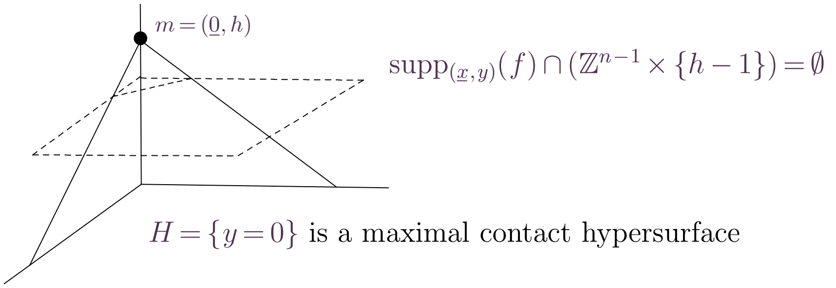

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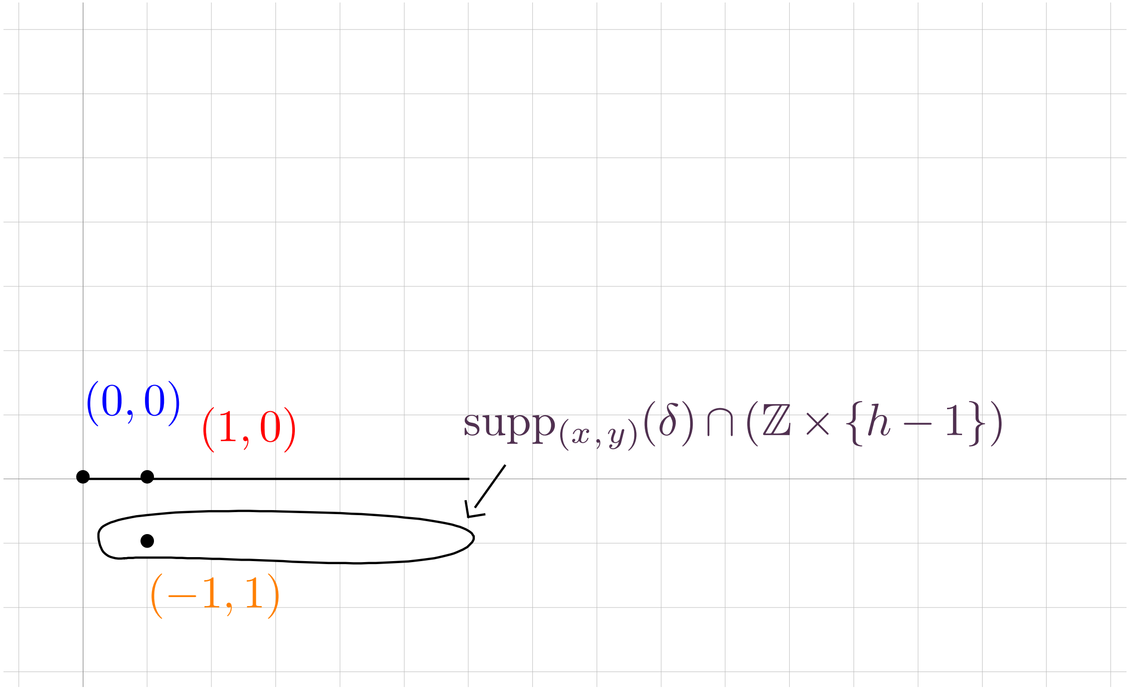

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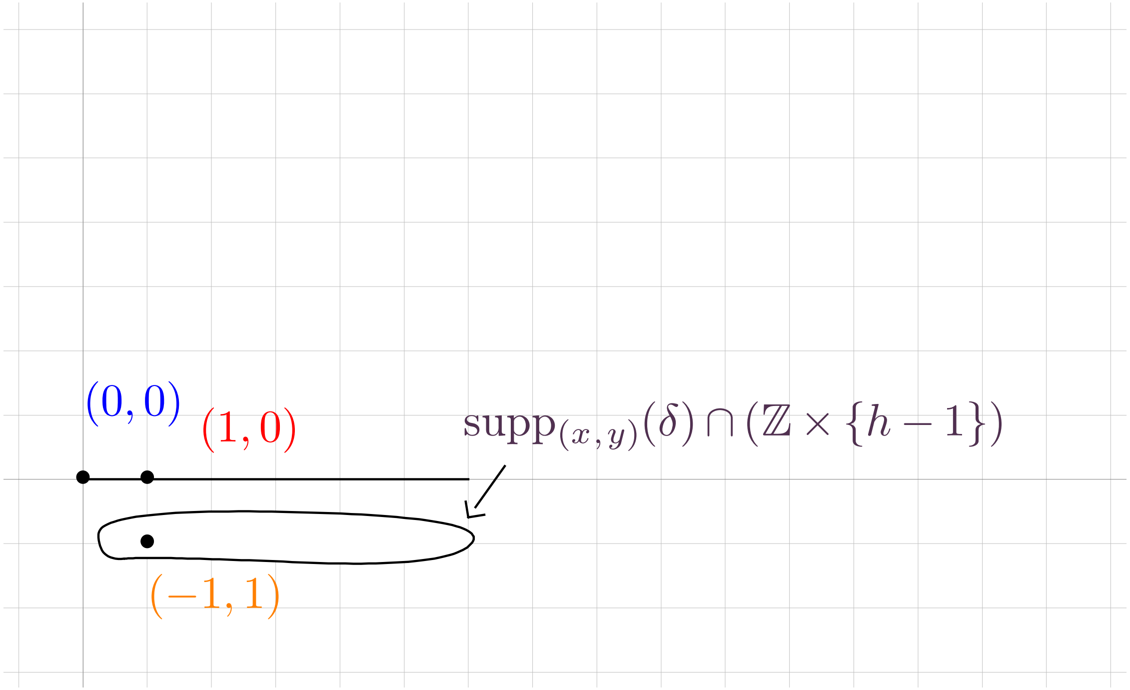

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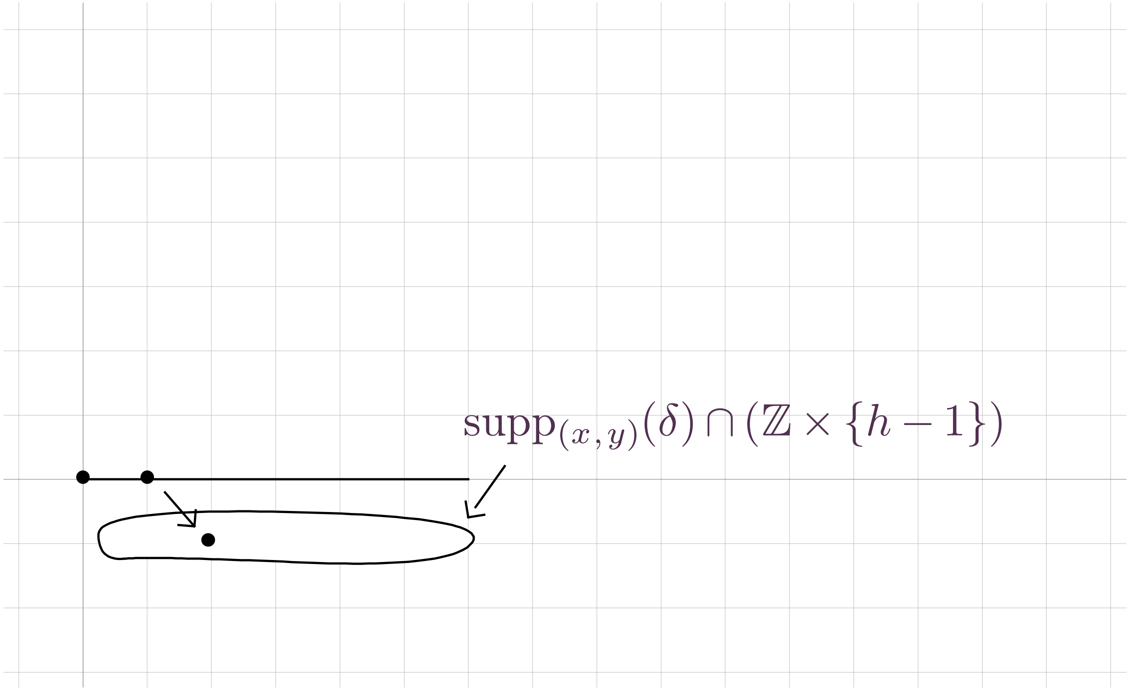

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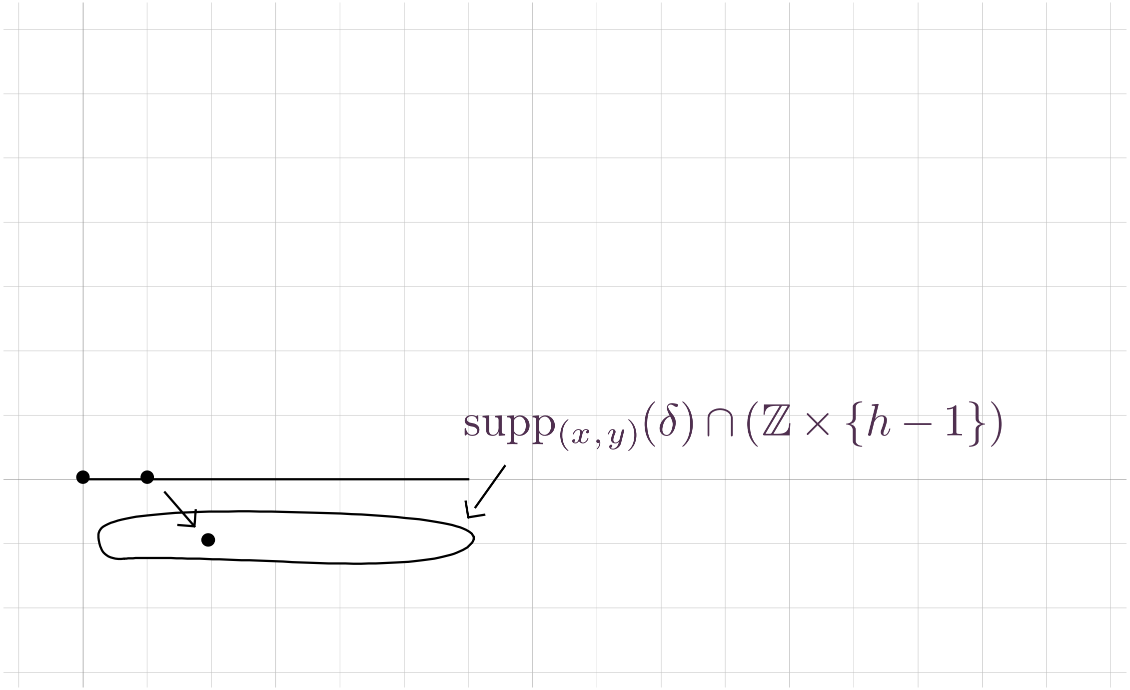

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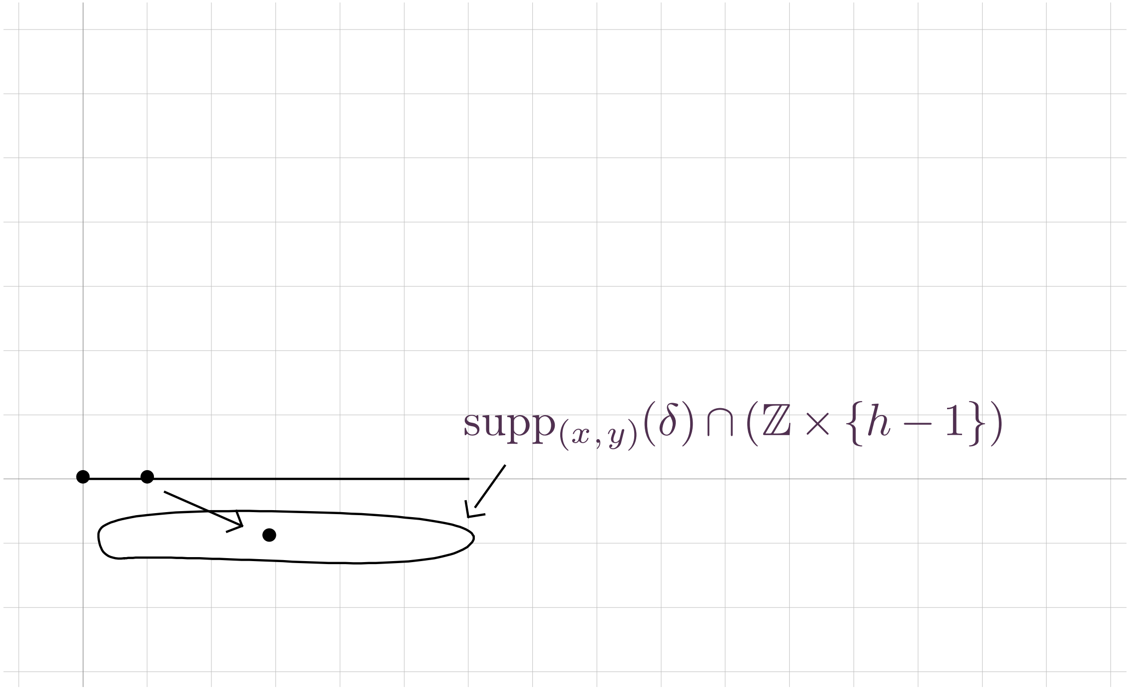

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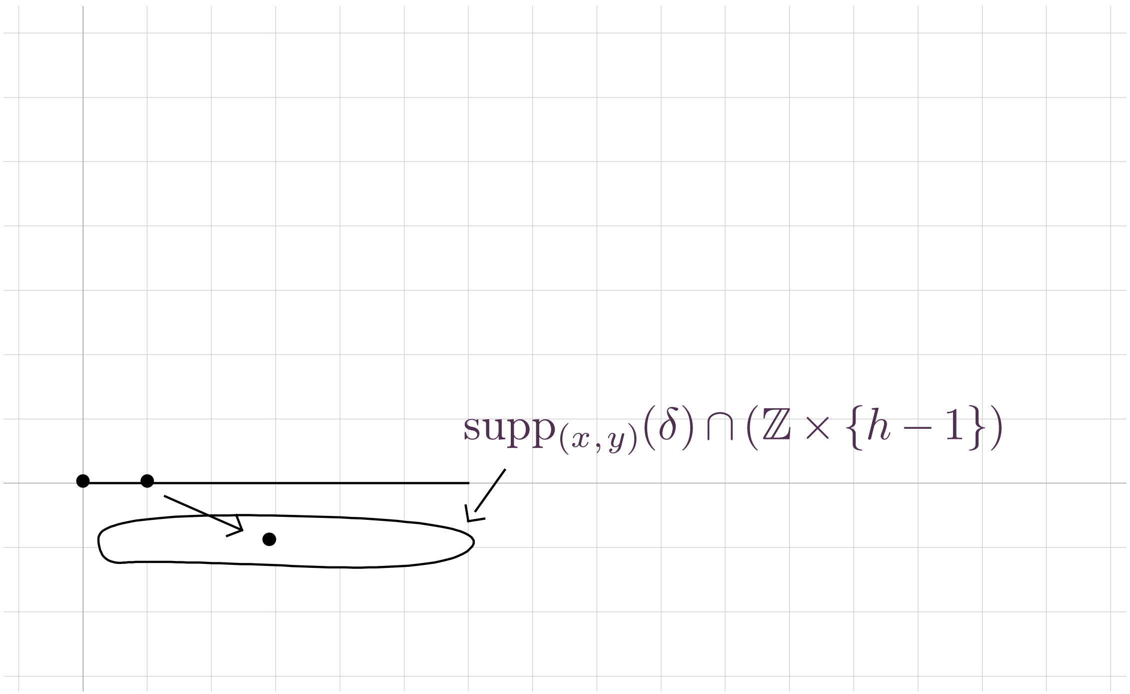

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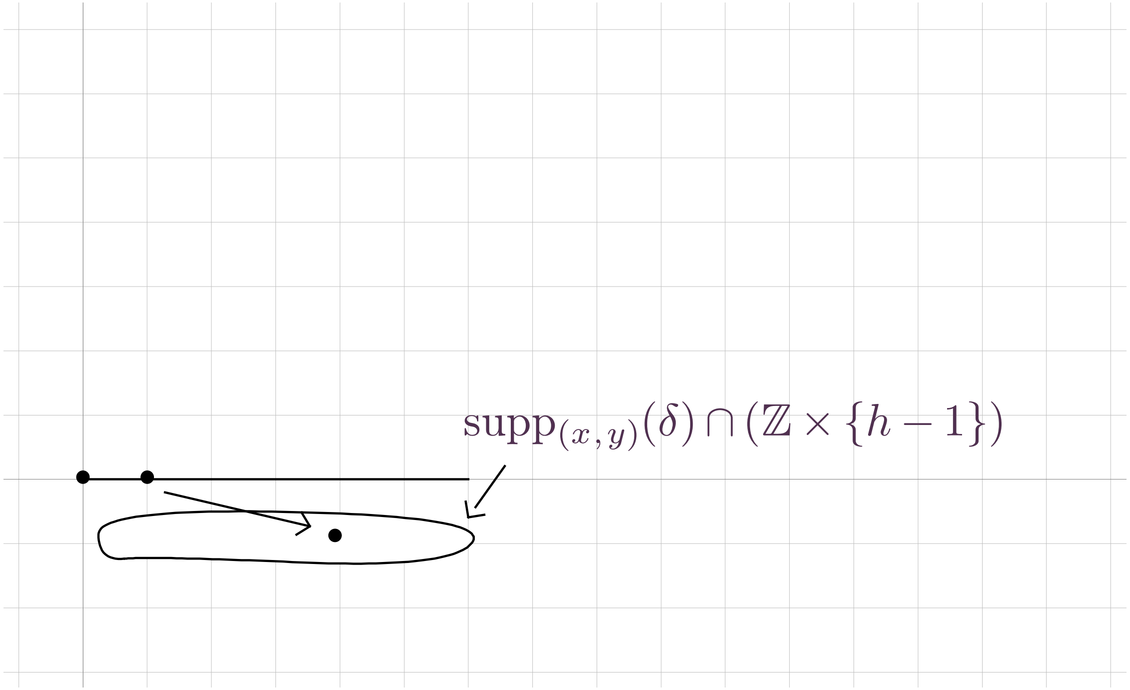

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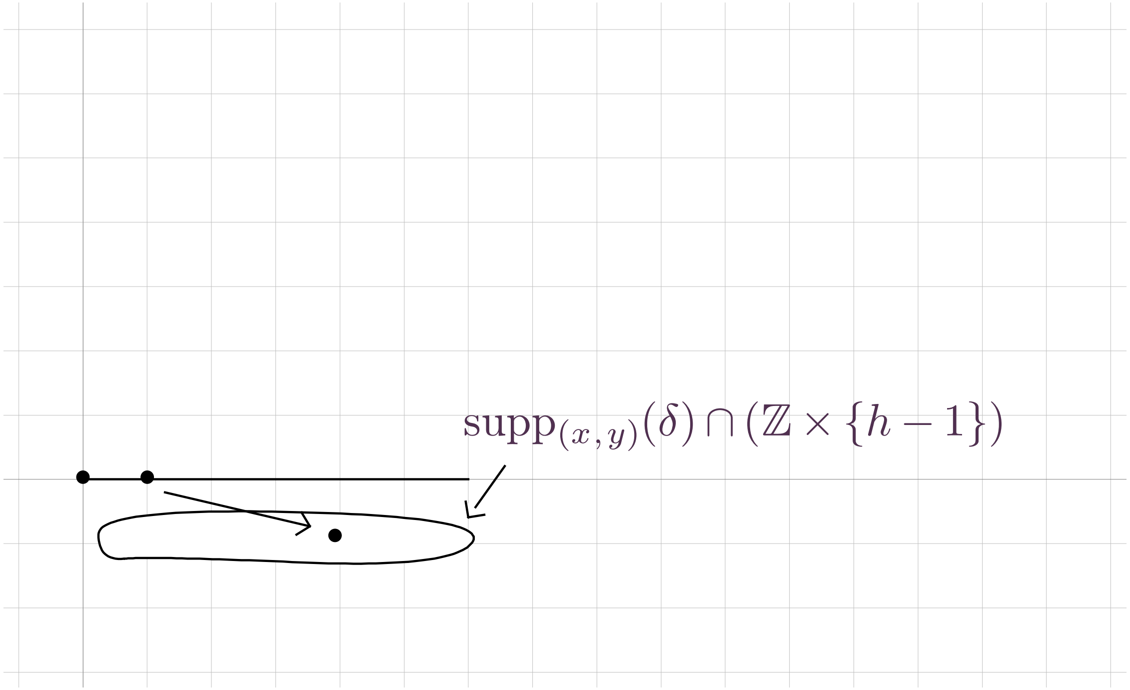

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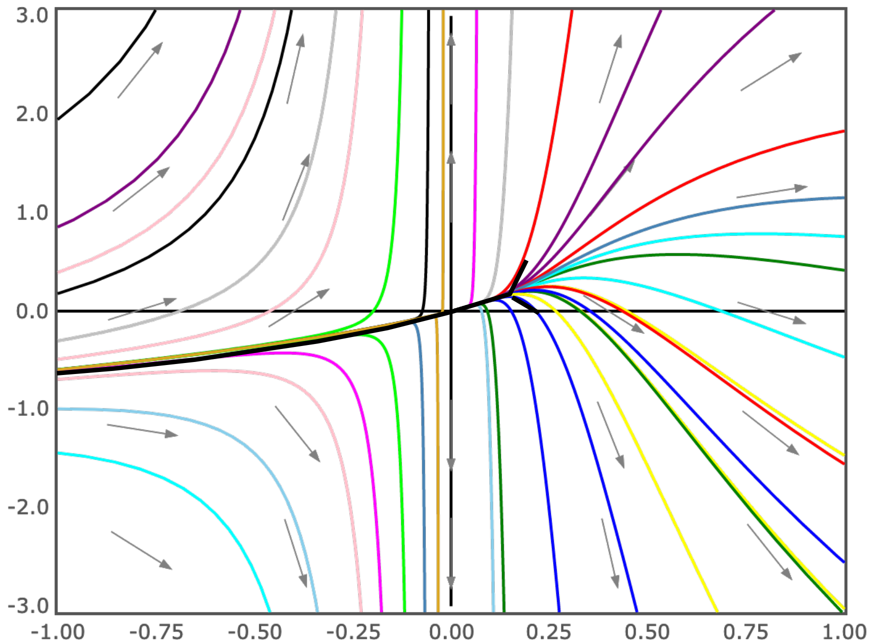

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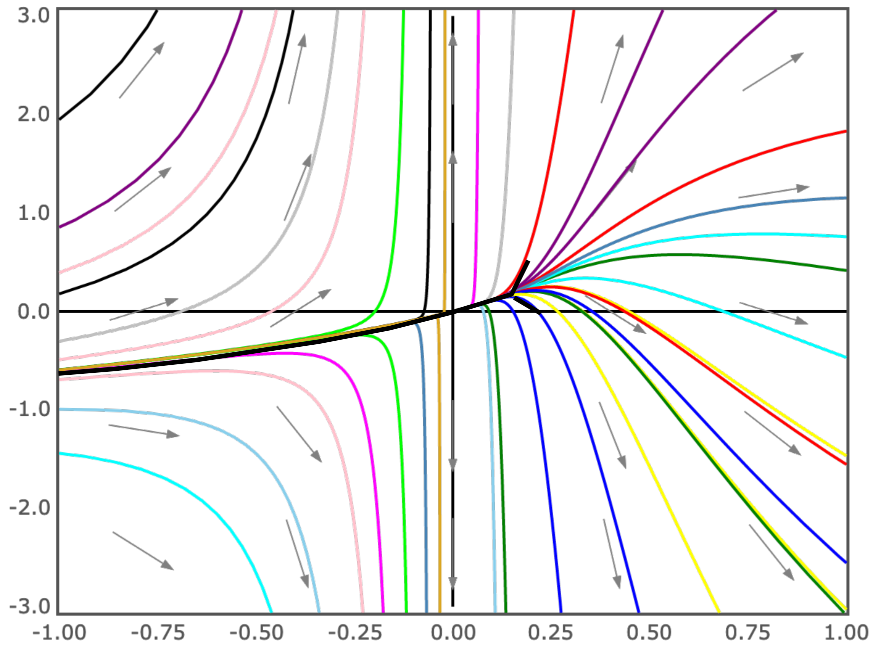

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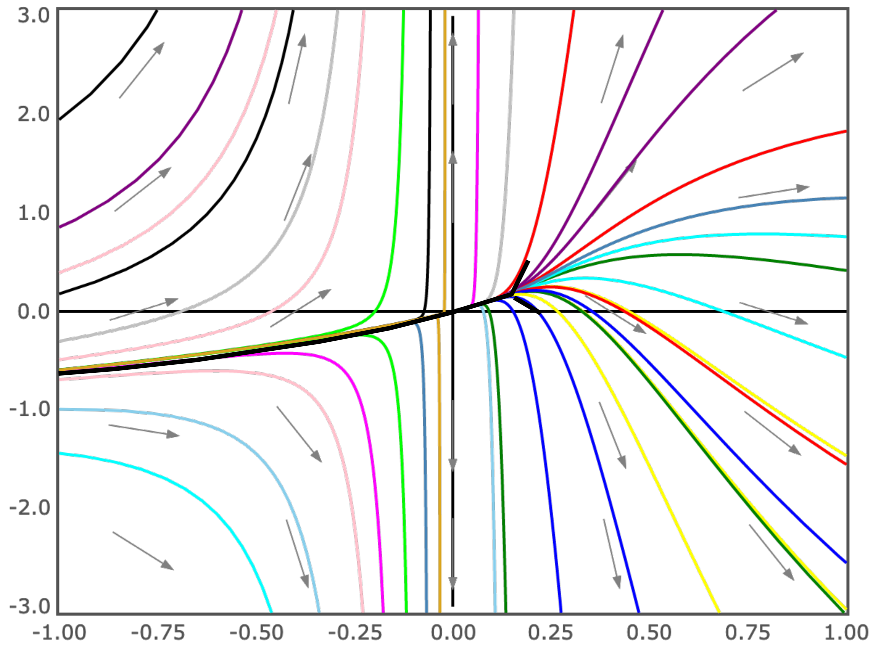

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4) New ideas for dimension greater or equal than four (The Kempf's unstability approach)

Some new phenomena in for final models in dimension three. . .

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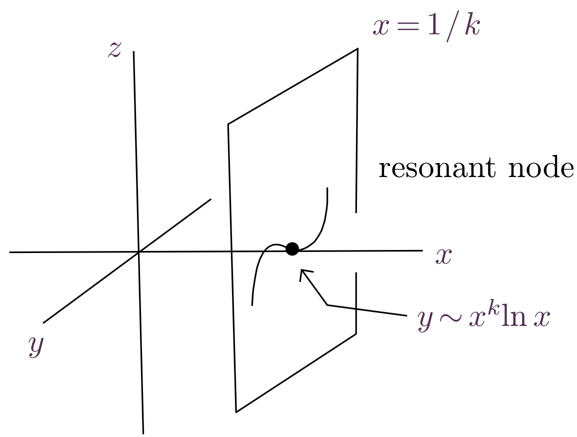

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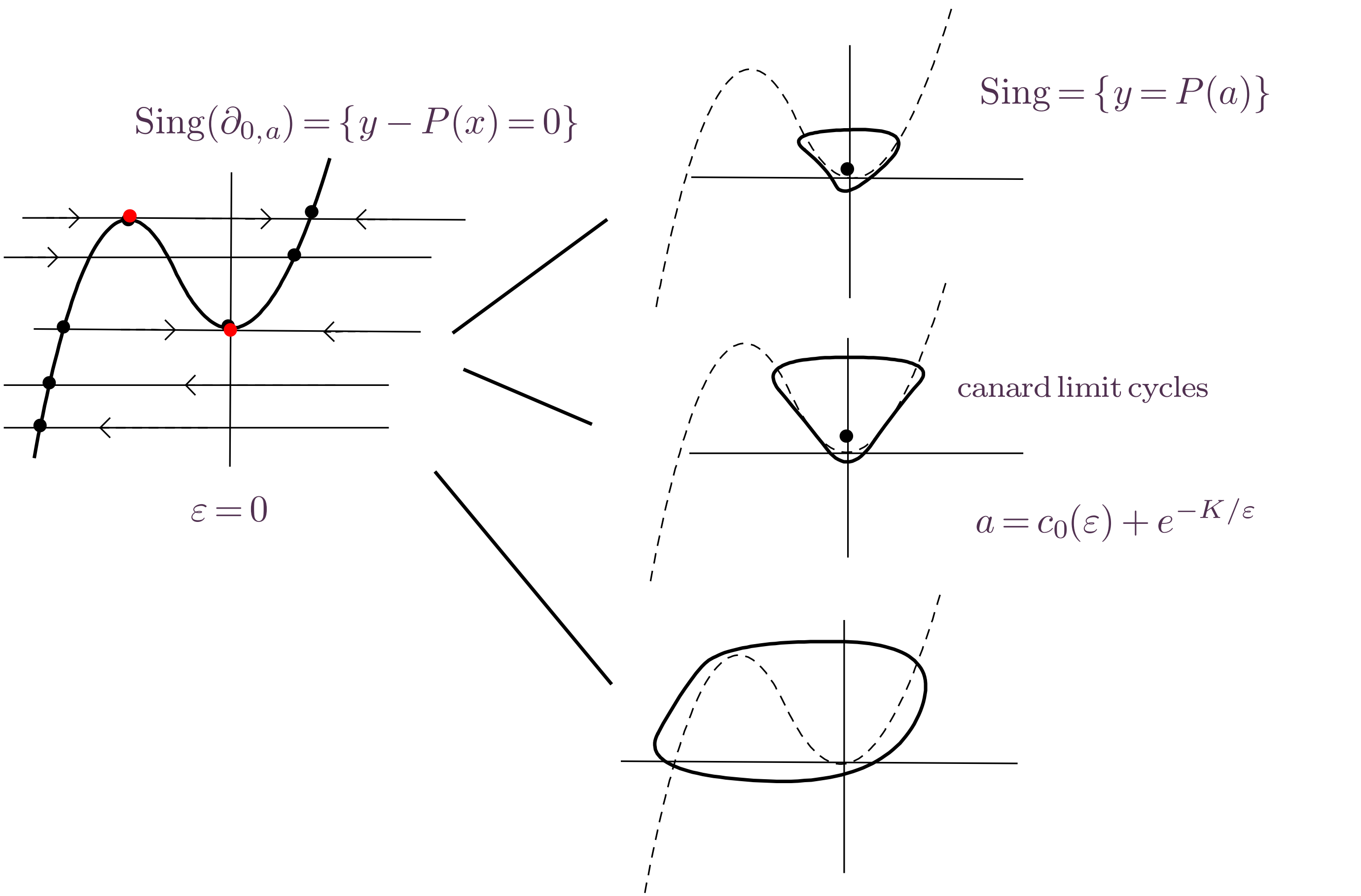

Example of resolution (in families). We assume $a = 0$ to simplify

$$\left(y - \frac{x^2}{2} - \frac{x^3}{3}\right)\frac{\partial}{\partial x} - \varepsilon x \frac{\partial}{\partial y} = \left(x^{-1}y - \frac{x}{2} - \frac{x^2}{3}\right)\left(x\frac{\partial}{\partial x}\right) - \varepsilon x y^{-1}\left(y\frac{\partial}{\partial y}\right)$$

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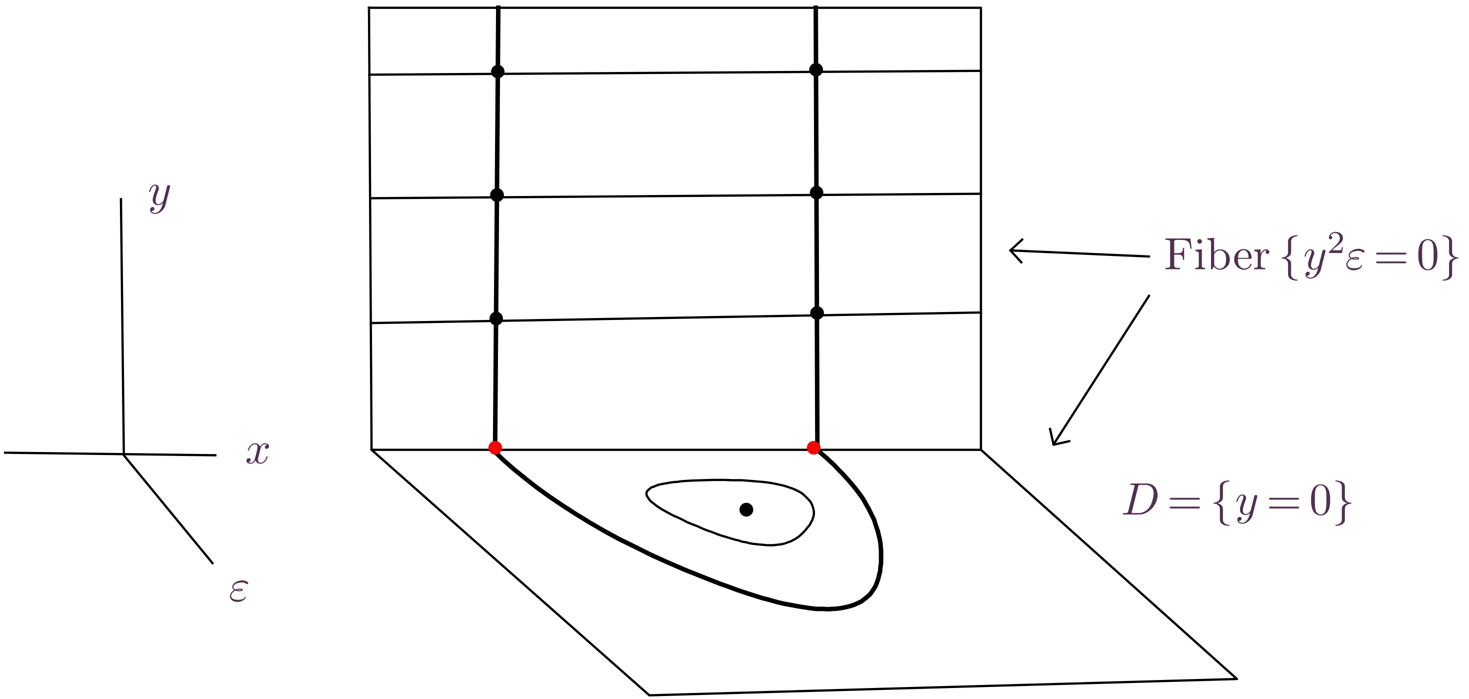

y
x
ε
Fiber $\{y^2\varepsilon = 0\}$
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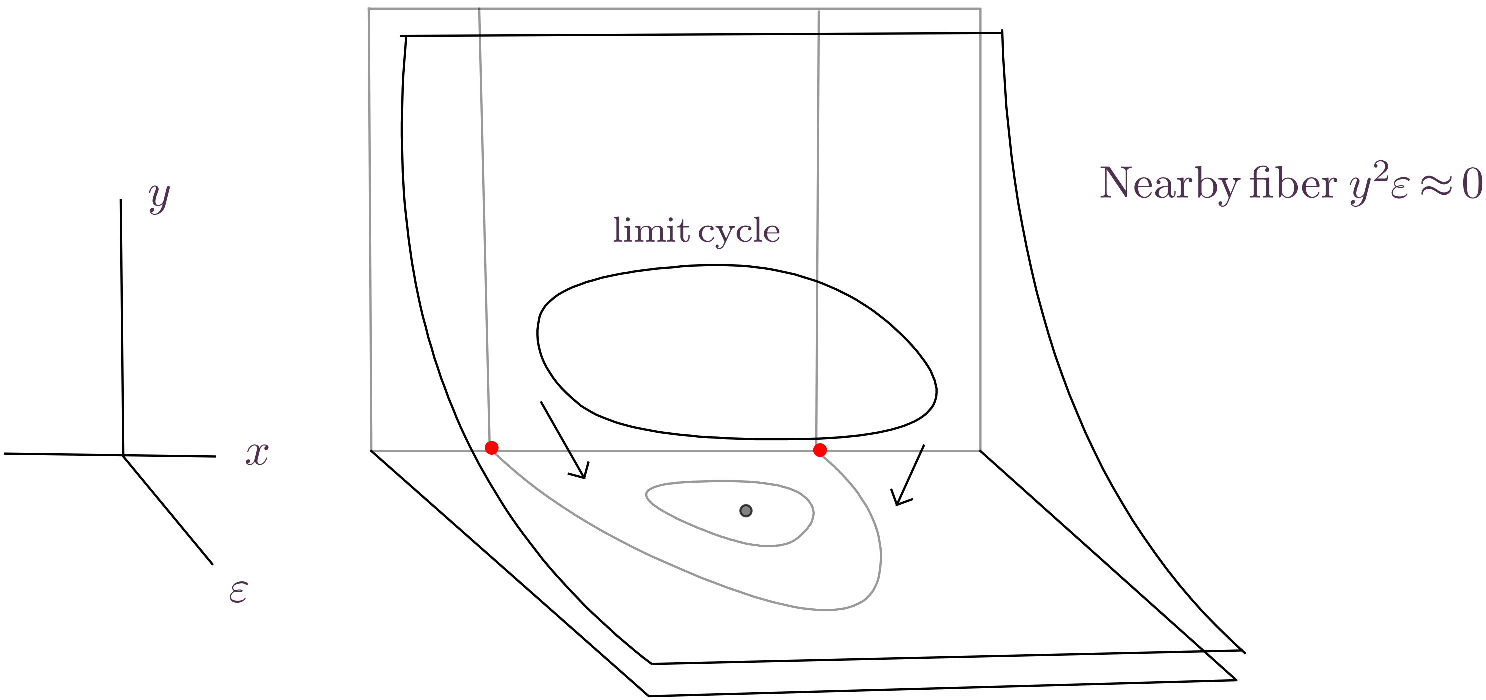
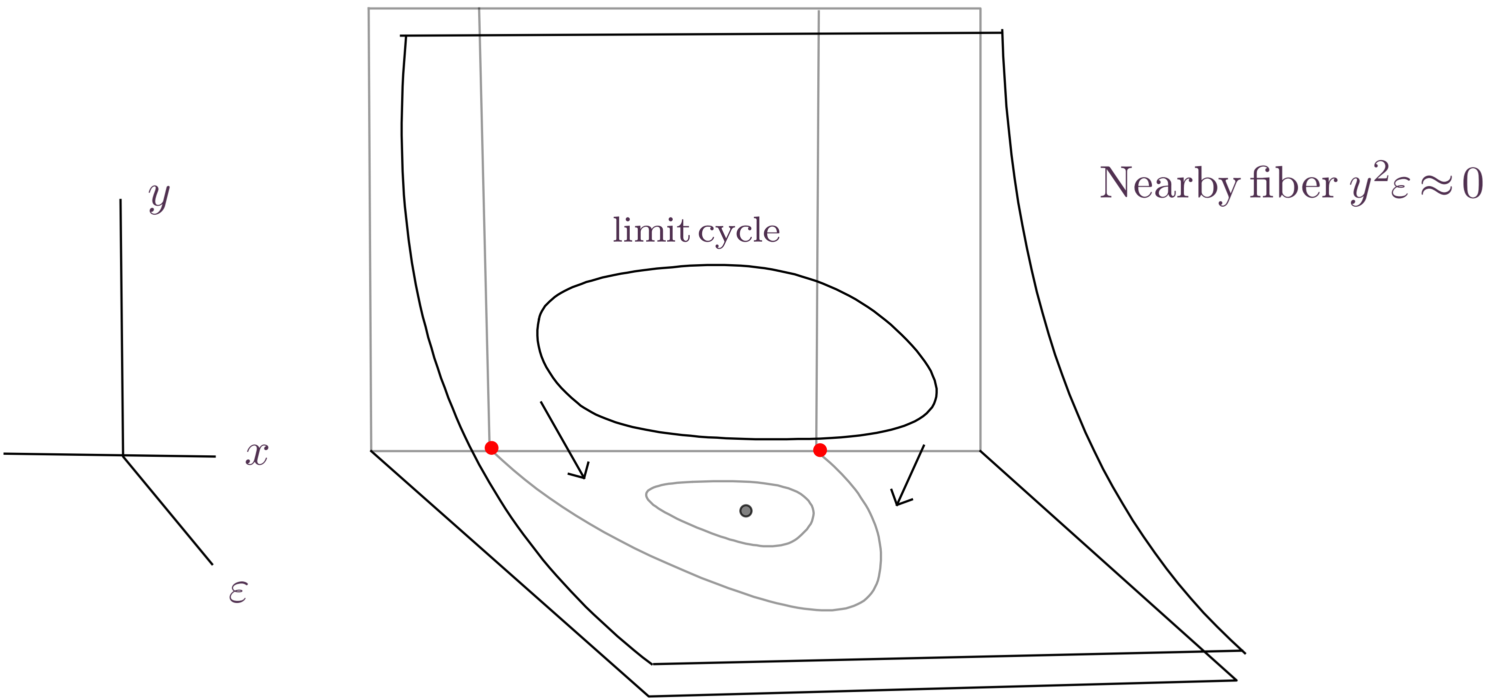


Nearby fiber $y^2\varepsilon \approx 0$
limit cycle
y
x
ε



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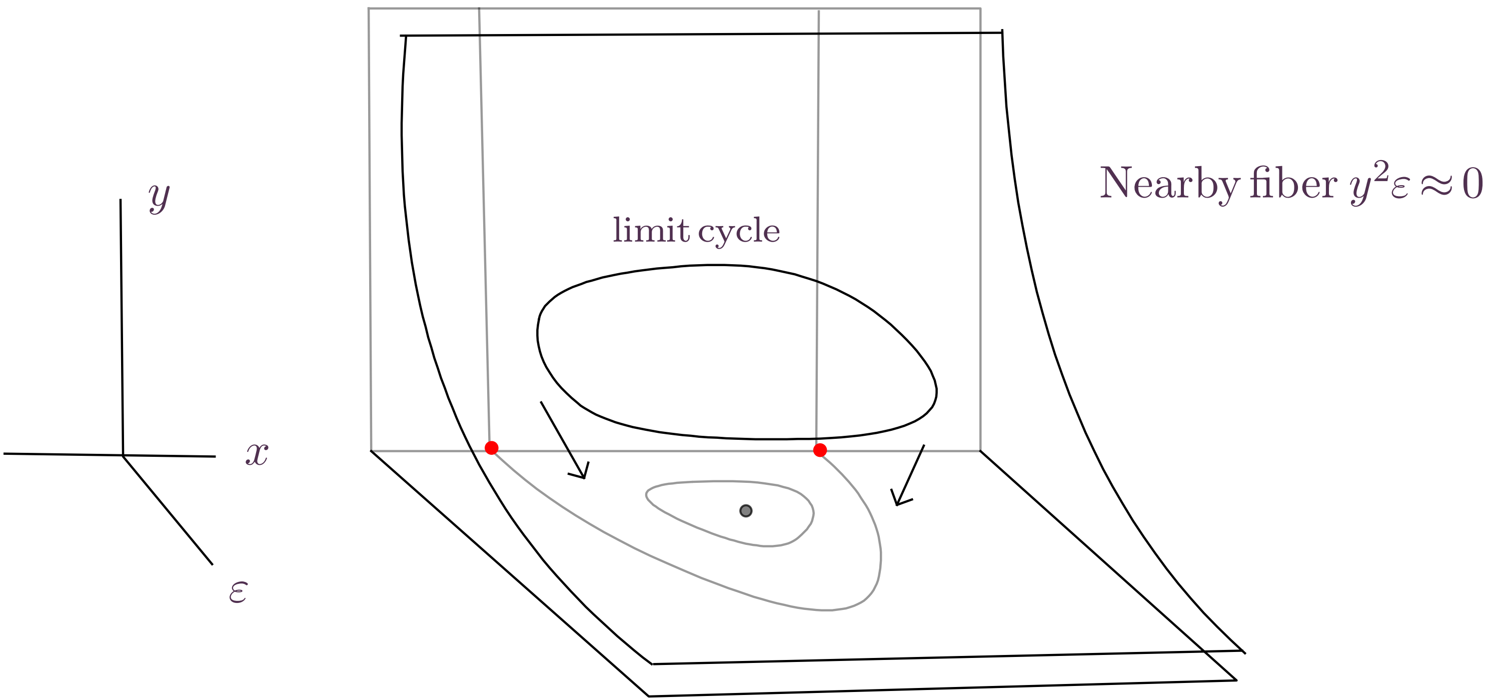

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(c.f. a very nice recent book of Maeschaalk, Dumortier, Roussarie - *Canard cycles:from birth to transition*).