Lecture 8

Theorem There is a derivation $\partial : \mathbf{No} \to \mathbf{No}$ s.t.

(SD0) $\partial \omega = 1$ (SD1) $x > \mathbb{R} \Rightarrow \partial x > 0$

(SD2) $\ker(\partial) = \mathbb{R}$ (SD3) $\partial e^f = e^f \partial f$

(SD4) (f_i) summable $\Rightarrow \partial\left(\sum f_i\right) = \sum \partial(f_i)$

Constr: • λ log-atomic $\Rightarrow D(\lambda) = \frac{\prod_{\nu<\omega} \log_\nu(\lambda)}{\prod_\alpha \log_\alpha(\omega)}$ for α ...

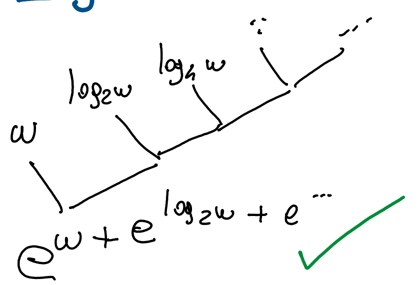
- $\mathcal{P}(x) := \{$paths of $x\} = \{(P_n)_n : P_0$ term of x, P_{n+1} of exp of $P_n\}$
- $\partial_D (P_n)_n := P_0 \cdots P_{\ell-1} \cdot \partial P_\ell$ if P_ℓ log-atomic; 0 otherwise
- $\partial x := \sum_{P \in \mathcal{P}(x)} \partial_D P$

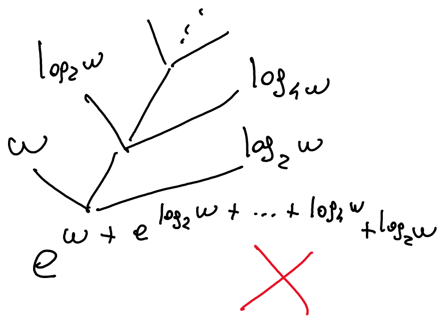
Thm "no Xmas trees" / every path is eventually rightmost

Exercise (*) Prove $\partial x = 0 \iff x \in \mathbb{R}$ (SD2)

- $x \in \mathbb{R} \Rightarrow \mathcal{P}(x) = \emptyset$, so $\partial x = \sum_{P \in \mathcal{P}(x)} \partial_D P = 0$
- $x \notin \mathbb{R} \Rightarrow$ take dominant path: P_0 $\preccurlyeq$-maximal not in $\mathbb{R}$, P_1 $\preccurlyeq$-maximal, etc. ① Prove $\exists \ell : P_\ell$ is log-atomic
- ② Prove $\partial_D P > \partial_D Q$ for any other $Q \in \mathcal{P}(x)$
- ③ Deduce (SD1) as well.

Why no Xmas trees?





Def Chopping $x = \sum_{i<\alpha} r_i e^{\gamma_i}$ (in N.F.!) means:

- drop a tail: $\sum_{i<\beta} r_i e^{\gamma_i}$ for some $1 \leq \beta < \alpha$; or
- $\alpha = \beta+1$: replace r_β with $\operatorname{sign}(r)$, maybe chop γ_β.

  (but: the last term must remain last)

Thm y chop of $x \Rightarrow y <_s x$

In particular, 'y chop of x' is well-founded.

Exercise x cannot be chopped $\Leftrightarrow$ $x \in \pm 1 \cup \pm \lambda_{No}^{\pm 1}$

Oversimplification If $(\partial_D P : P \in \mathcal{P}(x))$ not summable

$\Rightarrow$ can chop x to y s.t. $(\partial_D P : P \in \mathcal{P}(y))$ not summable.

Note In a Xmas tree, that family is not summable.

The theory of (No, ∂)

Alessandro: $\Psi = \{[\frac{\partial x}{x}] : x \in No^{>\mathbb{R}}\}$ has no inf.

(the inf would be $e^{-\sum_{\alpha \in On} \log_\alpha(\omega)} \notin No$!)

Cor (Rosenlicht) ∂ has asymptotic integration: $\forall f \exists g (\partial g \sim f)$

(+ FV Kuhlmann) ∂ is surjective.

Moreover:

- ∂ is λ-free: $\sum_{\beta<\alpha} \frac{\partial \log_\alpha \omega}{\log_\alpha \omega}$, for $\alpha \in On$, has no pseudo-limit
- ∂ is ω-free: $\sum_{\beta<\alpha} \left(\frac{\partial \log_\alpha \omega}{\log_\alpha \omega}\right)^2$, $\alpha \in On$, has no pseudo-limit

  See [ADH17, Ch. 11]

Trichotomy (Rosenlicht +) For an H-field (K, D), either

① there is asymptotic integration (Ψ has no inf)

② Ψ has an inf, but no minimum $\Leftarrow$ gap

③ Ψ has minimum $\Leftarrow$ grounded

surjective + functions exp, log $\Rightarrow$ Liouville closed

Thm (ADH) If D Liouville closed, $D(\mathcal{O}(1)) \subseteq o(1)$, and K is union of spherically complete grounded subfield, then

$$(K, D) \equiv (\mathbb{T}, \partial_{\mathbb{T}})$$

"d/dx" think $\mathbb{R}((\mathfrak{M}))$

Cor $(No, \partial) \equiv (\mathbb{T}, \partial_{\mathbb{T}})$

Proof For any $\mathfrak{M} < \Omega$ set, there is $\alpha \in On$ s.t. $\log_\alpha(\omega) < \mathfrak{M}^{\gamma}$!
$\mathbb{R}((\mathfrak{M} \cdot \log_\alpha(\omega)^{\mathbb{Z}}))$ is grounded & spherically complete $\square$

$\mathbb{R}\langle\langle\omega\rangle\rangle$ as functions

Call composition a function $\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times No^{>\mathbb{R}} \to No$:

1. $\omega \circ x = x$
2. $\left(\sum r_i e^{\alpha_i}\right) \circ x = \sum r_i e^{\alpha_i \circ x}$

Remark if it exists, it is unique

Example $\omega \circ x = x$, $e^\omega \circ x = x$, $(\omega^2 + \omega) \circ x = x^2 + x$.

Thm Such $\circ$ exists, and it satisfies:

3. $f \circ (g \circ x) = (f \circ g) \circ x$
4. $f < g \implies f \circ x < g \circ x$
5. $f \circ (x + \varepsilon) = f \circ x + \partial f \circ x \cdot \varepsilon + \partial^2 f \circ x \cdot \frac{\varepsilon^2}{2!} + \ldots$ when $\varepsilon \ll 1$
6. * $x < y$ & $\partial f > 0 \implies f \circ x < f \circ y$

Cor $\partial f \circ x = \lim_{\varepsilon \to 0} \dfrac{f \circ (x+\varepsilon) - f \circ x}{\varepsilon}$

Remark the ε's satisfying (5) lie in an open convex class

Beyond $\mathbb{R}\langle\langle\omega\rangle\rangle$

Is there a composition $No \times No^{>\mathbb{R}} \to No$? Add

7. $\partial(f \circ g) = (\partial f \circ g) \cdot \partial g$

Prop: no for 'simplest derivation':

$$\partial(\log_\omega(\omega)\circ\lambda) = (\partial\log_\omega(\omega)\circ\lambda)\cdot\partial\lambda$$
$$= \frac{1}{\prod_{n<\omega}\log_n(\lambda)}\cdot\partial\lambda$$

when $\lambda > \exp_n(\omega)$ for all n $= \frac{1}{\prod_{n<\omega}\log_n(\lambda)}\cdot\prod_{n<\omega}\log_n(\lambda) = 1$

Problem ∂ is aware of exp/log, not of $\exp_\omega/\log_\omega$.

Most recent progress: see Bagayoko-van der Hoeven-Kaplan.

Log$_\omega$: want $\log_\omega : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$

- $\log_\omega(\exp(x)) = \log_\omega(x) + 1$
- $\log_\omega(x+\varepsilon) = \log_\omega x + \frac{1}{x\log x\log_2 x\ldots}\cdot\varepsilon + \ldots$ if $\exists n\ \varepsilon \prec \frac{1}{\log_n(x)\log_n(x)\ldots}$
- $x < y \Rightarrow \log_\omega x < \log_\omega y$
- $\log_\omega x < \log_n x$ for all $n\in\mathbb{N}$.

Thm (Bagayoko-van der Hoeven-M): There is a 'simplest' such $\log_\omega$. It is surjective.
Its inverse $\exp_\omega$ is also 'simplest'.

Analogy. log expands monomials into series: $\log : \mathfrak{M} \to \mathbf{No}^{\uparrow}$
... but log does not expand log-atomics

- $\log_\omega$ expands log-atomics: $\log_\omega : \mathbb{A}_{\mathbf{No}} \to \mathbb{T}_r \simeq \mathbf{No}^{\uparrow}$
- $\log_{\omega^2}$ expands "$\log_\omega$-atomics" and so on.

See "Hyperseries"

Bag - vdH - Kap Construction of "hyperseries"

Bag - vdH Construction of $\log_\alpha$, $\exp_\alpha$ on No $\approx$ hyperseries.