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Integrability Theorem (Sussman): There exists a leaf of $\mathcal{F}$ through each point $p \in M$.

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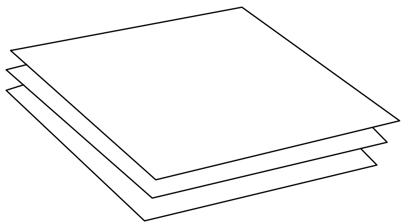

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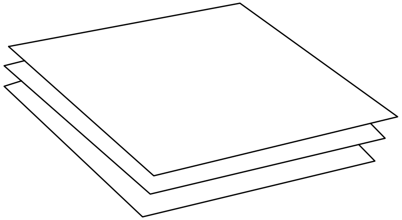

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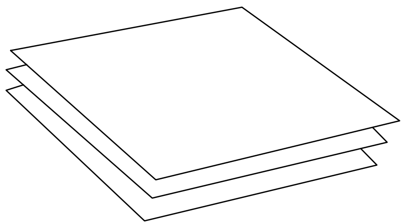

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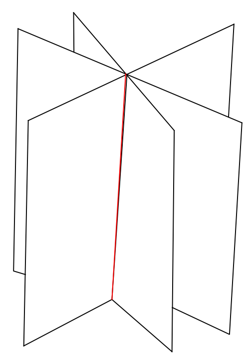

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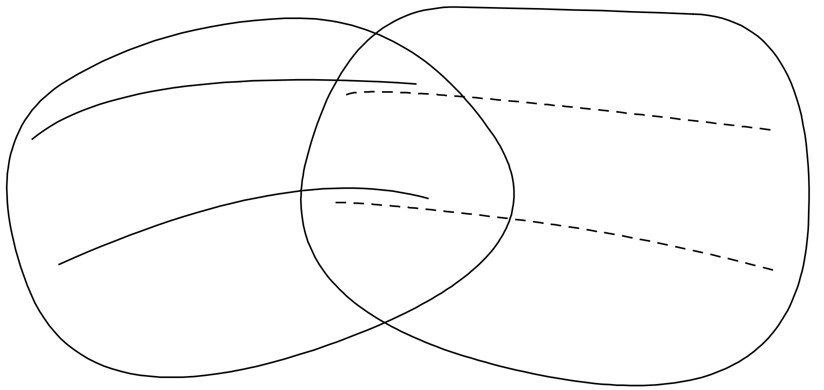
Remark: In general, we cannot expect to have a single global generator for a foliation.



We authorize reparametrizations of time
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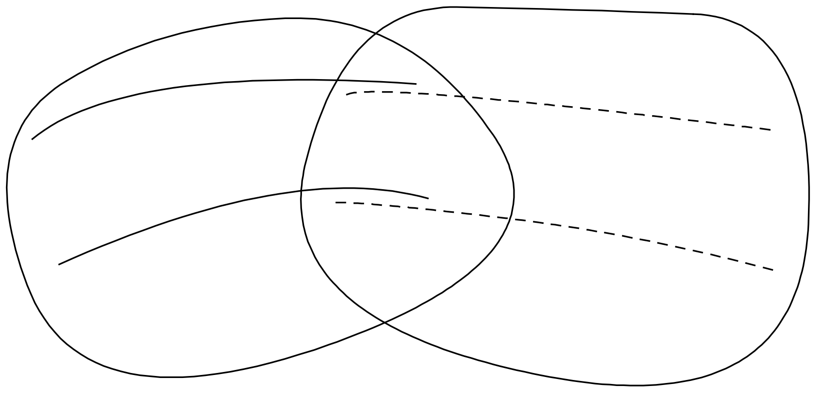

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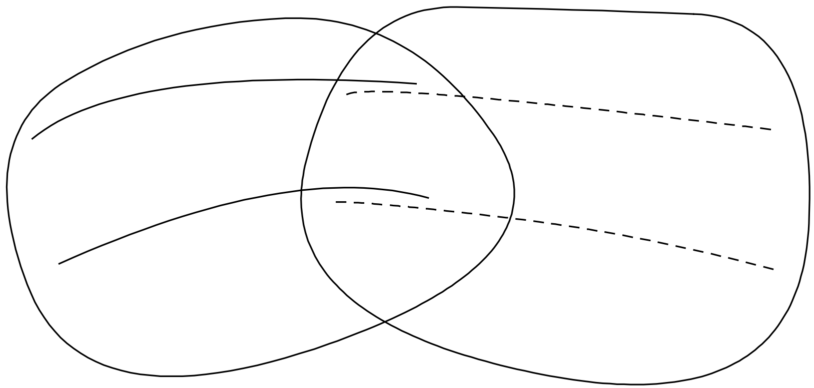

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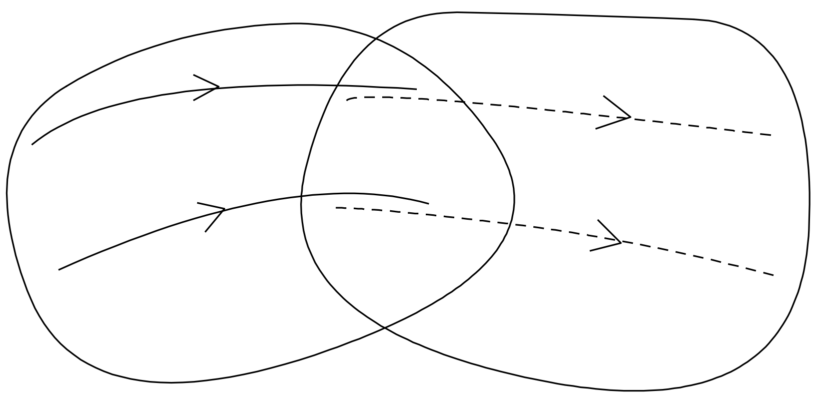

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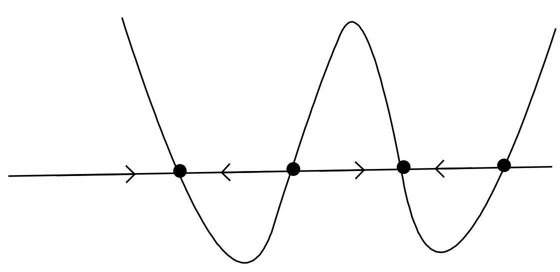

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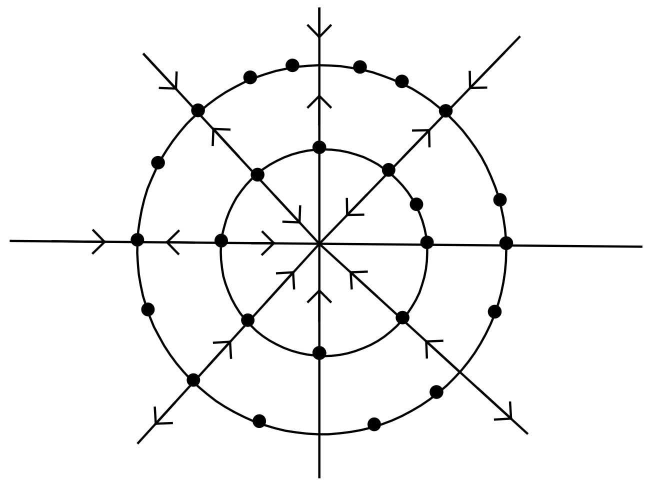

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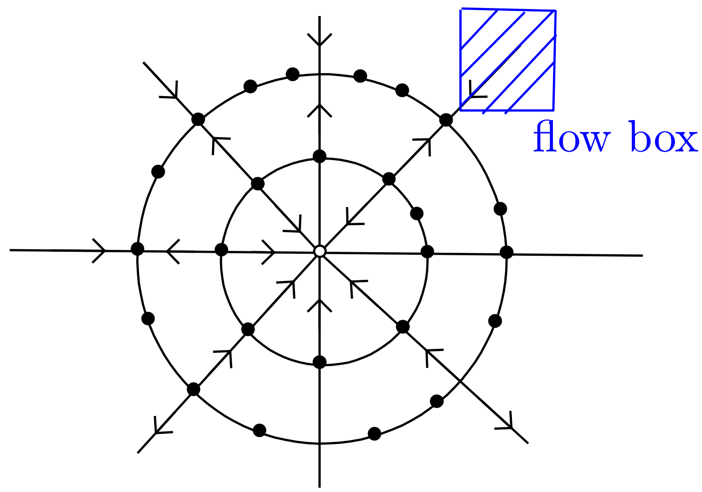

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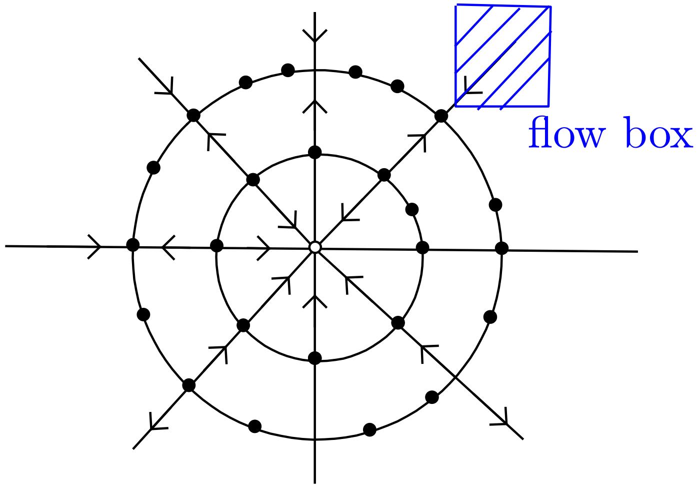

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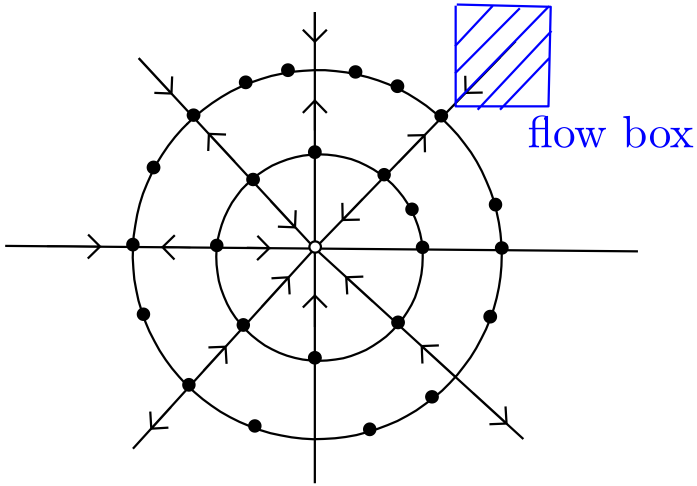

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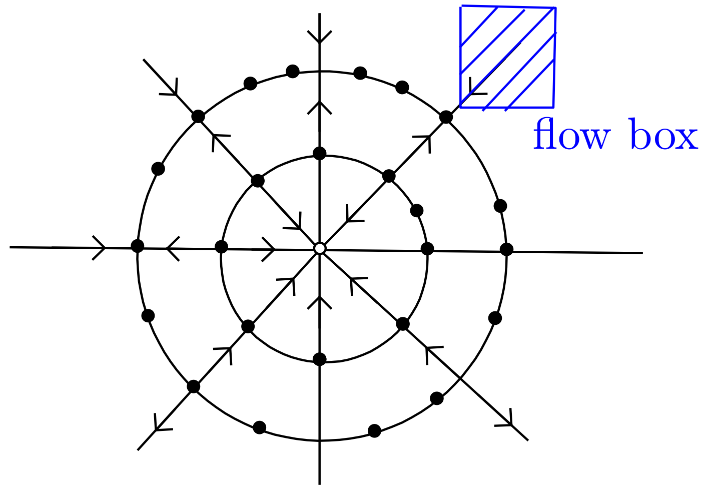

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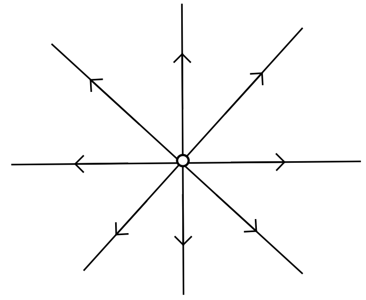

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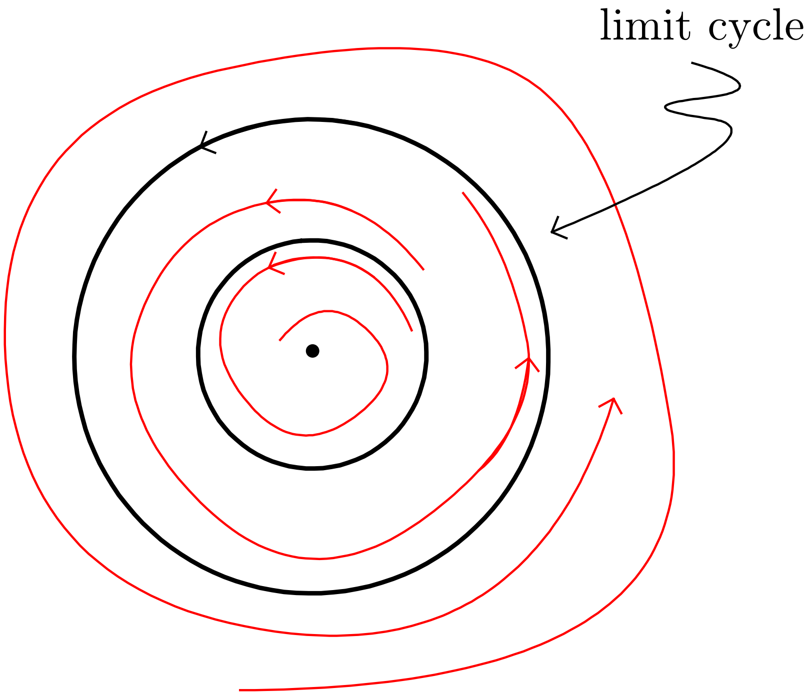

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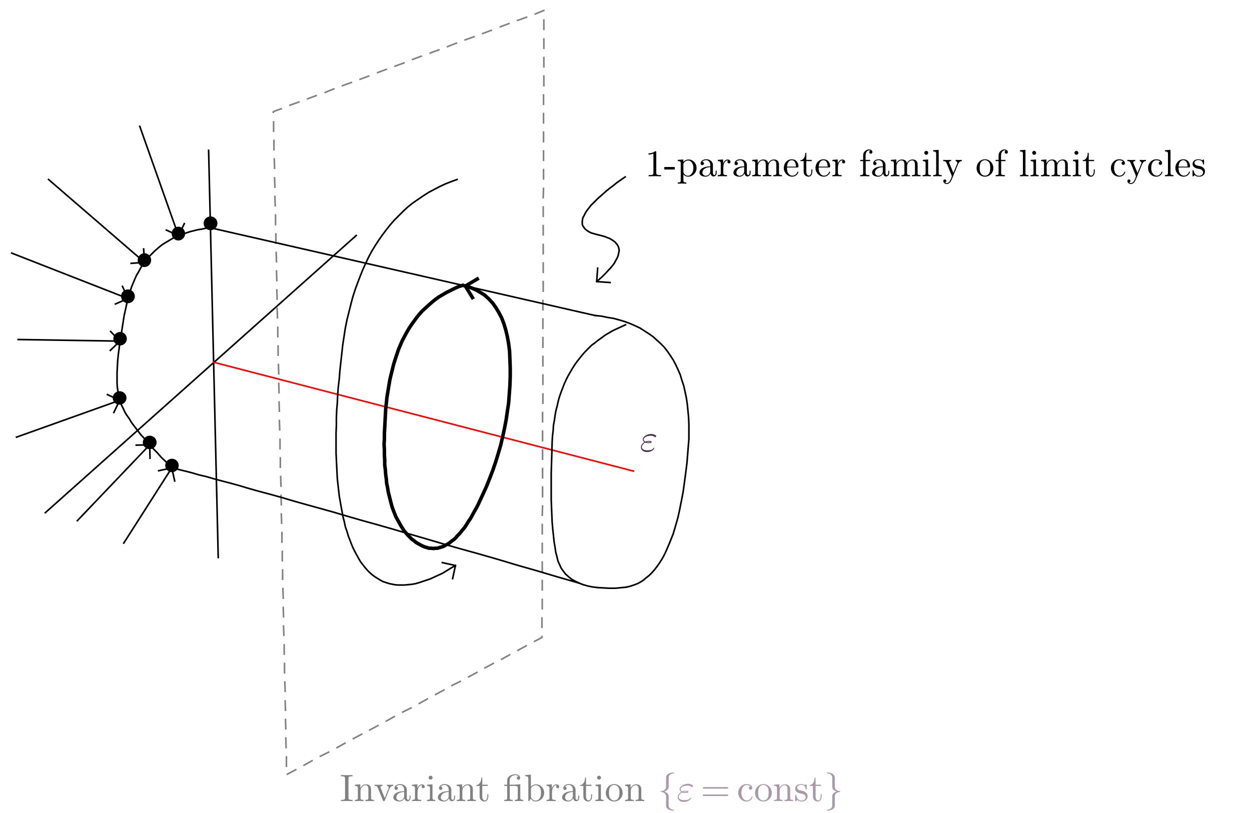

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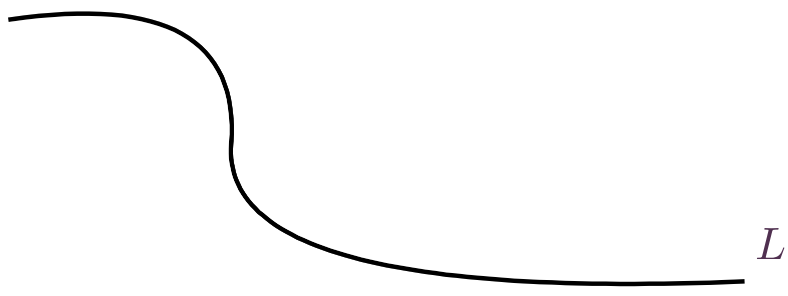
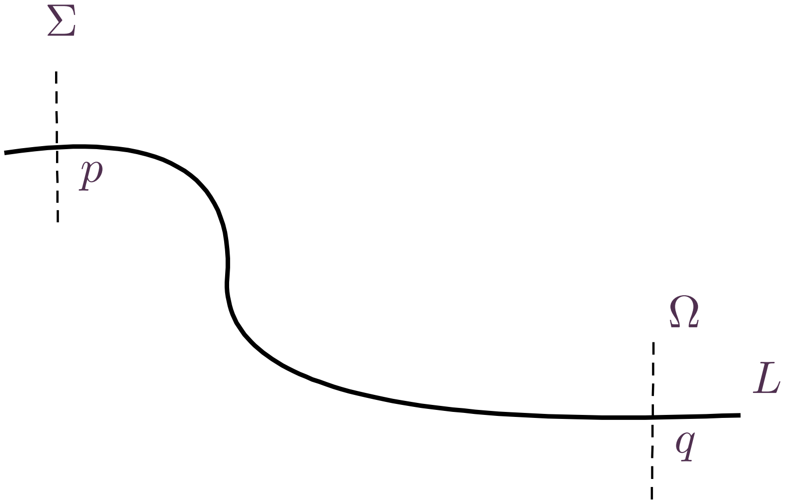
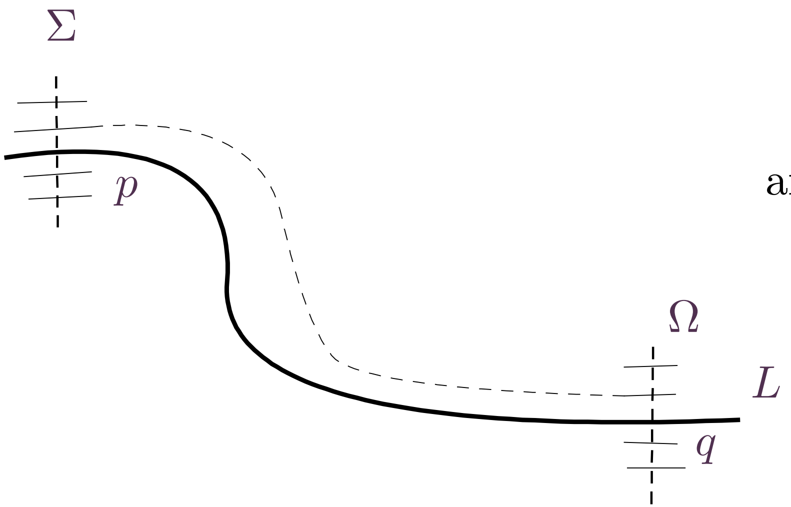
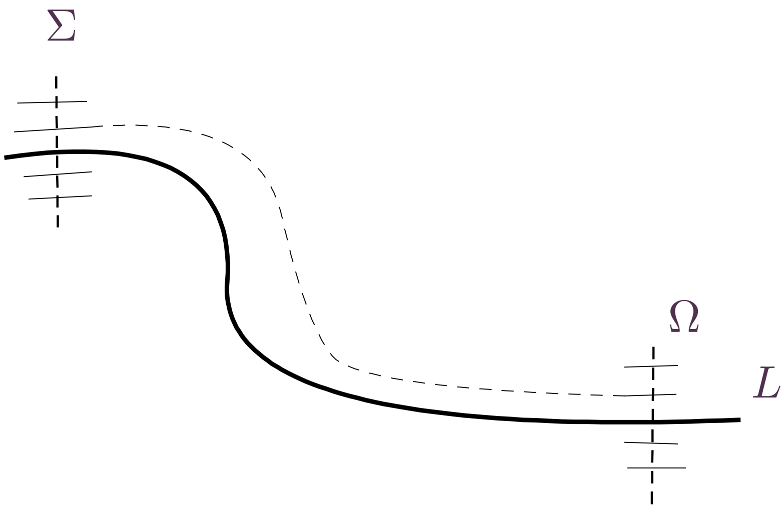

L



Σ
p
Ω
L
q



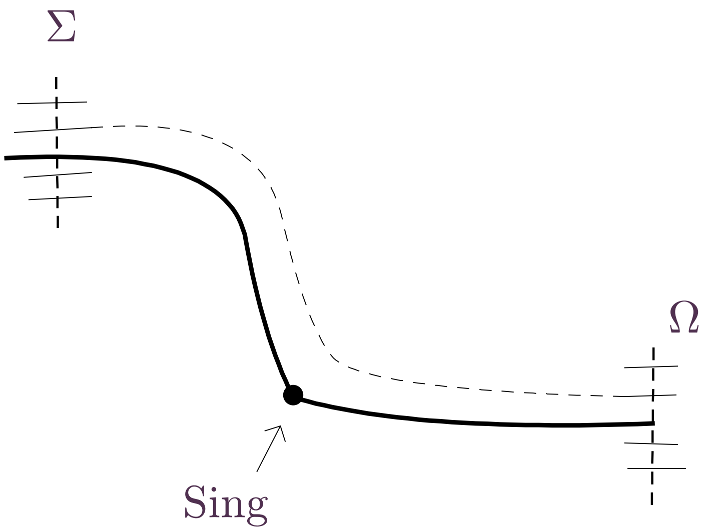
any path $p \longrightarrow q$ on L can be lifted to nearby leafs



$$\mathrm{hol}\colon (\Sigma, p) \to (\Omega, q)$$

$$\mathrm{hol} \in \mathrm{Diff}^{\omega}(\Sigma \to \Omega)$$

Adding a singularity on the path...



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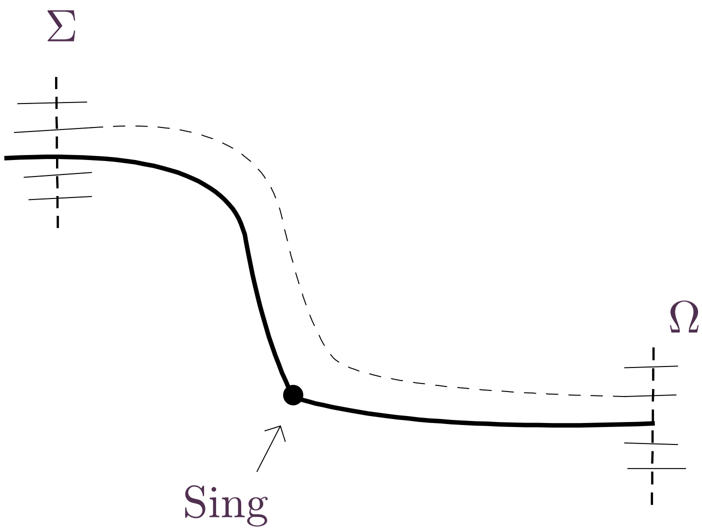

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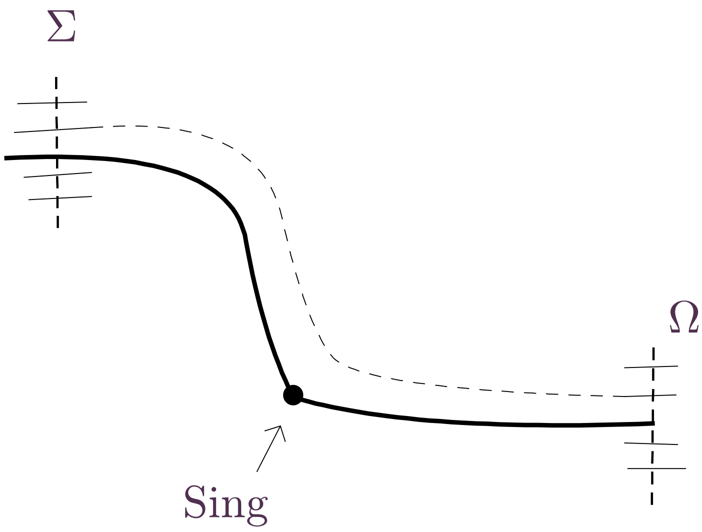

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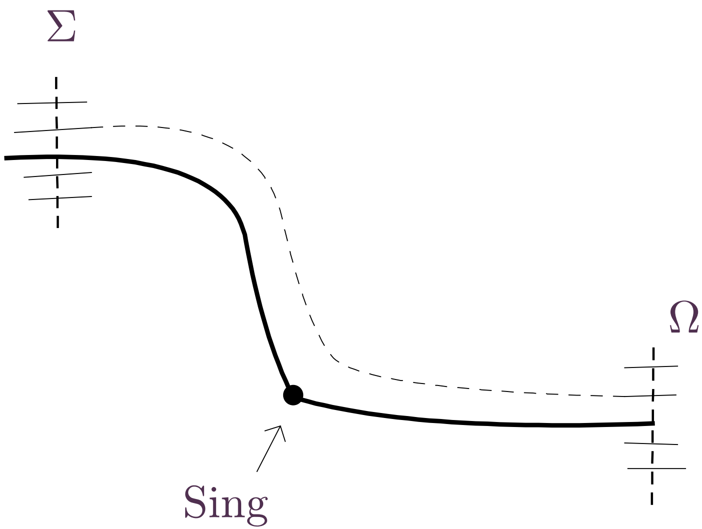

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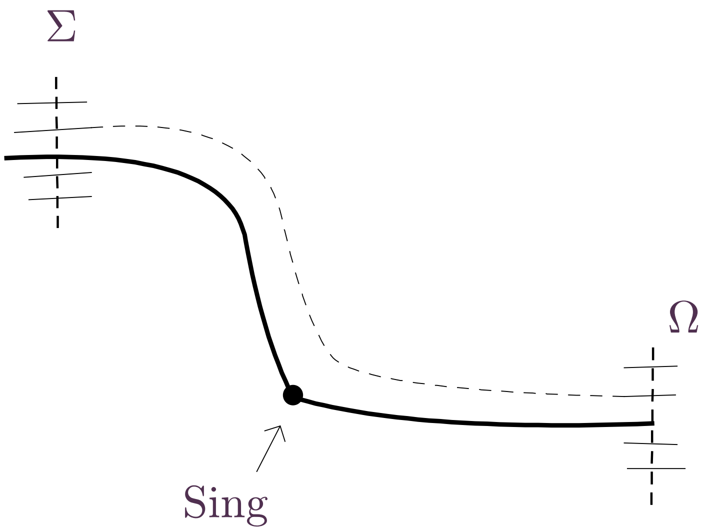

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Moreover, ∂_s and ∂_n are derivations of $\hat{\mathcal{O}} = \varprojlim J^k$ (see Jean Martinet - Exposé Bourbaki'81).

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Poincaré-Dulac normalisation: (over $\mathbb{C}$) Suppose that $\partial(\mathfrak{m}) \subset \mathfrak{m}$. Then, there exists formal coordinates $(x_1, \ldots, x_n)$ which diagonalize the semi-simple part of ∂, namely such that

$$\partial_s = \sum_i \lambda_i x_i \frac{\partial}{\partial x_i}$$

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Definition. A germ of vector field ∂ is ***elementary*** if:

- either $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ (i.e. in appropriate local coordinates $\partial = \frac{\partial}{\partial x}$)
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where k ranges over the subset $\mathbb{Z}^n \setminus \{0\}$ such that $\langle \lambda, k \rangle = 0$. These are the **resonant monomials**.

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The Poincaré-Dulac Theorem says that, up to a formal change of coordinates, we can write

$$\partial = \underbrace{\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)}_{\partial_s} + \underbrace{\sum_{k \geqslant 1} (x\,y)^k \left(a_k\,x\frac{\partial}{\partial x} + b_k\,y\frac{\partial}{\partial y}\right)}_{\partial_n}$$

where $u = x\,y$ is the generator of the subring $\ker(\partial_s)$. By further reductions, we can write

$$(1+F)\left(\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) + \frac{u^n}{1+\rho\,u^n}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right) \quad \text{or} \qquad (1+F)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

for some $F \in \mathbb{C}[[u]]$ of order $\geqslant 1$, $n \geqslant 1$ and $\rho \in \mathbb{C}$.

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is a first integral of the vector field (namely, $\partial I = 0$). It is an element of $\mathbb{R}_{\mathrm{an,exp}}$.

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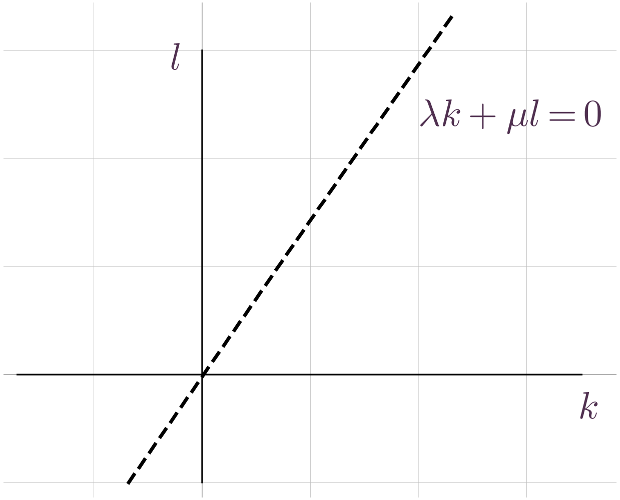

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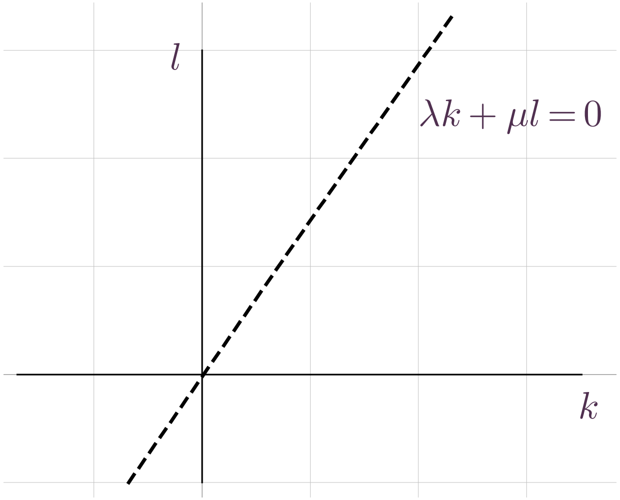

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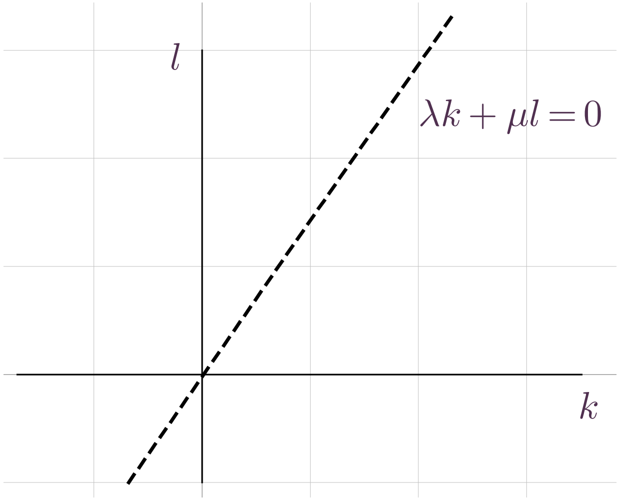

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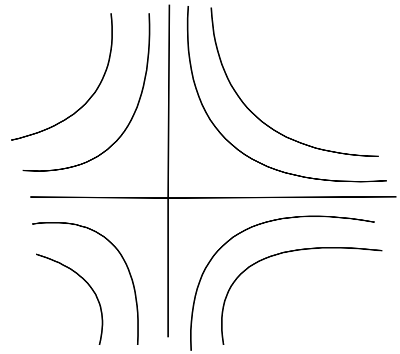

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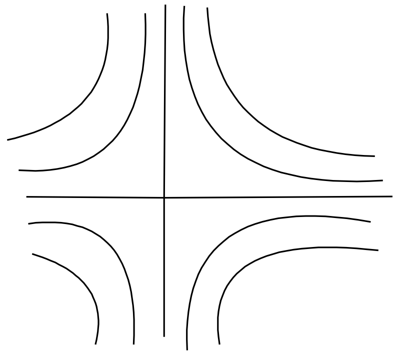

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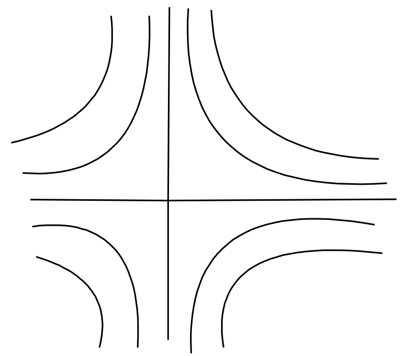

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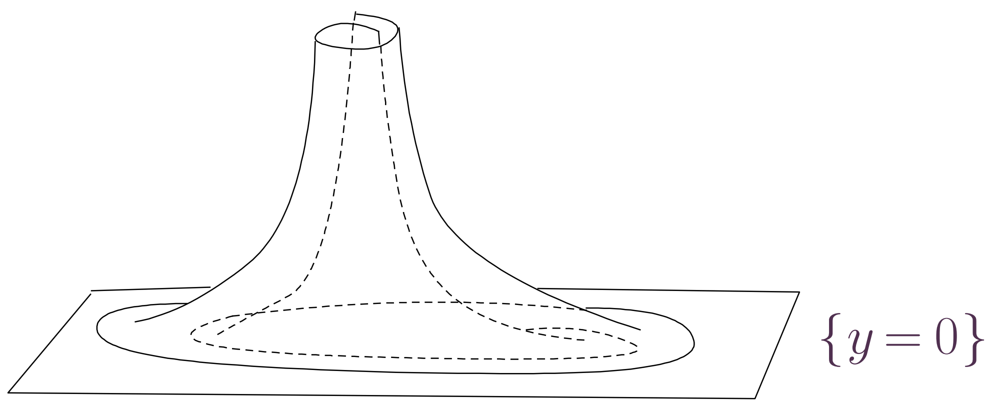

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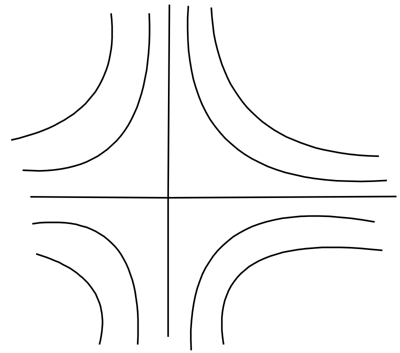

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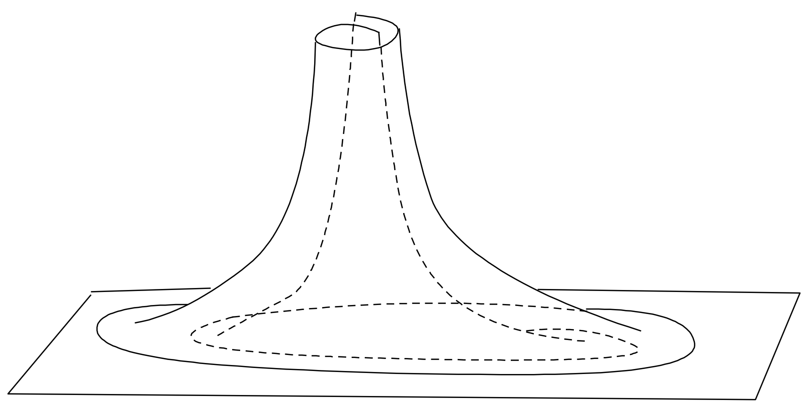

$\{y=0\}$

Over $\mathbb{C}^2$: There are several **rigidity phenomena**

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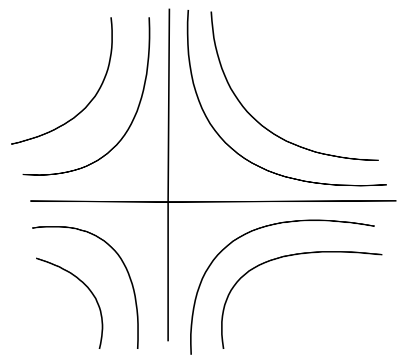

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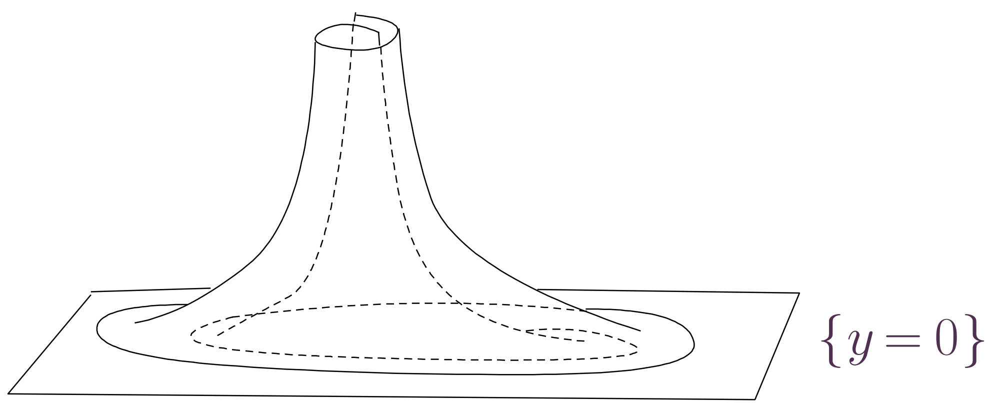

Over $\mathbb{C}^2$: There are several **rigidity phenomena**

E.g. Some analytic invariants are topologically determined (for instance, linearizability).

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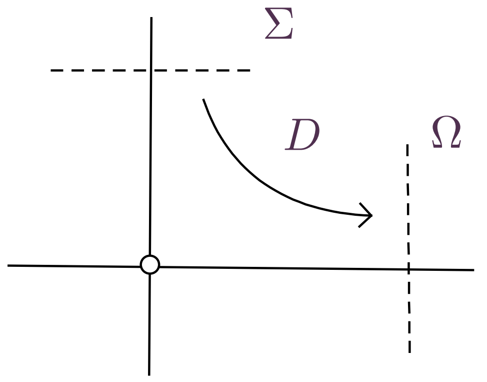

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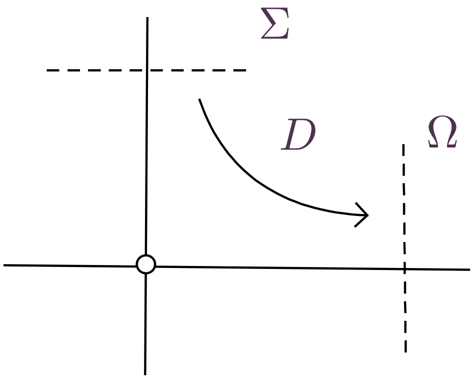

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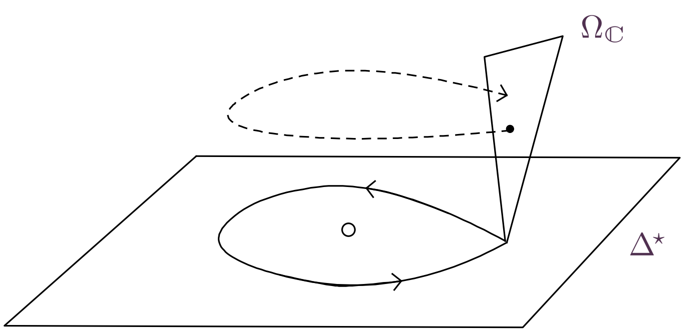

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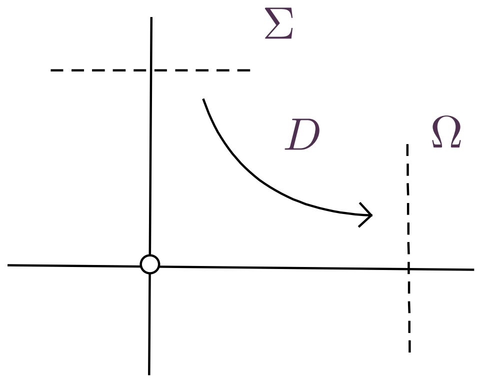

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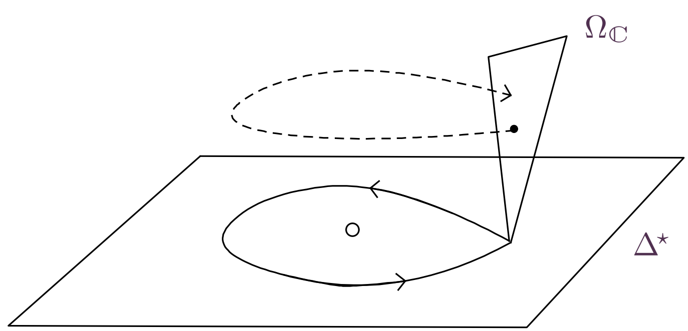

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We can recover the (orbital) analytic class of the saddle from the analytic class of one of these maps (once we fix the ratio μ/λ)

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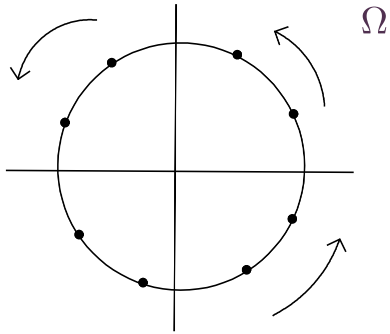
Definition: Two germs of vector fields $\partial, \tilde{\partial}$ are **orbitally analytic equivalent** if there exists a unit $u \in \mathbb{C}\{x\}$ such that ∂ is analytically conjugated to $u\,\tilde{\partial}$.

Dynamics of the complex holonomy map as an element of $\mathrm{Diff}(\mathbb{C}, 0)$



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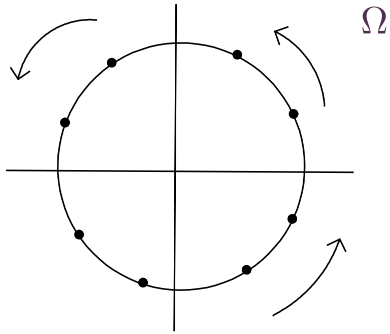

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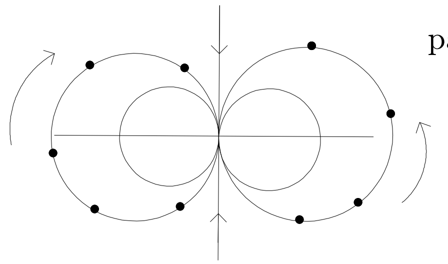

parabolic fixed point

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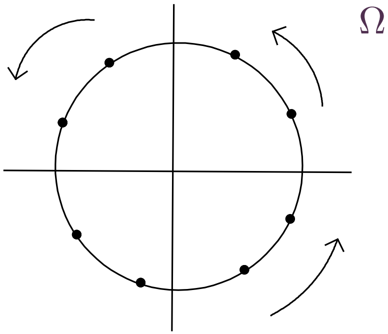

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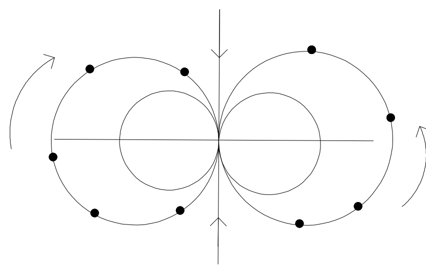

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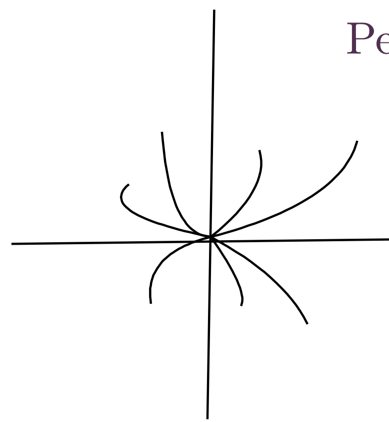

Perez – Marco's
hedgehogs

Classification Problem: “Describe” the orbits of the action of $\mathrm{Aut}(\mathbb{C}\{x\})$ on $\mathrm{Der}(\mathbb{C}\{x\})$ by conjugation

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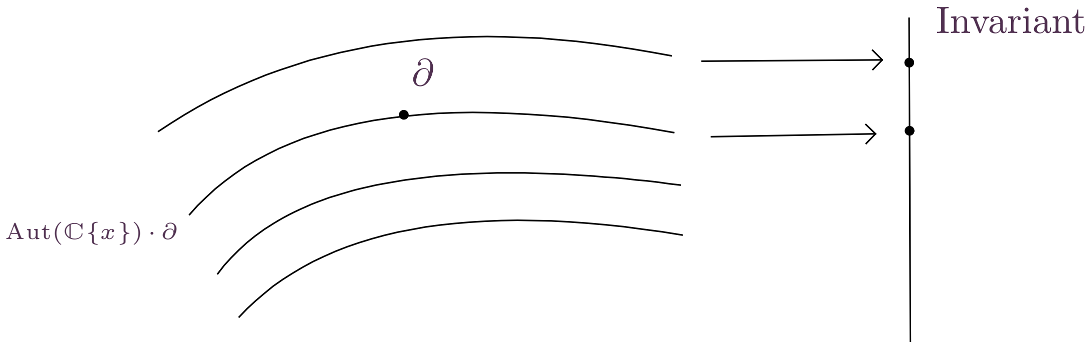

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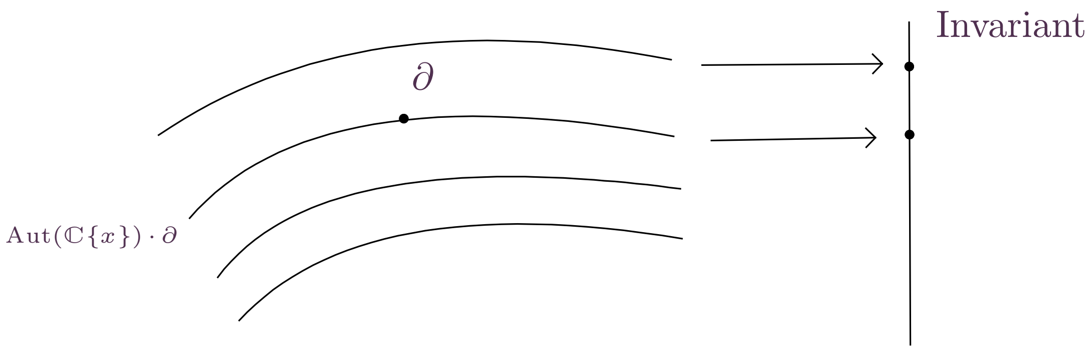

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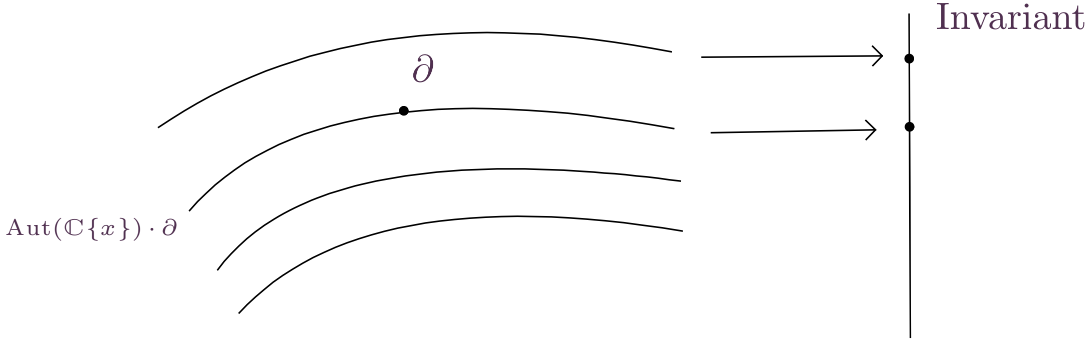

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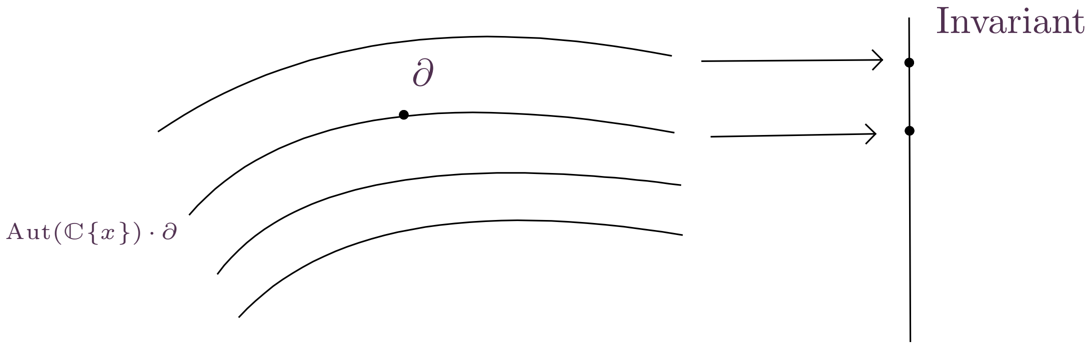

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This problem is much less understood for vector fields higher dimensions.

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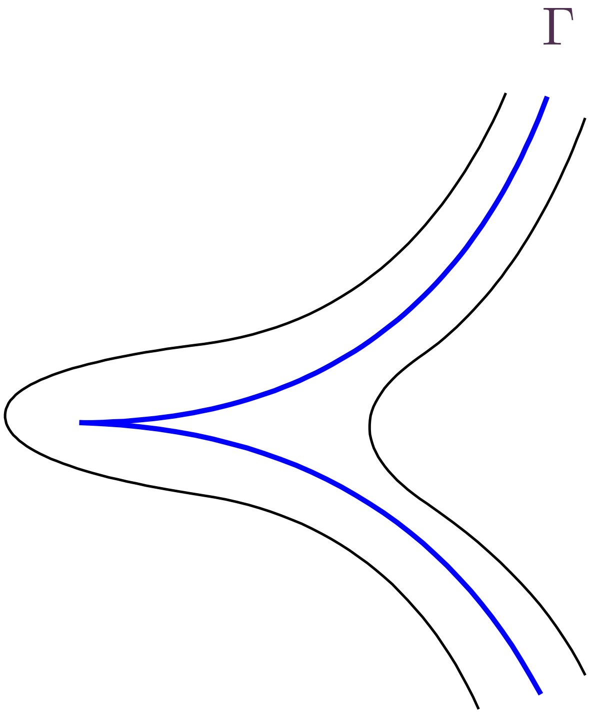
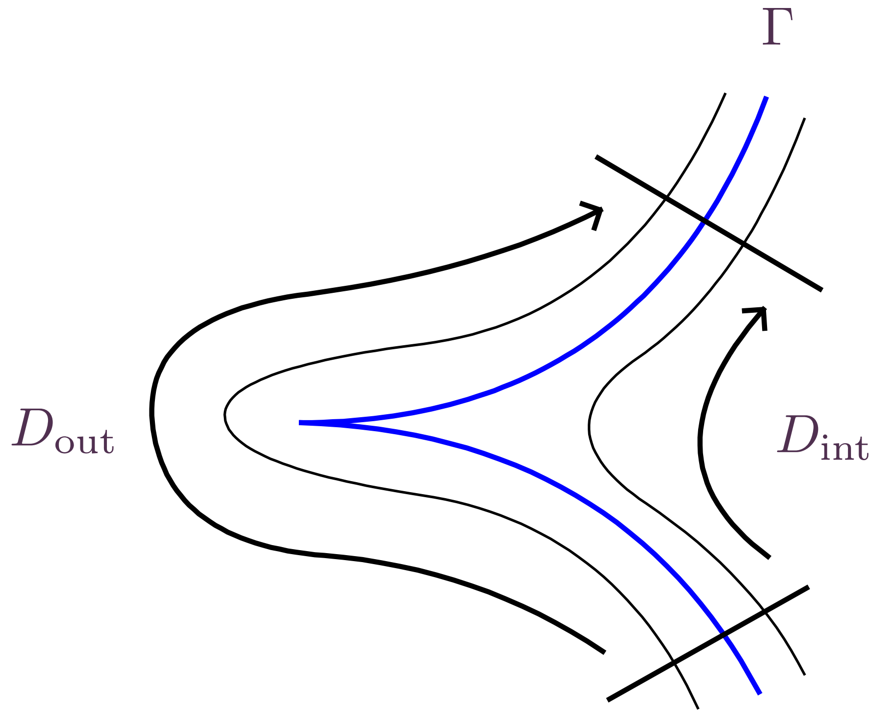
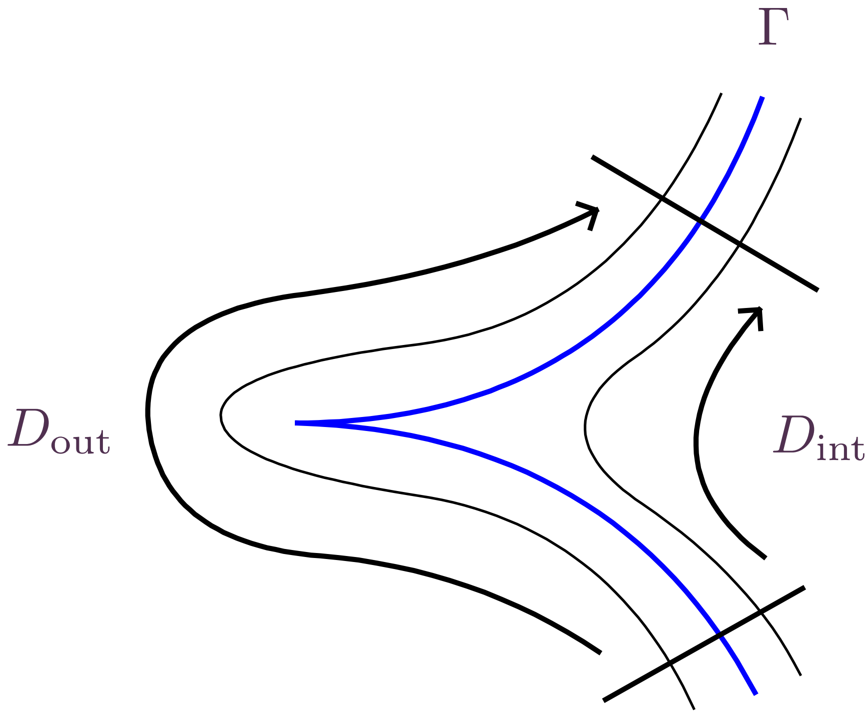
The cusp $\Gamma = \{f = 0\}$ is an invariant curve.

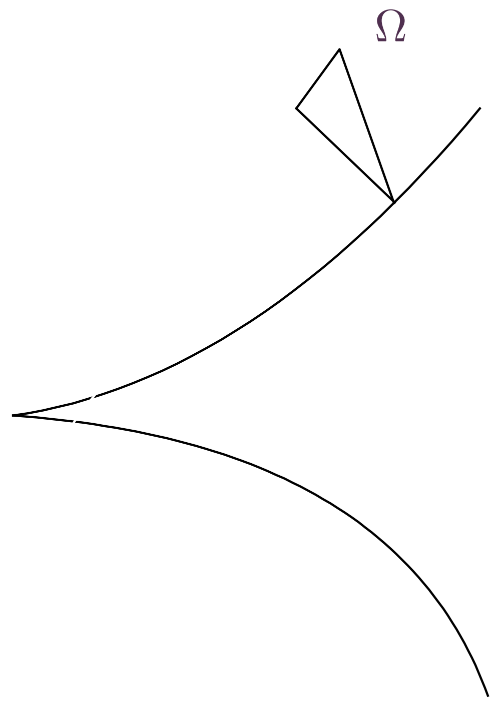

Γ

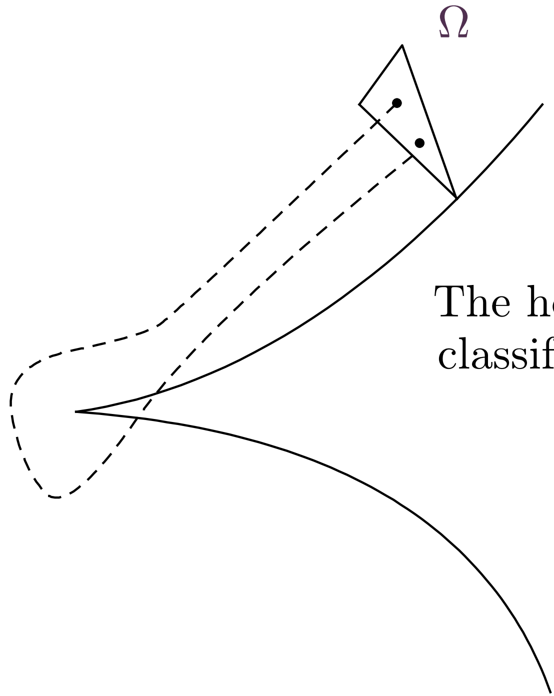


Γ
D_{out}
D_{int}



There are two **distinct** corner transition maps.


Ω



The holonomy map **does not** classify the singularity

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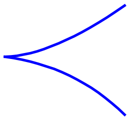

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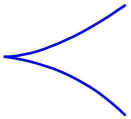

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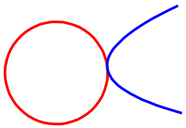

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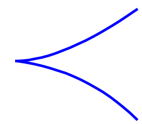

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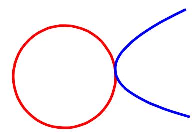

Blow-up 2: $x \longrightarrow x\,y, \qquad y \longrightarrow y$

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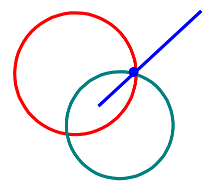

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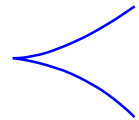

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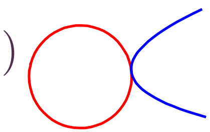

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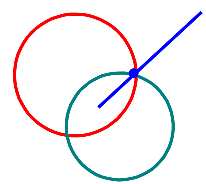
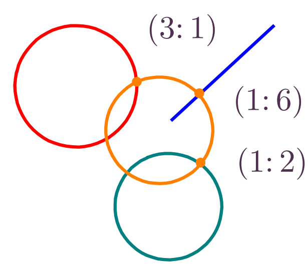


Blow-up 3: $x \to x\,, \qquad y \to x\,y$

$$d(x^6 y^3 (y - 1))$$



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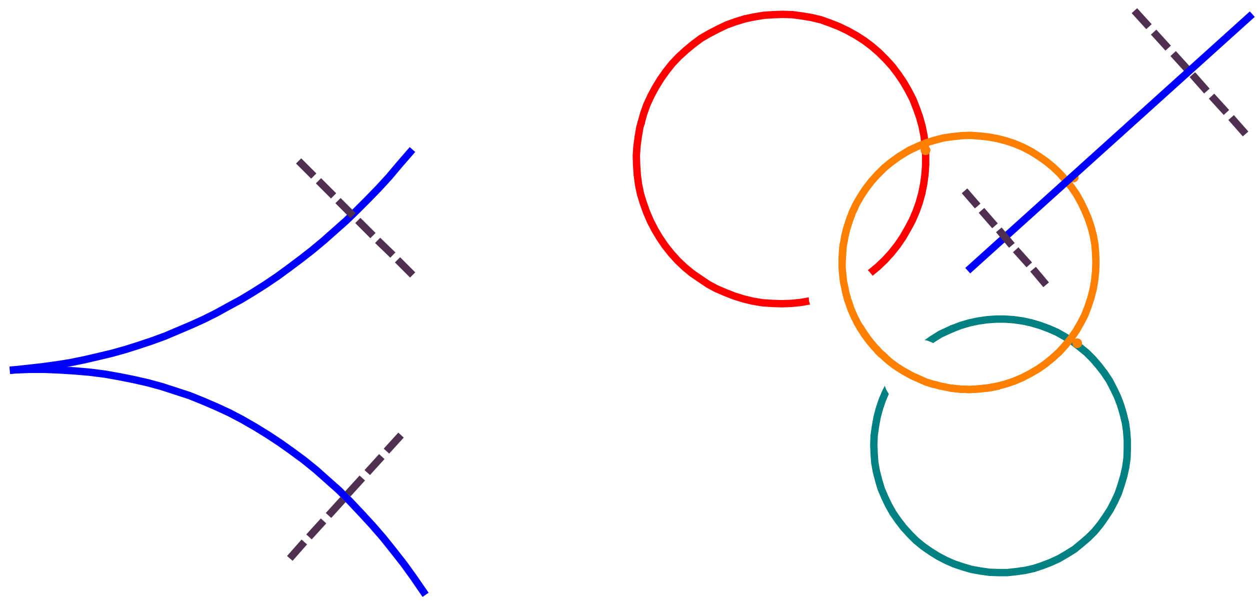

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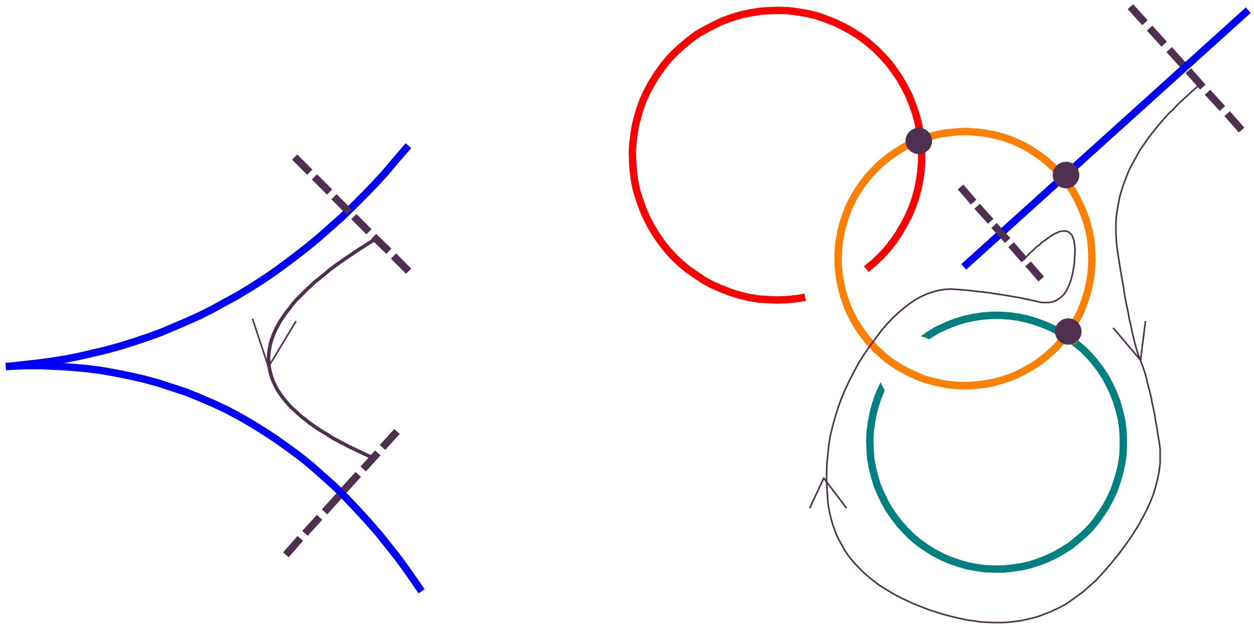
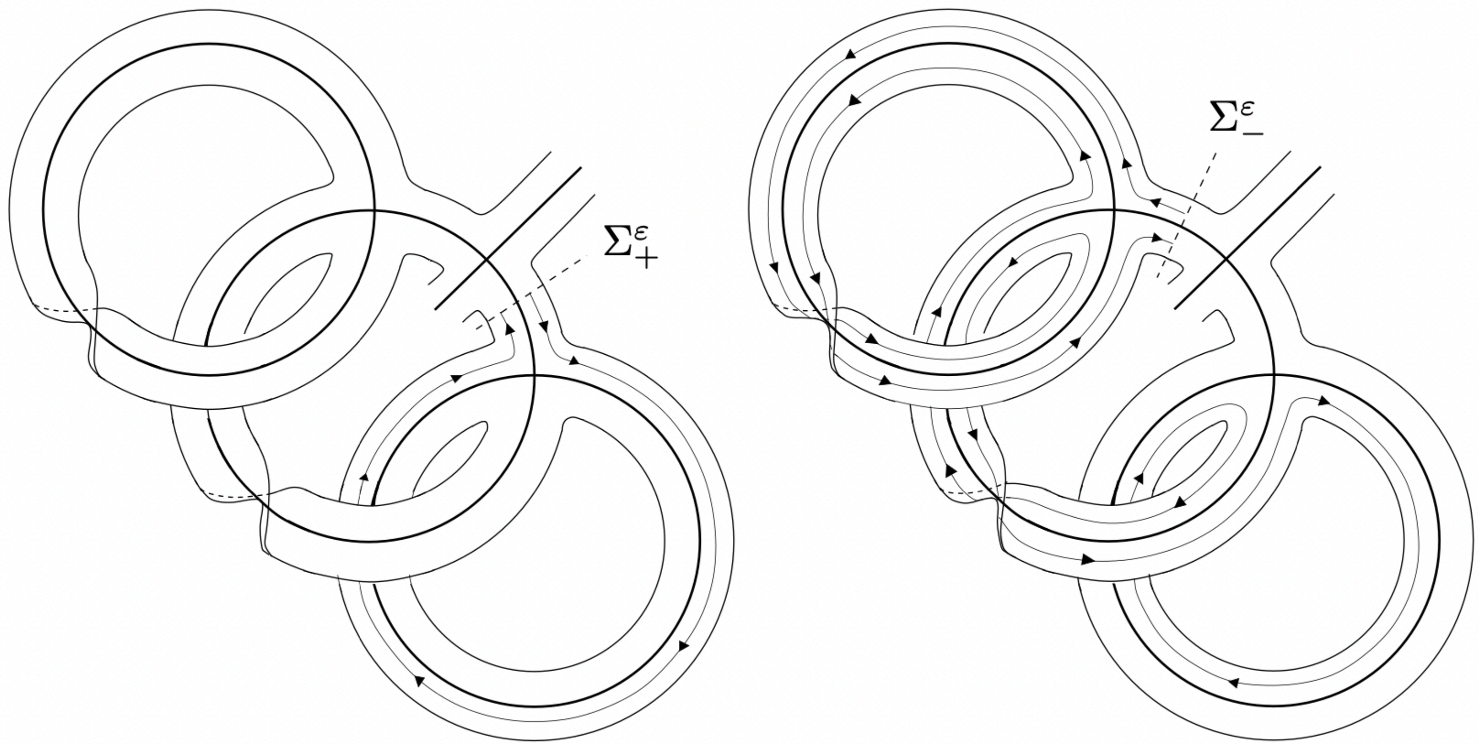
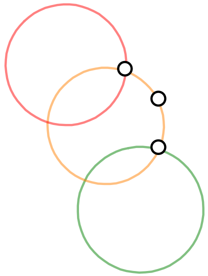
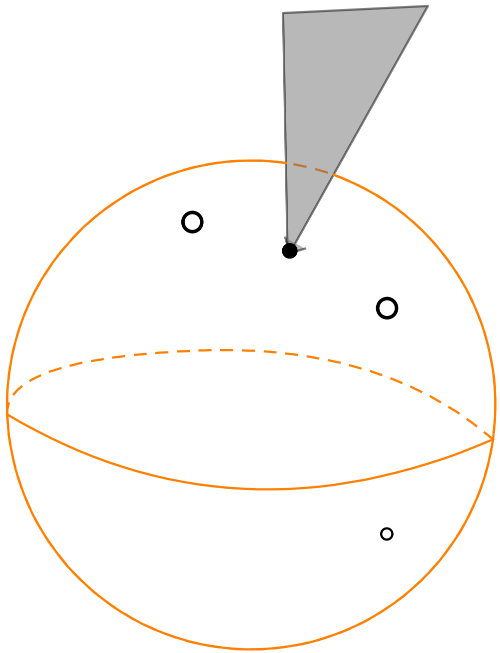


The foliation is now organized in a neighborhood of the exceptional divisor..


Σ_+^ε
Σ_-^ε

Can we recover the analytic moduli from the transverse behaviour?





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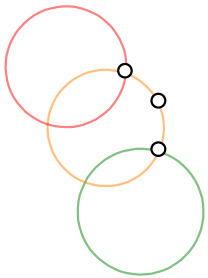

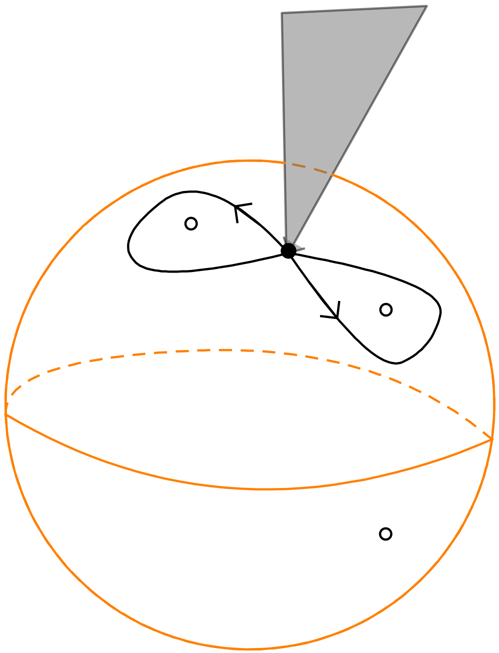

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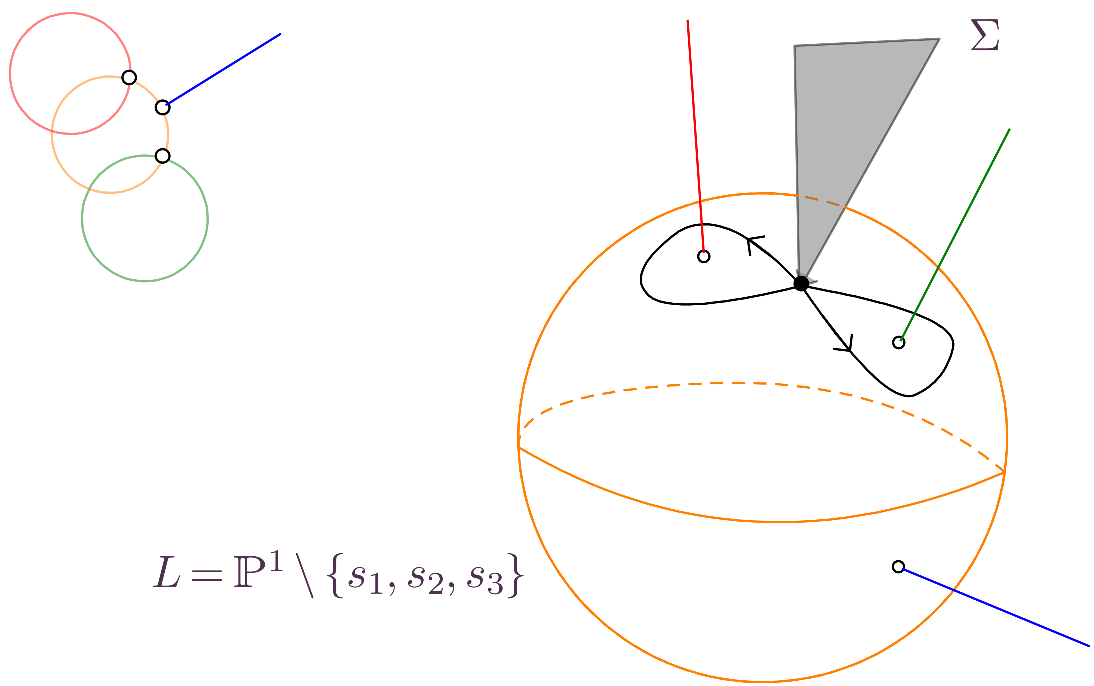

(Moussu) The vanishing holonomy $\mathrm{Hol}(\mathcal{F}, L) = \langle f, g \in \mathrm{Diff}(\mathbb{C}, 0) \mid \; f^2 = g^3 = \mathrm{id} \rangle$ characterizes the analytic class of the germ of foliation.

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Alternatively,

$$p \in \mathrm{Nilp}(M,\mathcal{F}) \iff \forall k \in \mathbb{N} \exists n \in \mathbb{N} : (\partial_k)^n = 0$$

where $\partial_k \colon J^k \to J^k$ is the induced derivation on the k^{th} jet.

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$$\partial(\langle f^i \rangle) \not\subset \langle f^{i+1} \rangle$$

In other words, for $E = (x_1 \dots x_k = 0)$,

$$\partial = \sum_{i=1}^{k} a_i \left(x_i \frac{\partial}{\partial x_i} \right) + \sum_{i=k+1}^{n} a_i \frac{\partial}{\partial x_i}$$

with $a_1, \dots, a_n \in \mathbb{C}\{x\}$ such that $\langle a_1, \dots, a_n \rangle \not\subset \langle x_i \rangle$, for each $i = 1, \dots, k$.

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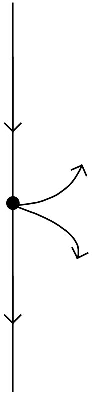

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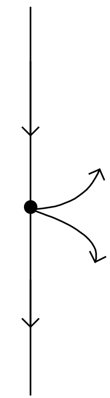

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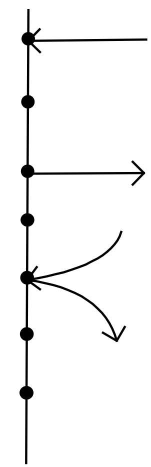

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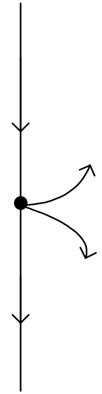

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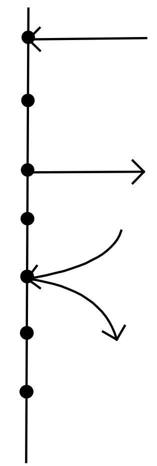

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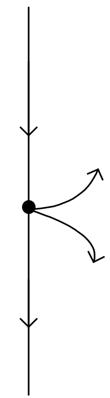

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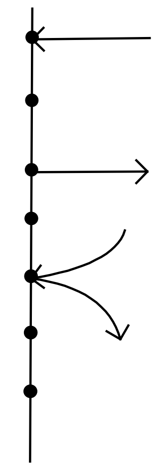

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How to compute the transform of a foliation by blowing-up?

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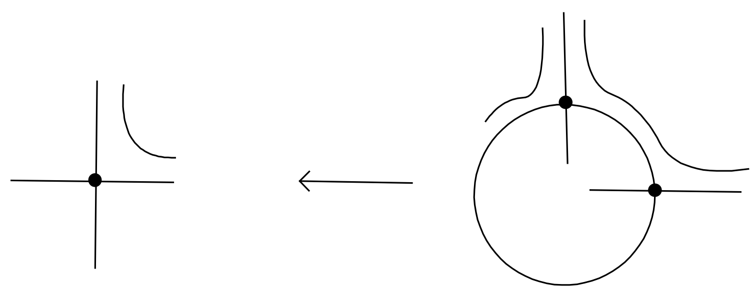

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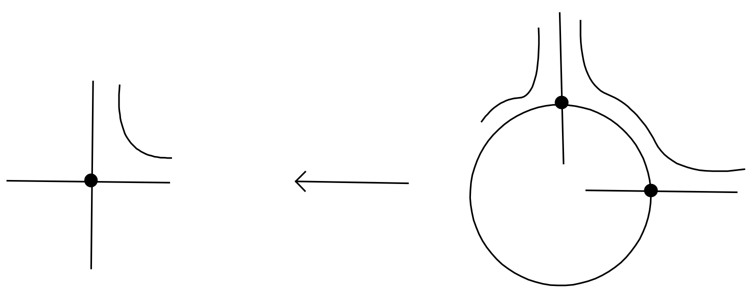

We can never get rid of saddle points...

Example: node

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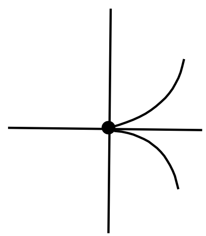
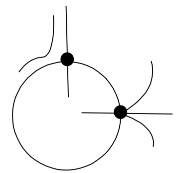


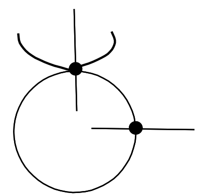
$\rho > 1$



$\tilde{\rho} = \rho - 1$

$$\rho = \rho_0 + \frac{1}{\rho_1 + \frac{1}{\cdots}}$$

$\rho < 1$



$\tilde{\rho} = \frac{1}{\rho} - 1$

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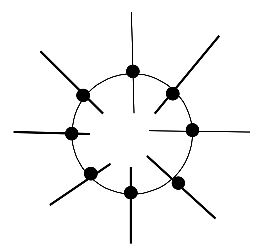

(dicritical situation)

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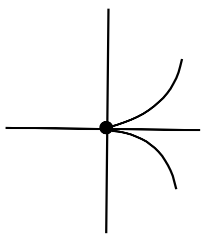

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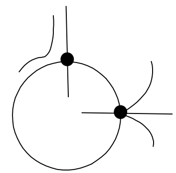

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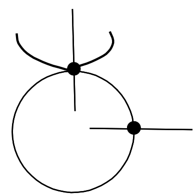

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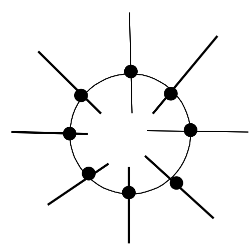

(dicritical situation)

We can never get rid of a node if $\rho\notin\mathbb{Q}$.

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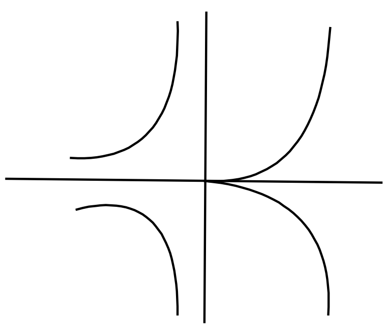
$$\text{First integral} \qquad h = (x^m y)\exp\left(\frac{1}{k\,x^k}\right)$$



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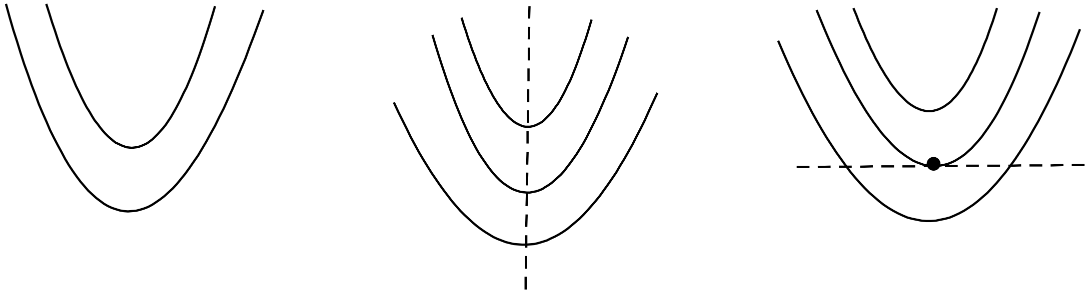

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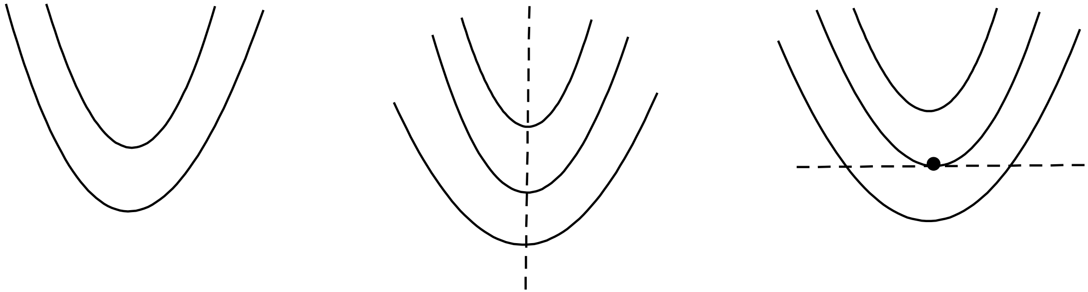

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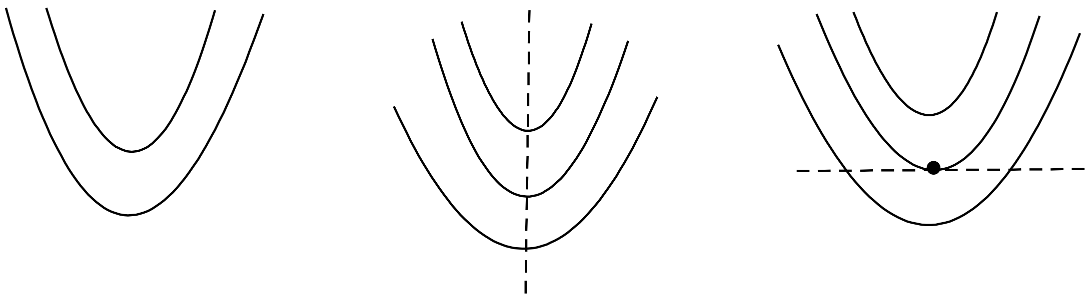

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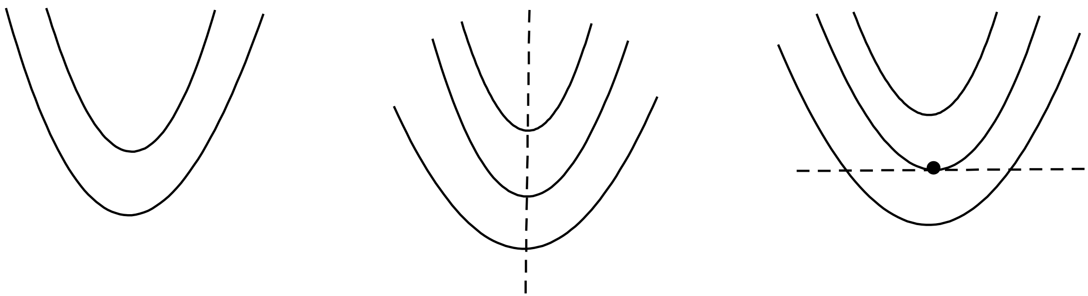

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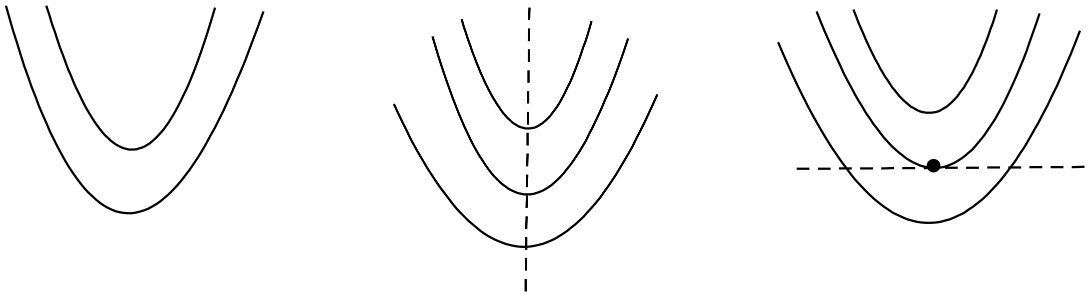

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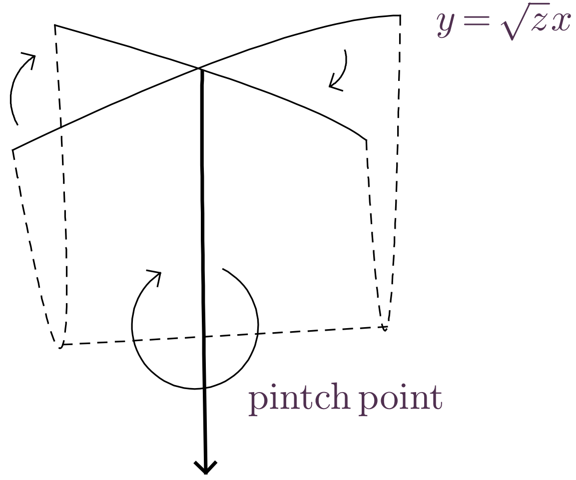

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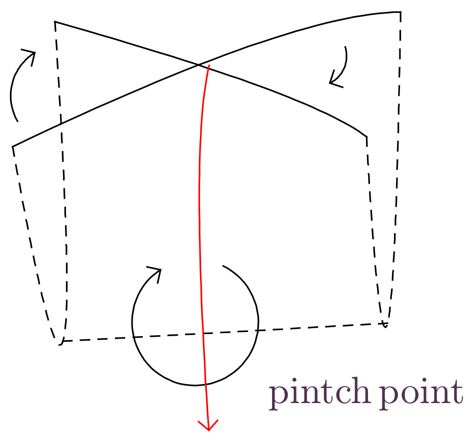

Formal expansion of the "handle"

$$y = \tau(z) = \sum \tau_n z^n, \quad \tau_n \sim \lambda\,(n!)^2$$

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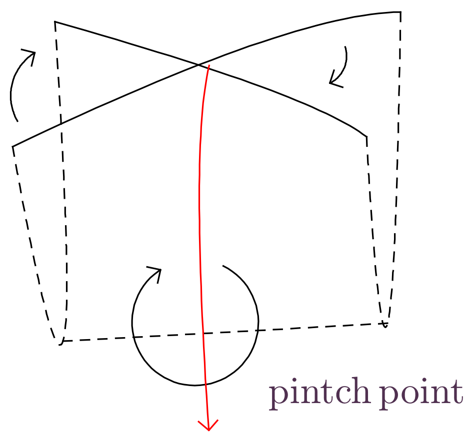

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We cannot take the handle as a blowing-up center because it is non-analytic.

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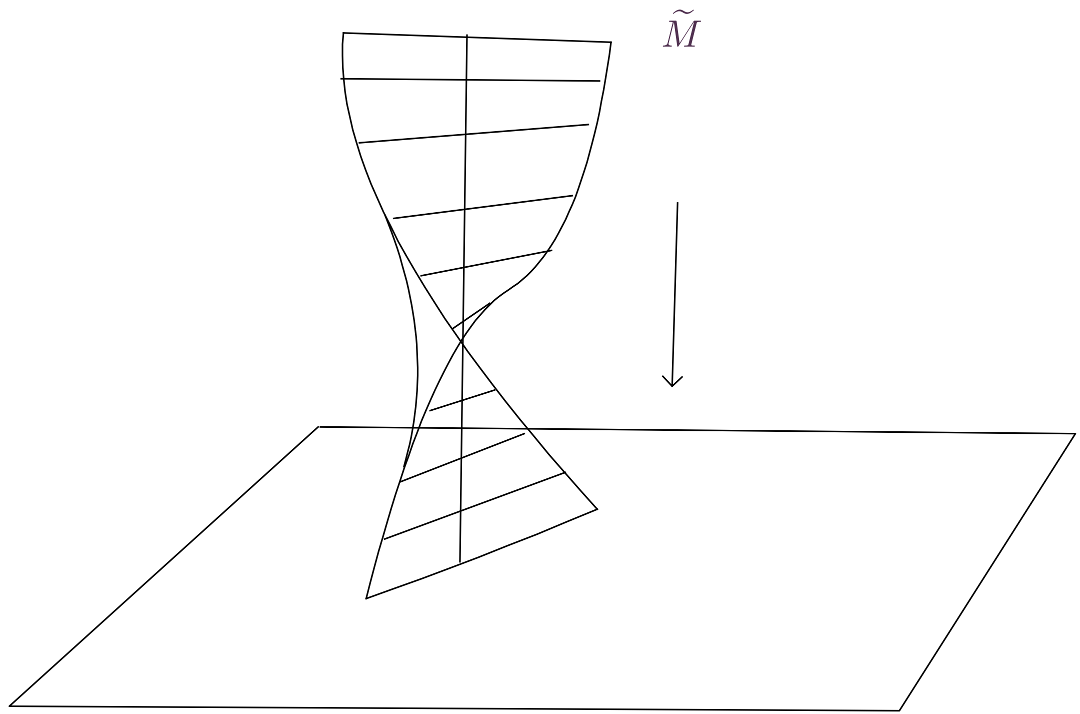
and the projection $\pi \colon \widetilde{M} \to \mathbb{C}^n$ is the weighted blowing-up of the origin in $\mathbb{C}^n$.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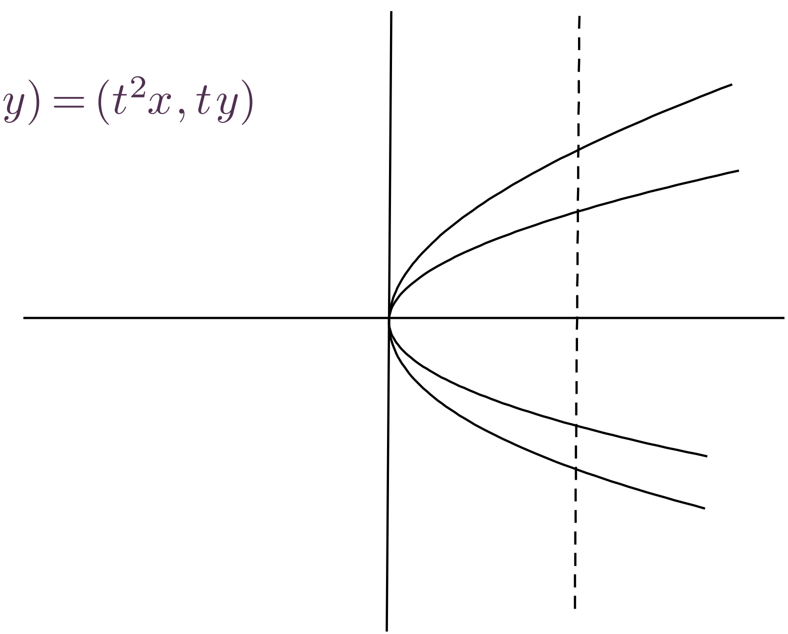
Example

$\omega = (2, 1)$

slice $\{x = 1\}$

$t \cdot (x, y) = (t^2 x, t y)$



$y \sim -y$

$\mathbb{Z}/2\mathbb{Z}$

We have to take into account the quotient by $\mathbb{Z}/2\mathbb{Z}$.

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We interpret $(y_{1,..}, y_n)$ as an **orbifold chart** on $\widetilde{M}$. Namely the affine space $\mathbb{C}^n$ equipped with an action of the cyclic group $\mathbb{Z}/\omega_1\mathbb{Z}$, defined by

$$y_1 \to \xi y_1, \qquad \text{For } 2 \leqslant k \leqslant n\text{:} \quad y_k \longrightarrow \xi^{-\omega_k} y_k$$

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An **orbifold** is a pair $(M, \mathcal{U})$ where M is paracompact Hausdorff topological space and $\mathcal{U}$ is a maximal orbifold atlas on M.

A sub-variety $Y \subset M$ is a **sub-orbifold** if for each point $p \in Y$ there exists a local chart (U, G, ϕ) such that $\phi^{-1}(Y \cap U)$ is a G-invariant submanifold of U.

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X is the quadratic cone.

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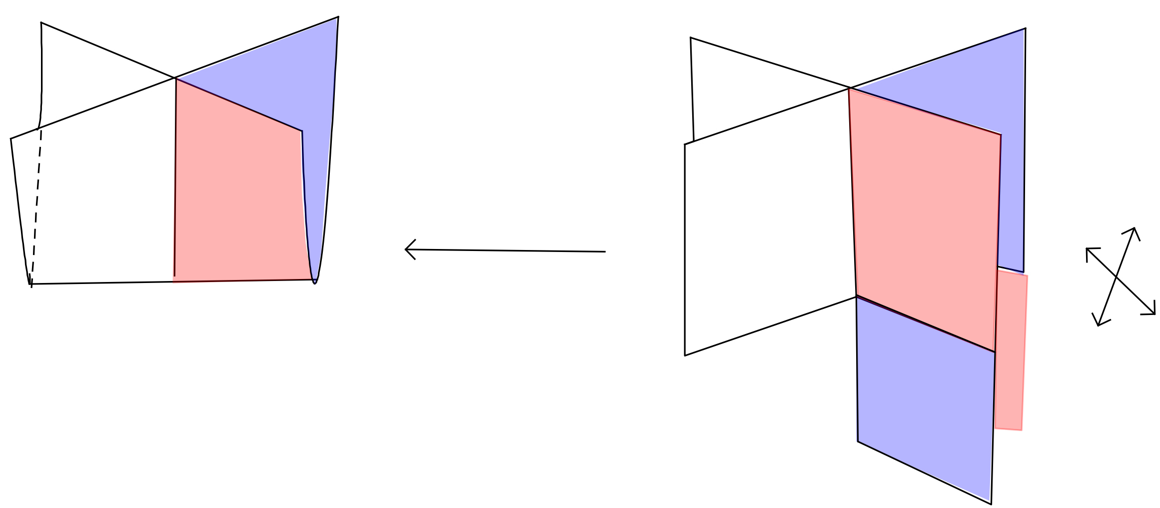

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The **spherical blowing-up** of $\mathbb{R}^n$ at the origin with weight ω is the real analytic map

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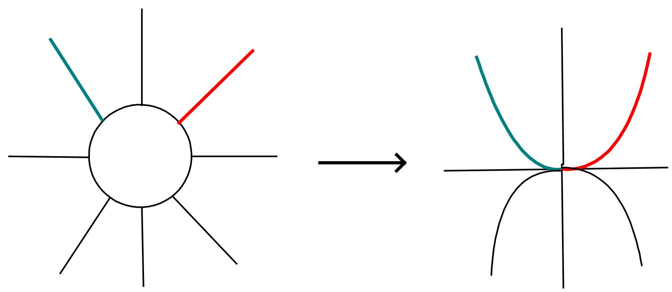

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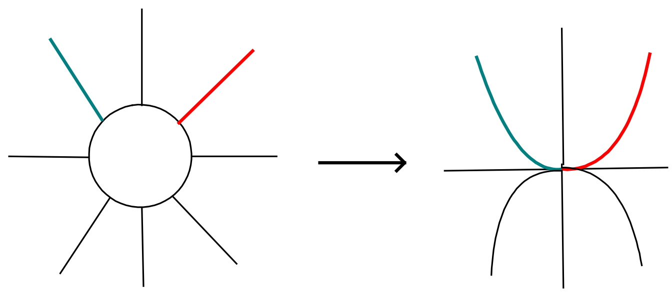

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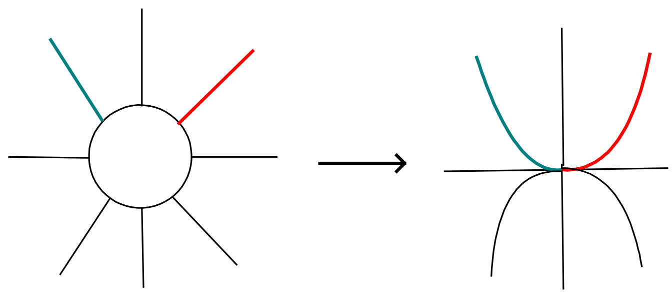

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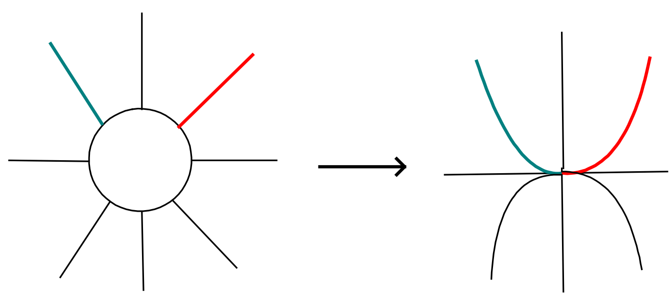

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(c.f. Melrose's "Analysis on manifolds with corners" - online)

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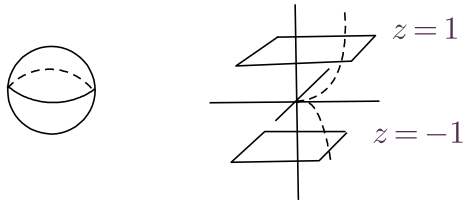

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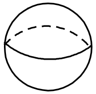

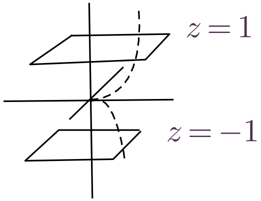

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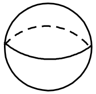

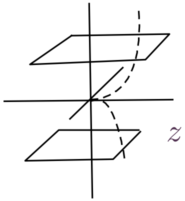

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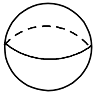

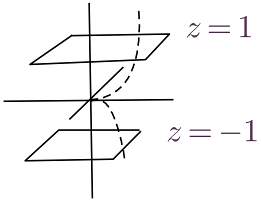

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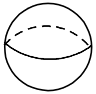

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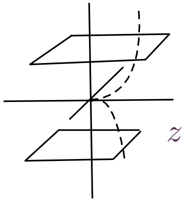

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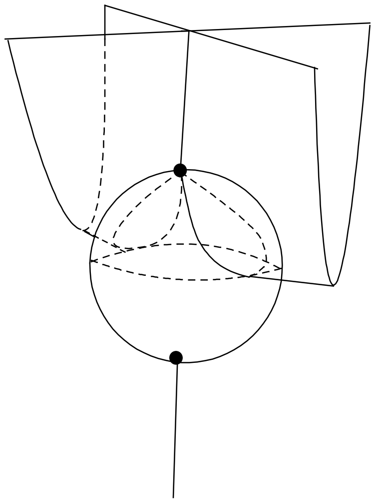

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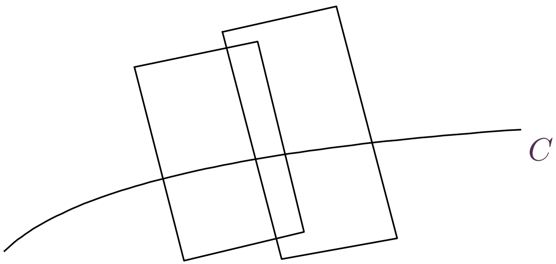

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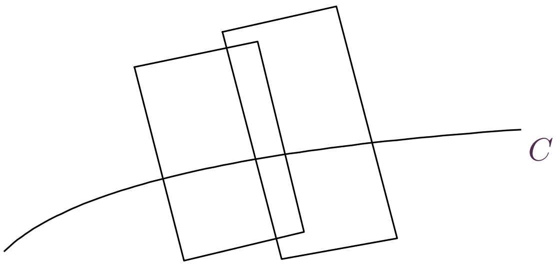

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such that $F_i\,F_j\subset F_{i+j}$ and such that, for each point p on the support, the stalk of this filtration coincides with a quasi-homogeneous filtration as defined above.

Example: $C = Z(x, y) \subset \mathbb{C}^3$

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More generally, all automorphisms obtained by integrating the Lie algebra (over $\mathbb{C}$) generated by

$$\left\{ x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^l\frac{\partial}{\partial y}, y^m\frac{\partial}{\partial x} \quad | \quad m \geqslant 1, l \geqslant \beta \right\}$$

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The solution curves of ∂ are precisely the orbits of the torus action $t \cdot (x, y) = (tx, t^n y)$.

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$$\partial = x\,y\left(x\,\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y}\right) + 3x\,y^{-1}\left(y\frac{\partial}{\partial y}\right) + x^2\Delta = x\left(x\,y\,\frac{\partial}{\partial x} + 3(1-y^2)\frac{\partial}{\partial y}\right) + x^2\Delta$$

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The divisor $\{x=0\}$ is contained in the nilpotent locus. We factor out x and write

$$\partial_1 = xy\frac{\partial}{\partial x} + 3(1-y^2)\frac{\partial}{\partial y} + \Delta_1$$

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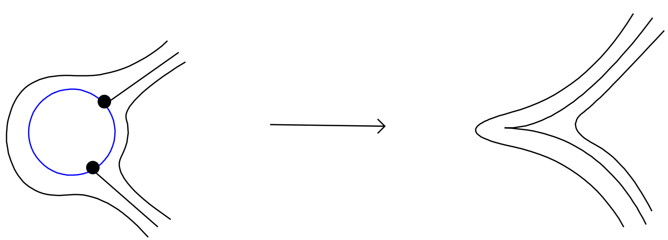

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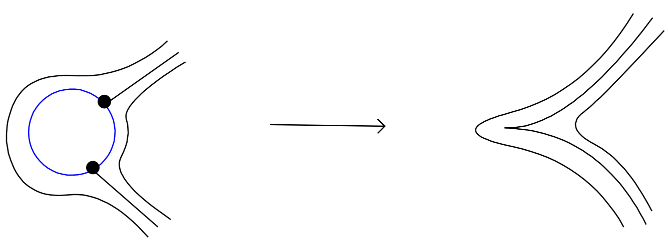

The resulting perturbation Δ is of quadratic order along E (does not change the eingenvalues)

Local symmetries of the foliated orbifold

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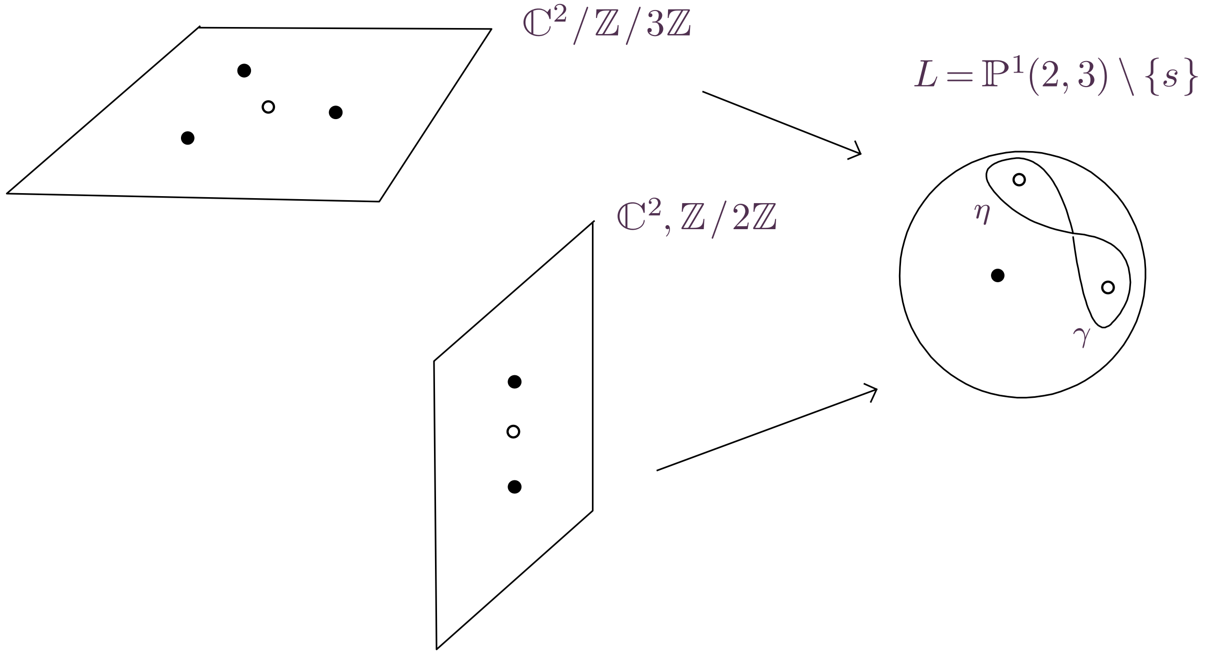

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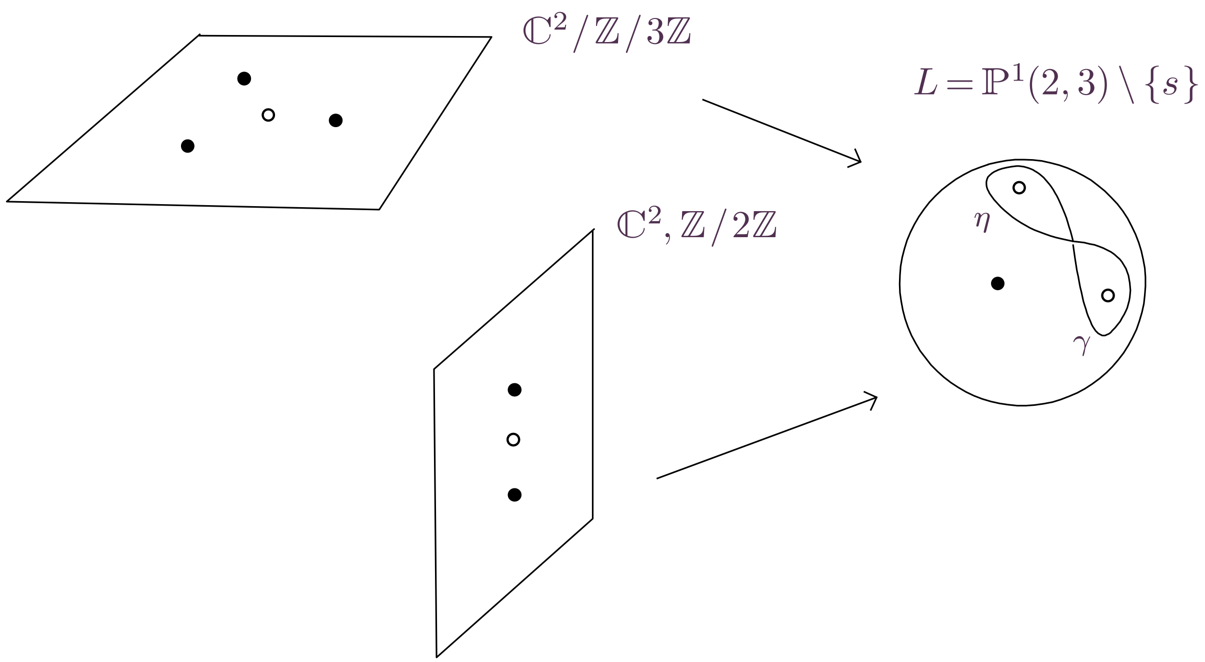

$$\pi_1(L) = \{\gamma, \eta, \rho |\, \gamma^2 = \eta^3 = 1, \rho = \gamma\eta\}$$

$$\partial_1 = x\,y\,\frac{\partial}{\partial x} + 3(1 - y^2)\frac{\partial}{\partial y} \quad \circlearrowleft \quad \mathbb{Z}/2\mathbb{Z}$$

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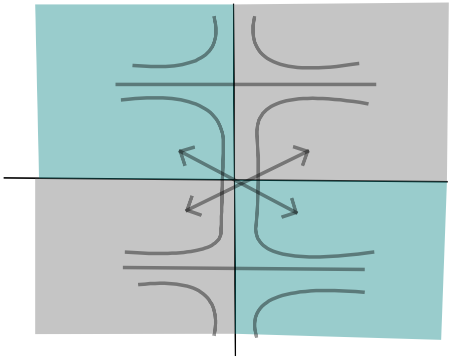

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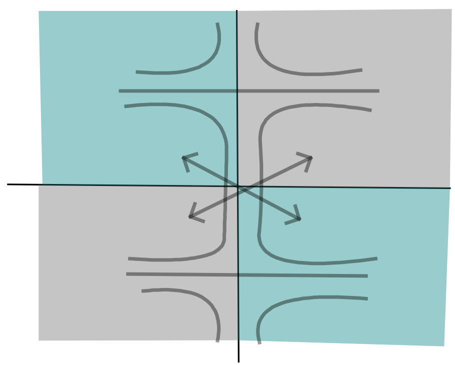
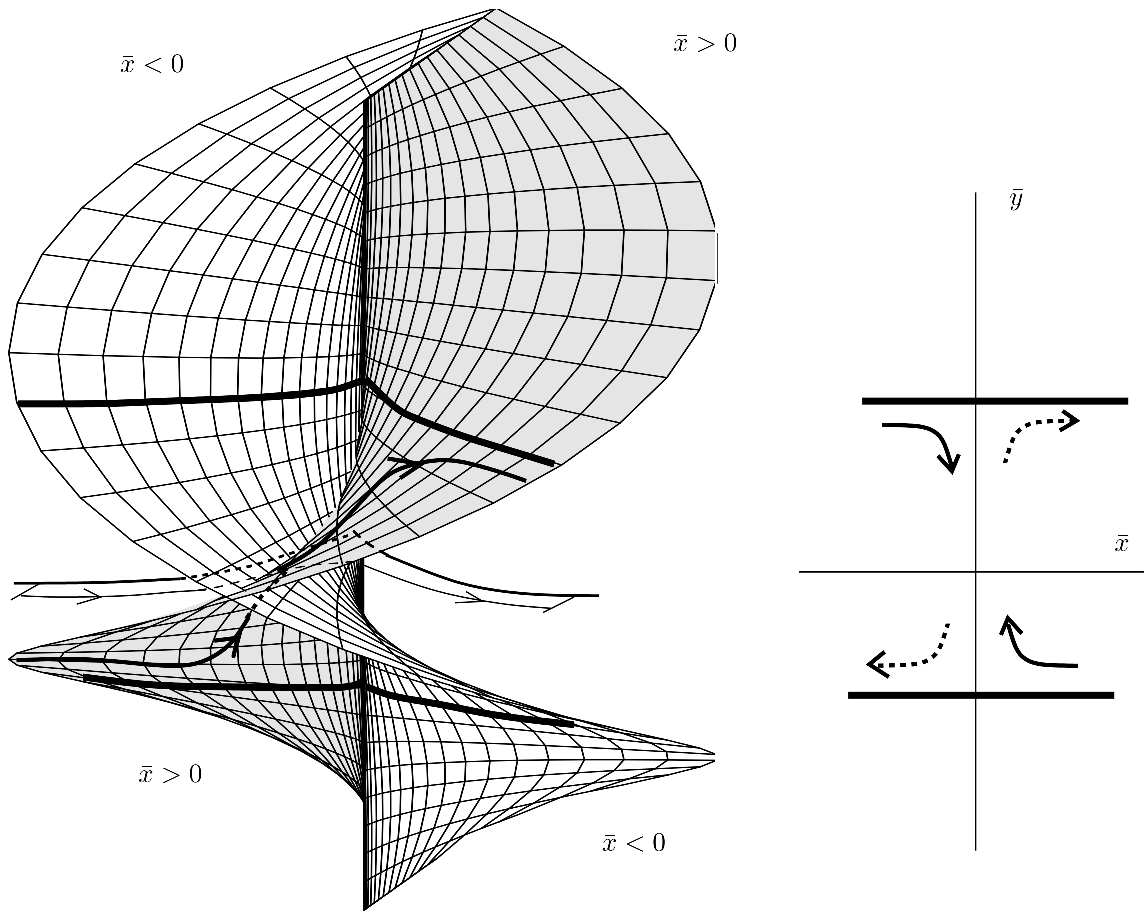


Other chart

$$\partial_2 = 2(1-x^3)\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y}$$

$$g\cdot x = \xi^{-2}\,x, \quad g\cdot y = \xi\,y, \qquad (\xi^3 = \mathrm{id})$$

$$g\cdot \partial_2 = \xi^2 \partial_2$$



$\bar{x} < 0$
$\bar{x} > 0$
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$\bar{x} < 0$
$\bar{y}$
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Elimination of nilpotent points in dimension two - Classical proof

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Suppose that the germ is singular. We can assume that $a, b \in \mathbb{C}\{x, y\}$ have no common factor and consider

$$m(0) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(a, b)} \geqslant 1, \quad \mu(0) = \min_k \{(J^k a, J^k b) \neq (0, 0)\}$$

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- If $l(0) \geqslant 2$ then $m(p_j) < m(p)$

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$$l = \begin{cases} \mu(a, b) & \text{if } \partial \text{ is non-dicritic} \\ \mu(a, b) + 1 & \text{if } \partial \text{ is dicritic} \end{cases}$$

- If $l(0) \geqslant 2$ then $m(p_j) < m(p)$
- If $l(0) = 1$ then this is a special case which has to be treated separately...

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$\mu=1, m=M$

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The center is obviously p, but we have to choose the appropriate quasi-homogeneous filtration…

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We can reorder the expansion and write

$$\partial = \sum_{k\in\mathbb{Z}^n} x^k L(\mu_k)$$

where, we recall, each $L(\mu)=\sum \mu_i x_i\frac{\partial}{\partial x_i}$, is an element of the $\mathbb{C}$-maximal toral subalgebra

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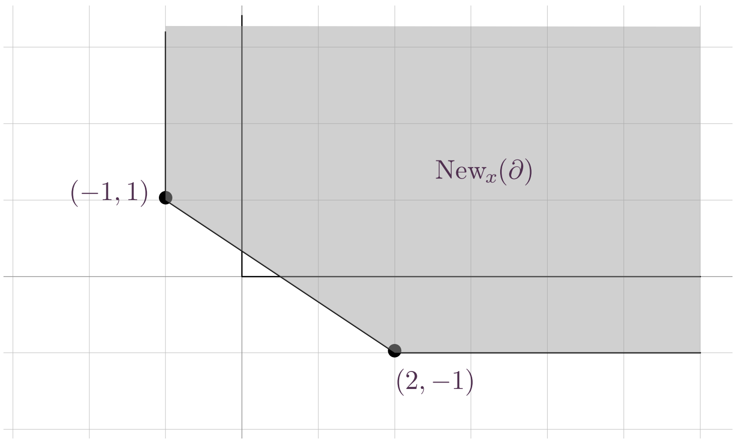

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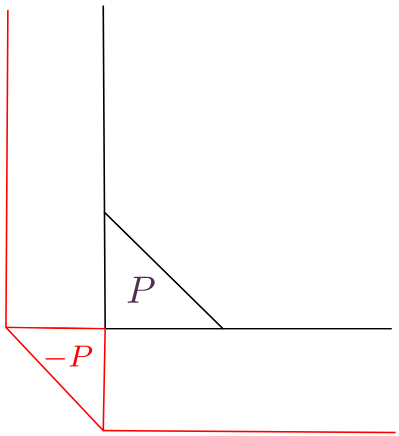

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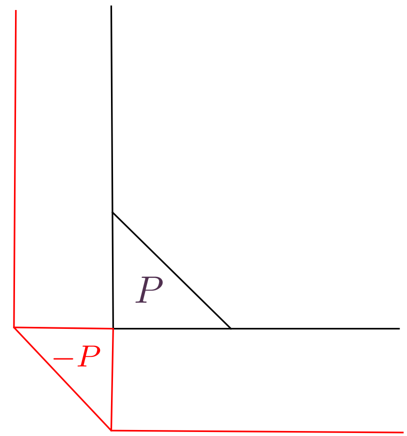

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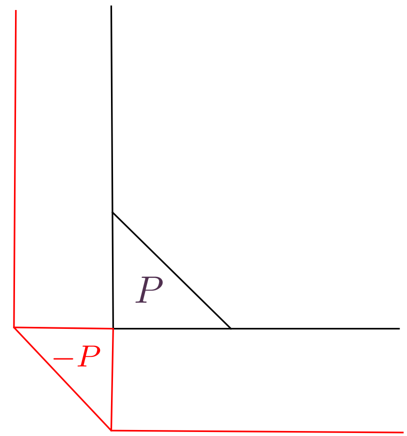

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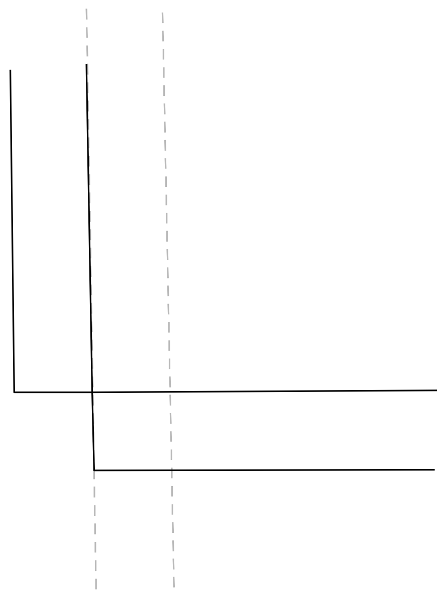

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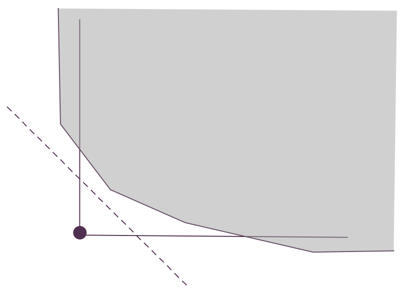
Proposition: $\partial \in \mathrm{Der}(\mathcal{O})$ is a nilpotent germ if and only if there exists a local system of coordinates $x = (x_1, \ldots, x_n)$ such that $0 \notin \mathrm{New}_x(\partial)$.



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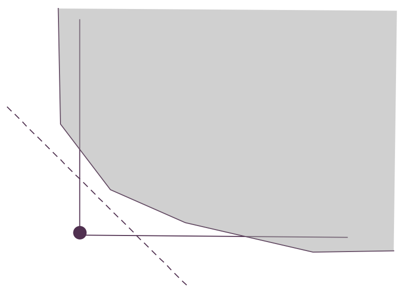

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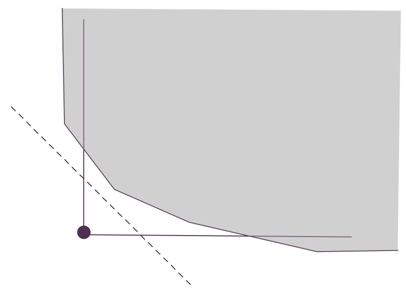

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$$\mathrm{New}_x(\partial) \subset H = \{\langle \omega, \cdot \rangle \geqslant \alpha\}$$

(indeed, if some $\omega_i < 0$ then for $v \in \mathrm{supp}_x(\partial)$, $\langle \omega, v + t e_i \rangle \to -\infty$ as $t \to +\infty$).

We can assume that $\omega \in \mathbb{Z}^n_{\geqslant 0} \setminus \{0\}$ and consider the quasi-homogeneous quasi-homogeneous graduation of $\mathcal{O}$ associated to the torus action $\lambda\colon \mathbb{C}^\star \to \mathrm{Aut}(\mathcal{O})$

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And, naturally

$$\partial \in \operatorname{Gr}_\alpha, f \in \operatorname{Gr}_\beta \Longrightarrow \partial f \in \operatorname{Gr}_{\alpha + \beta}$$

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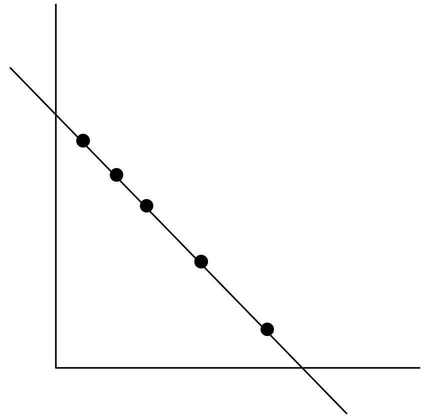

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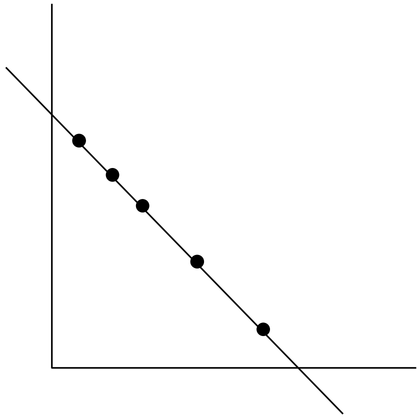

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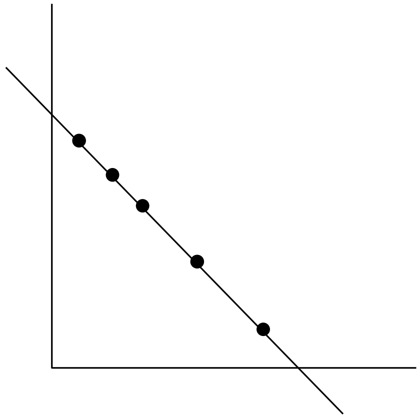

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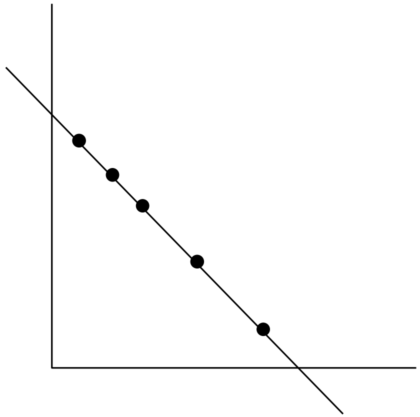

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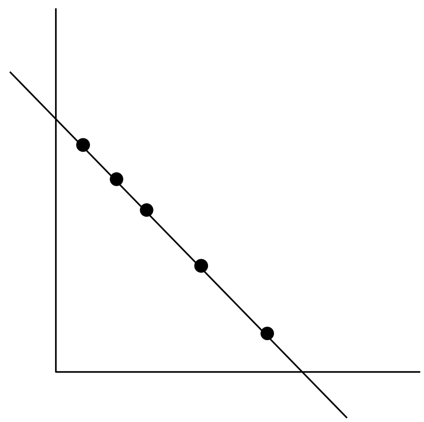

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and if if $f \in \mathrm{Gr}_{\geqslant \beta}(\mathcal{O}, \lambda)$ then $\partial^r(f) \in \mathrm{Gr}_{\geqslant r\alpha + \beta}(\mathcal{O}, \lambda)$.

As a consequence, for $\mathfrak{m} = \langle x_1, \ldots, x_m\rangle$ the maximal ideal, for each s there exists a $r \geqslant 1$ such that

$$\partial^r(\mathfrak{m}^s) \subset \mathfrak{m}^{s+1}$$

$(\text{because for } k \in \mathbb{Z}^n_{\geqslant 0}, |k| \geqslant \langle \omega, k\rangle / \max \{\omega_i\})$. Hence, ∂ is nilpotent.

Reciprocally, assume that ∂ is nilpotent. Then, $\partial(\mathfrak{m}) \subset \mathfrak{m}$ and $\partial_S = 0$. There exists a local coordinate system such that $\partial|_{J^1} = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & & & \\ & & & 1 & 0 \end{pmatrix}$, i.e. such that

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We now consider the weight-vector $\rho = (-n/2, \ldots, n/2)$, or any other rational vector satisfying.

$$\langle h, \rho \rangle = 0, \qquad \langle \rho, e_{i+1} - e_i \rangle > 0, \qquad e_i = (0, \ldots, 1, \ldots 0)$$

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(because for $|k| \geqslant 2$, $\langle \omega, k \rangle \geqslant 2 - n\varepsilon |k|$, and $\mathrm{New}_x(\partial)$ has finitely many vertices)

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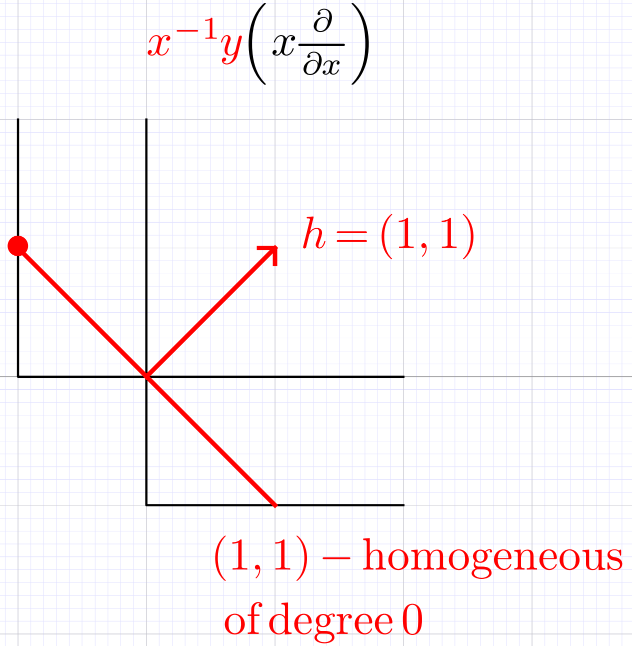

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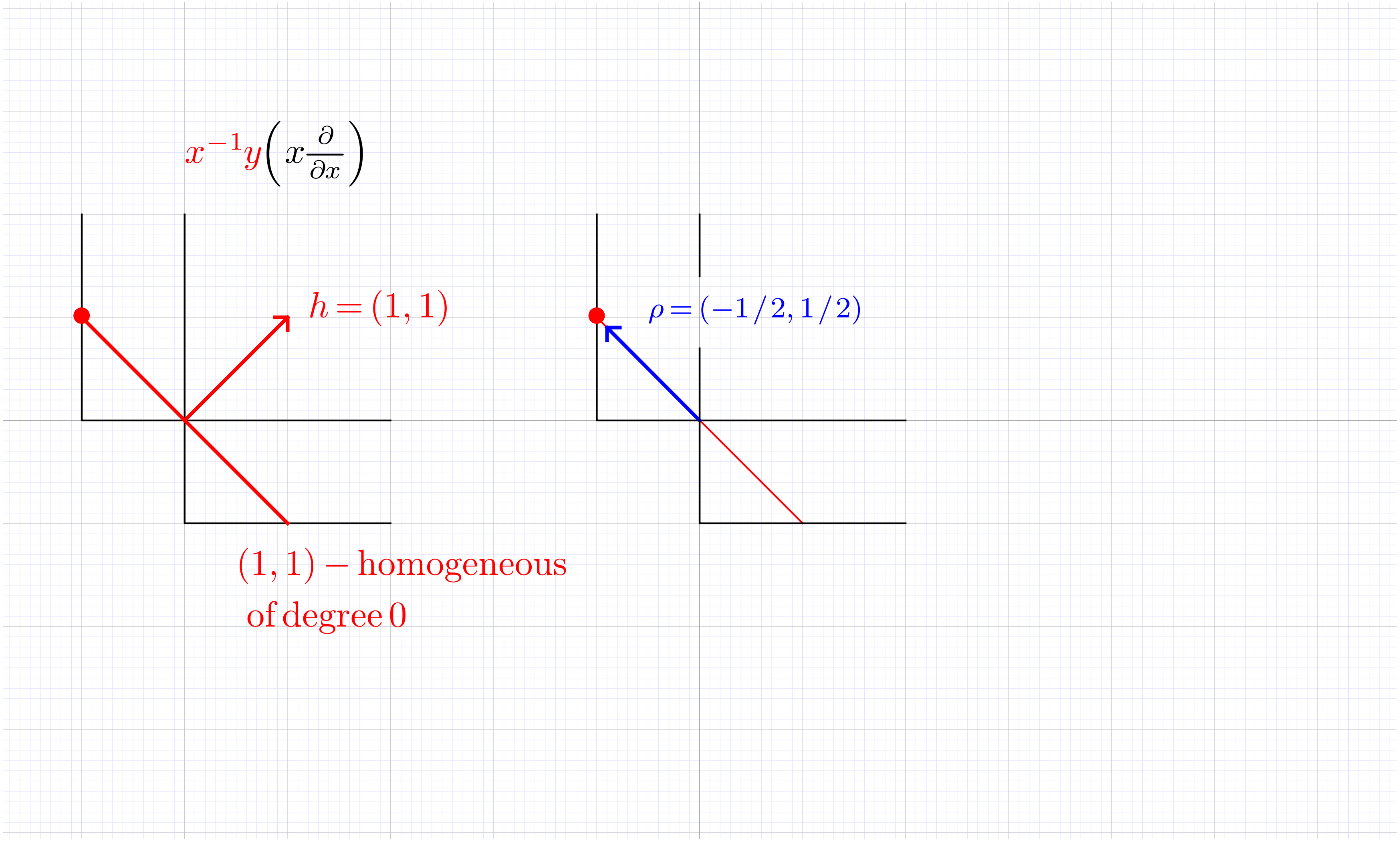

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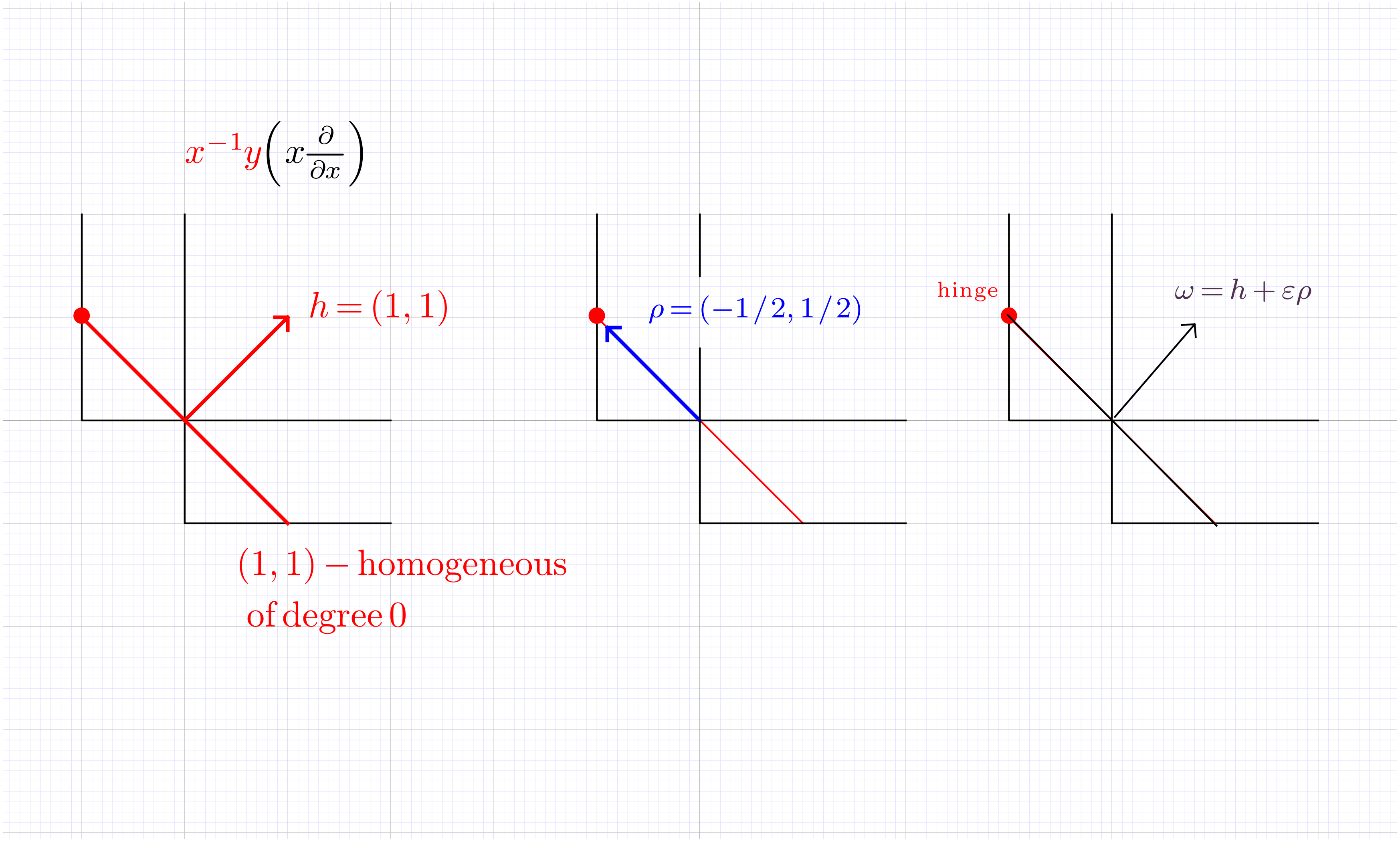

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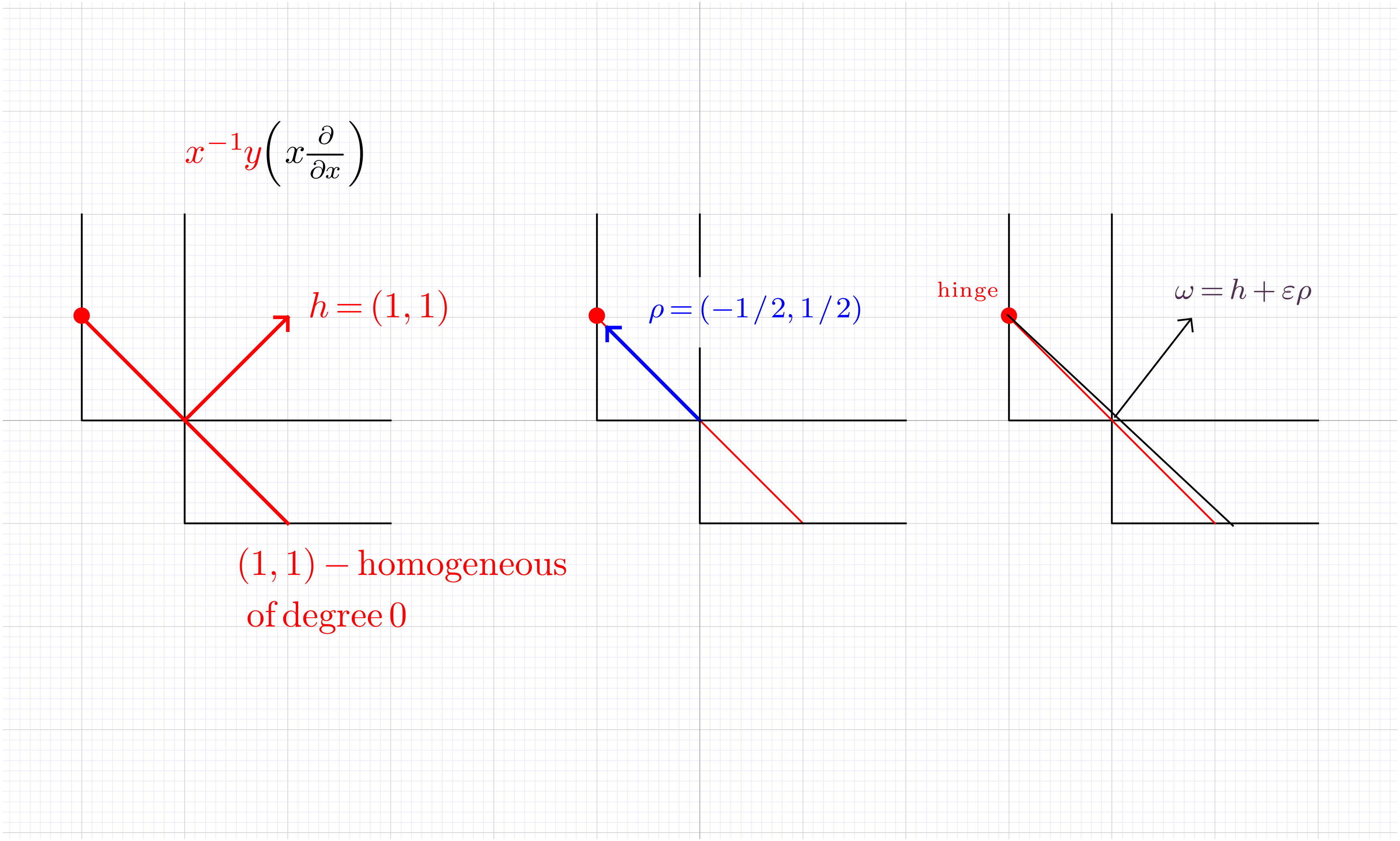

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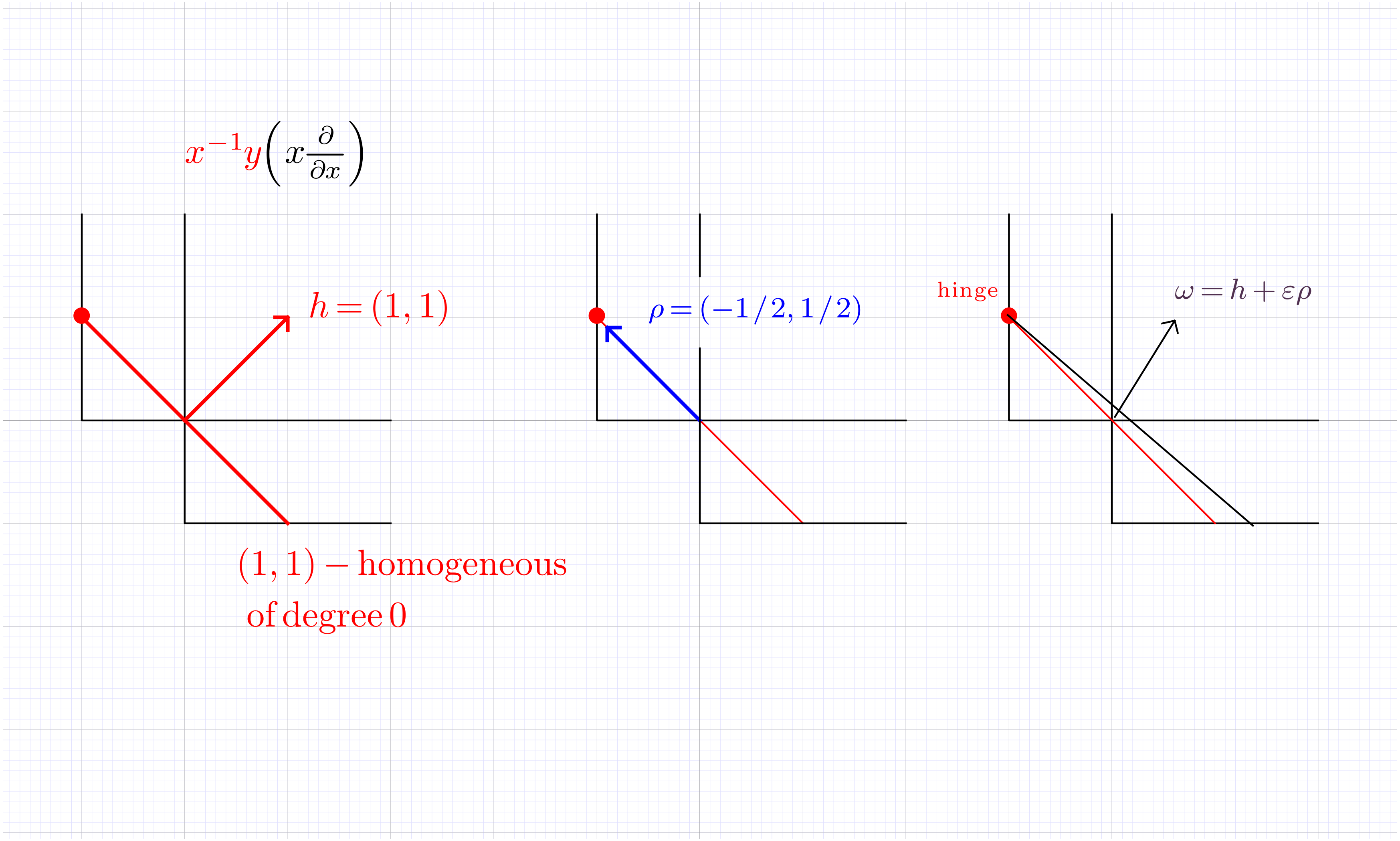

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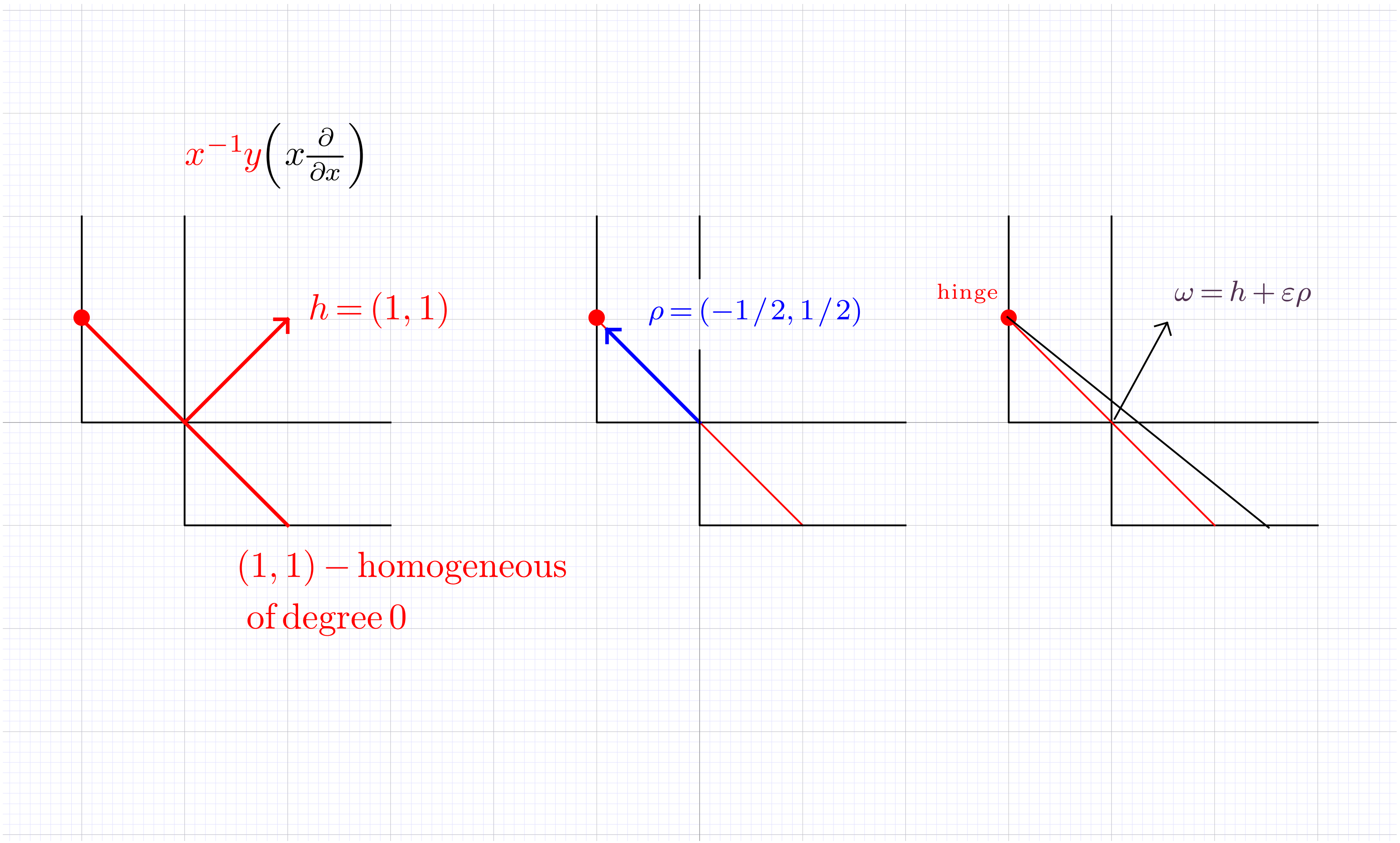

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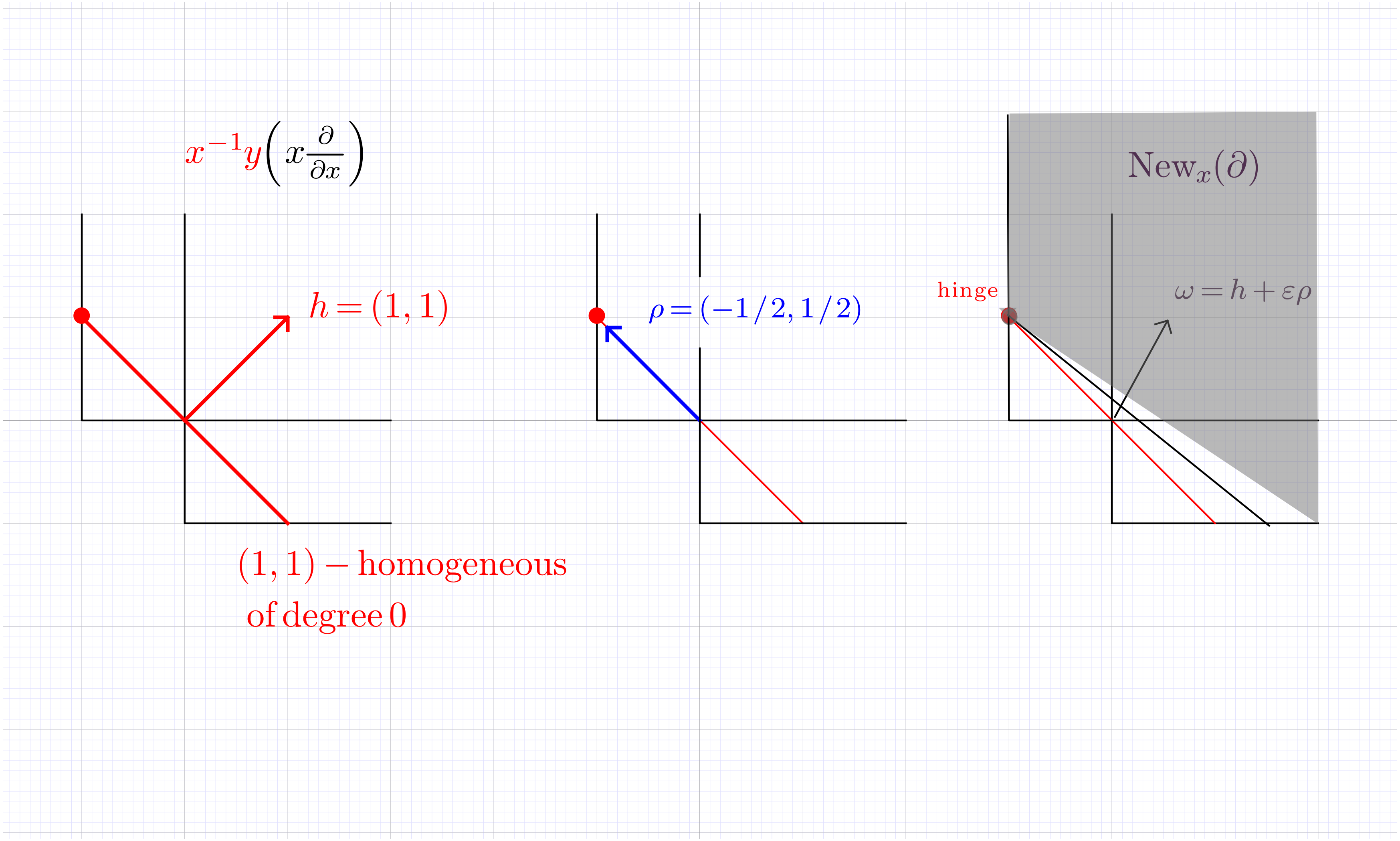

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Suppose that ∂ is elementary (i.e. not-nilpotent). Then, for all choices of coordinate systems $(x_1, \ldots, x_n)$,

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Indeed, the hypothesis means that either $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ or that $\partial(\mathfrak{m}) \subset \mathfrak{m}$ and $\partial_s \neq 0$.

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$$\partial(f) = \mu f$$

for some $\mu \neq 0$. Let Gr be the graduation Gr defined by an arbitrary one-parameter group λ : Then $f \in \mathrm{Gr}_{\geqslant \alpha}$ then and $\partial \in \mathrm{Gr}_{\geqslant \beta}$ implies that $\partial(f) \in \mathrm{Gr}_{\alpha+\beta}$.

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In fact: $\partial(\mathfrak{m}) \not\subset \mathfrak{m}$ if and only if

$$\Longleftrightarrow \exists i \in \{1, \ldots n\}: \quad -e_i = (0, \ldots, -1, \ldots, 0) \in \mathrm{New}_x(\partial)$$

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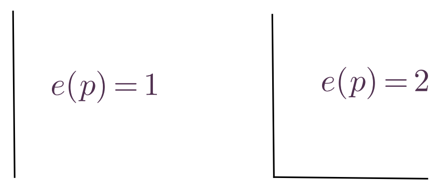

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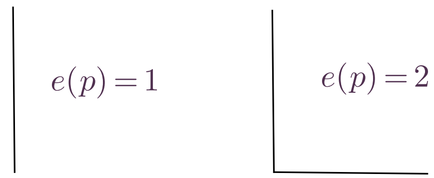

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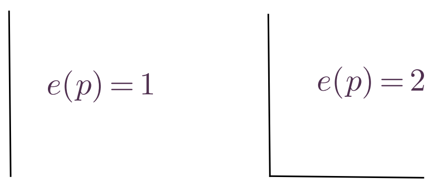

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The center is obviously p, but we have to choose the appropriate quasi-homogeneous filtration…